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NONAUTONOMOUS ULTRAPARABOLIC EQUATIONS APPLIED TO POPULATION DYNAMICS

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Abstract. We prove the existence and positivity of solutions to nonautonomous ultraparabolic equations using a perturbation method. These equations come from population dynamics, namely from a fish larvae model.

1. Introduction

In the present work, we investigate a model for the dynamics of the fish larvae of certain species, namely the equation

$$\frac{\partial w}{\partial t} + \operatorname{div}(Vw) - \frac{\partial}{\partial x_3}\big(h\frac{\partial w}{\partial x_3}\big) + \mu w = 0, \tag{1.1}$$

where h, V, μ depend on the time and space variables.

The main characteristic of this equation is that it has mixed parabolic-hyperbolic type, due to the directional separation of the diffusion and convection effects. Such problem is called also nonautonomous ultraparabolic equation, that is parabolic in many directions. In a previous work by Ghouali and Touaoula [6], a simplified version of the model of the larvae had been investigated. It was assumed that the horizontal current V_i, $i = 1, 2$ is uniform throughout the water column, i.e. does not depends of the vertical variable x_3. Under this assumption, it was possible to uncouple the vertical and the horizontal components in the following sense: The study was restricted to each of the horizontal streamlines. Such restriction to a line reduces the functions of time horizontal components to functions of time alone so that the full model reduces on such a line to a diffusion equation in the vertical variable coupled with a first order growth equation. Our purpose in this work is to extend this method to the more realistic situation where the horizontal current depends of all its variables.

The lack of the coercivity of the operator can be handled by using a convenient perturbation argument. Monotone operator theory [7, p. 316] can be applied which gives us existence and uniqueness of the solution of the perturbed problem. After that, we establish the positivity of our solution. Then passing to the limit, in a suitable way, we obtain the existence of a solution of the main model.

The paper is organized as follows: Section 2 is devoted to recall some important results. In section 3 we formulate the perturbed problem. In section 4 we prove

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existence, uniqueness and positivity of the solution. Section 5 is devoted to prove the convergence result and the existence of the exact solution of the model.

2. Notation and preliminary results

We recall here some definitions and results that will be used latter. Let X be a separable and reflexive Sobolev space with norm $\|\cdot\|$ and its dual X' with norm $\|\cdot\|_*$. We denote by $\langle , \rangle$ the duality bracket of $X' \times X$. For $v \in L^2(0,T;X)$, we define the norm

$$\Big(\int_0^T \|v\|^2 dt \Big)^{1/2}.$$

We denote by $\mathcal{D}(0,T;X)$ the space of infinitely differentiable functions with compact support in $(0,T)$ and with values in X. We denote by $\mathcal{D}'(0,T;X)$ the space of distributions on $(0,T)$ with values in X. We set also $W(0,T;X,X') := \{v, v \in L^2(0,T;X), \frac{\partial v}{\partial t} \in L^2(0,T;X')\}$.

Definition 2.1. We say that an operator A from X to X' is monotone, if

$$\langle A(u) - A(v), u - v\rangle \geq 0 \quad \forall u, v \in X. \tag{2.1}$$

The operator A is strictly monotone if we have a strict positivity in (2.1) for all $u, v \in X$ and $u \neq v$.

Remark 2.2. If A is a linear operator, then the monotonicity is equivalent to

$$\langle Au, u\rangle \geq 0 \quad \forall u \in D(A).$$

Definition 2.3. Let A be a monotone operator from X to X'. We say that A is a maximal monotone operator if its graph is a maximal subset of $X \times X'$ with respect to set inclusion.

Lemma 2.4 ([7])**.** *Let L be a unbounded linear operator, with a dense domain $D(L)$ in X taking its values in X'. Then L is maximal monotone if and only if L is a closed operator and such that*

$$\langle Lv, v\rangle \geq 0 \quad \forall v \in D(L),$$
$$\langle L^*v, v\rangle \geq 0 \quad \forall v \in D(L^*).$$

where L^* is the adjoint operator of L.

Theorem 2.5 ([7])**.** *Let X be a reflexive Banach space. Let L be a linear operator of dense domain $D(L) \subset X$ and take its values in X'. Assume that L is maximal monotone and suppose that A is a monotone, coercive operator from X to X', i.e.*

$$\frac{\langle A(v), v\rangle}{\|v\|} \to \infty \quad \textit{as } \|v\| \to \infty.$$

Then, for all $f \in X'$, there exists $u \in D(L)$ such that $Lu + A(u) = f$.

Remark 2.6. If we assume in addition that the operator A is strictly monotone then there exists a unique solution $u \in D(L)$ such that $Lu + A(u) = f$.

Remark 2.7. One can easily see that in the case of a linear operator A, the coercivity implies strictly monotonicity.

Remark 2.8. Let u be a solution to the problem

$$\begin{gathered} \frac{\partial u}{\partial t} + Au = f, \\ u(0) = u_0, \end{gathered} \tag{2.2}$$

where A is a linear operator. We set $u = ve^{kt}$, $k \in \mathbb{R}$, then v is a solution of the problem

$$\begin{gathered} v'(t) + (A + kI)\, v(t) = f_1, \\ v(0) = u_0. \end{gathered} \tag{2.3}$$

Hence, proving existence, uniqueness and positivity of solutions of problem (2.3) is equivalent to prove the same properties to problem (2.2). Throughout this paper we will deal with problem (2.3), where k is a real constant that we will choose later.

Remark 2.9. We consider two Hilbert spaces V,H with $V \hookrightarrow H$, the continuous injection $\hookrightarrow$ having dense image in H. Then we can identify H with its dual H', and therefore

$$V \hookrightarrow H \hookrightarrow V'.$$

From Remarks 2.8 and 2.9 we obtain the following Lemma.

Lemma 2.10 ([3]). *For $u_0 \in H$ there exists v in $W(0, T; V, V')$, such that $v = u_0$ in H. Thus $w = u - v$, solves the problem*

$$\begin{gathered} \frac{\partial w}{\partial t} + (A + kI)w = f_2, \\ w(0) = 0, \end{gathered} \tag{2.4}$$

where u is solution of problem (2.3)*.*

Therefore, we will consider the case where $u_0 \equiv 0$.

3. The model

Our model takes into account both the physical and biological effects. For the physical part, the model stresses two main factors: 1) Transport entailed by the currents: The currents are computed using Navier-Stokes equations and are introduced in the equations of the larvae as functions of space and time with sufficient regularity to allow existence and uniqueness of stream lines. 2) Vertical diffusion induced by vertical mixing in the upper part of the water column. For the biological part the main parameters are a function which gives the instantaneous rate of progression within the stages from the egg fertilization to the end of the yolk-sac period.

The model is expressed in a generality which encompasses a large variety of situations. The motivation for this work is the study of the dynamics on the Bay of Biscay anchovy; that is, a region of the Atlantic ocean close to the French coast, bordered eastward by the continental shelf. The Bay of Biscay goes from the Northern Spanish coast up to about 46° in "latitude". In this region at the end of May, a thermocline establishes. The top of the thermocline is roughly at the same distance z_{therm} from the surface. The thermocline divides the water column into three regions: The upper part, from the surface to z_{therm} meters deep, is the so called mixed layer and its where the larvae grow. Below this there is the thermocline, a rather thin layer where the temperature loses rapidly a few degrees, water surface

$$z_{\text{therm}}\left\{\begin{array}{l}\text{mixed layer,}\\ \text{where larvae grow}\end{array}\right.$$

thermocline

another mixed layer

FIGURE 1. Water column divided into three regions

and the vertical mixing coefficient is negligibly small. Below the thermocline is another well mixed layer where the temperature changes very little with depth. This region is of no concern to us for this study. We will be confined to the mathematical issues related to the above model, and we study only the upper layer, the mixed layer of the water column; see Figure 1.

The domain under consideration is $\Omega = D \times (0, z^*)$, where D is an open subset of the surface, that is D is a portion of the plane, and z^* is the distance from the surface to a region above the thermocline.

We denote by Q the product space $\Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$ the boundary of Q. The state variable for the dynamics of the larvae is the density of larvae. For the part of the larval cycle which goes from fertilization to the end of stage, the density $w = w(t, s, P)$, where s denotes the position within the stages, which we take specifically of the Bay of Biscay anchovy in $[1, 12[$ see for example [1] and $P = (x_1, x_2, x_3)$ represents a generic point in the physical space.

The region of observation is assimilated to the product of the horizontal plane and a vertical line. The origin is a point of the surface in the sea, the x_1 axis is oriented westward, the x_2 axis is oriented northward, and the x_3 axis is oriented downward. Of course t is the chronological time. w is a density with respect to the stage and the position. The larvae are characterized by their density, that is to say, at each time $t \in [0,T]$, where T is the maximal time of observation, $w(t, s, P)$ can be thought of as the larvae biomass per unit of volume evaluated at the point P, at that time. The full model is as follows

$$\begin{gathered}
\frac{\partial w}{\partial t} + \frac{\partial (fw)}{\partial s} + div\,(Vw) - \frac{\partial}{\partial x_3}(h\frac{\partial w}{\partial x_3}) + \gamma w = 0, \\
w = 0, \quad \text{in } \Sigma \\
w(t, 1, x, y, z) = l_0(t, P).
\end{gathered} \tag{3.1}$$

The significance of the parameters in the model is as follows:
The **velocity** vector $V(t, P) = (V_1(t, P), V_2(t, P), V_3(t, P))$ describes the sea current which is supposed to be known.
The **mixing coefficient** $h = h(t, P)$ gives the diffusion rate, supposed to be essentially vertical.
The **growth function** is the main biological parameter, $f(t, s)$, which gives the instantaneous rate of progression within the stages from the egg fertilization to the end of the yolk-sac period. For the principle of determination see [8, 1].
The **mortality of larvae** is modelled by the expression $\gamma = \gamma(t, s, P)$.

The **Demographic boundary conditions** are given at $s = 1$, at any time during the spawning period, the variable s takes its values in the interval $[1, 12[$, where $s = 1$ corresponds to the newly fertilized eggs, and $s = 12$, to the end of the yolk sac period.
The **boundary condition** is zero, we assume that there is no larvae on the boundary.
The **Time of observation** is restricted to time interval when the larvae remains in the domain Ω.

4. Existence, uniqueness and positivity of solution of the perturbed problem

The objective of this section is to study existence, uniqueness and positivity of solution of the associated perturbed problem (4.3). For this, we start by using the method of characteristics to reduce the number of variables. We assume that

(H1) f is in $C^1((0,T) \times (1, 12))$.

We introduce the flow generated by the size growth, that is

$$\phi := \phi(\tau, t_0, 1),$$

and for each initial value $\tilde{\zeta} \equiv (t_0, 1)$, $\phi(\tau, \tilde{\zeta})$ is the solution of the equation

$$\Big(\frac{dt}{d\tau}, \frac{ds}{d\tau}\Big) = (1,\ f(t, s)), \tag{4.1}$$

that satisfies $t(0) = t_0$, $s(0) = 1$, since the theory of ordinary differential equations guarantees that a unique characteristic curve passes through each point $\tilde{\zeta}$. Let

$$t = T(\tau, t_0), \quad s = S(\tau, t_0),$$

be a solution of the characteristic system (4.1) emanating from the point $\tilde{\zeta}$. We assume that

$$\frac{\partial S}{\partial t_0} - f \neq 0$$

at $\tau = 0$. Without loss of generality we can assume that $t_0 = 0$, otherwise we replace V and h by those restrictions along the characteristic line. So to each $\tilde{\zeta}$, we have associated the following problem, see for instance [6],

$$\begin{gathered} \frac{\partial l}{\partial t} + \operatorname{div}(Vl) - \frac{\partial}{\partial z}\big(h\frac{\partial l}{\partial z}\big) + (\mu + k)l = 0, \\ l = 0, \quad \text{in } \Sigma, \\ l(0, P) = l_0(P), \end{gathered} \tag{4.2}$$

where μ is function of order zero, and k is a real constant that we will chose later. We will use a perturbation method to get a time dependent parabolic equation whose resolution will yield to the solution of equation (3.1). Namely we consider for the perturbed problem

$$\begin{gathered} \frac{\partial l}{\partial t} + \operatorname{div}(Vl) - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\big(a_i^{\varepsilon}\frac{\partial l}{\partial x_i}\big) + (\mu + k)l = 0, \\ l = 0, \quad \text{in } \Sigma, \\ l(0, P) = l_0(P), \end{gathered} \tag{4.3}$$

where

$$a_i^\varepsilon(t,P)=\begin{cases}\varepsilon & \text{if } i=1,2\\ h(t,P)+\varepsilon & \text{if } i=3.\end{cases}$$

Let

$$Lu_\varepsilon=\frac{\partial u_\varepsilon}{\partial t}+\operatorname{div}(Vu_\varepsilon)+(k+\mu)u_\varepsilon,$$

with

$$D(L)=\{v\in L^2(0,T;W_0^{1,2}(\Omega));\ \frac{\partial v}{\partial t}\in L^2(0,T;W^{-1,2}(\Omega)),v(0)=0\},$$

and

$$Au_\varepsilon=-\sum_{i=1}^3\frac{\partial}{\partial x_i}\big(a_i^\varepsilon\frac{\partial u_\varepsilon}{\partial x_i}\big),$$

defined by

$$\langle Au_\varepsilon,v\rangle=\sum_{i=1}^3\int_Q a_i^\varepsilon\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial v}{\partial x_i}\,dP\,dt,$$

for each $v\in L^2(0,T;W_0^{1,2}(\Omega))$. We now state the assumptions of this section.

- (H2) $h\in C^1(\bar Q))$, $V_i\in C([0,T]\times\bar\Omega)$, $i=1,2,3$ and $\gamma\in C([0,T]\times[1,12]\times\bar\Omega)$.
- (H3) $h\geq c_0>0$ in $[0,T]\times\bar\Omega$.

The main result of this section is the following theorem that gives conditions under which problem (4.3) has a unique positive solution.

Theorem 4.1. *Assume (H2)–(H3) hold. Let $l_0\in L^2(\Omega)$, be such that $l_0\geq 0$. Then problem* (4.3) *has a unique non negative solution $u_\varepsilon\in D(L)$.*

Proof. The main idea is to use Theorem 2.5. In the first step we will see that L is a closed operator with a dense domain; indeed, let u_n in $D(L)$ be such that $u_n\to u$ in $L^2(0,T;W_0^{1,2}(\Omega))$ and $Lu_n\to y$ in $L^2(0,T;W^{-1,2}(\Omega))$, hence

$$u_n\to u\quad\text{in } \mathcal{D}'(0,T;W^{-1,2}(\Omega))$$

and

$$Lu_n\to y\quad\text{in } \mathcal{D}'(0,T;W^{-1,2}(\Omega)).$$

It follows that

$$Lu_n\to Lu\quad\text{in } \mathcal{D}'(0,T;W^{-1,2}(\Omega)).$$

Therefore, $y=Lu$ and $u\in D(L)$. Hence L is a closed operator. It is not difficult to see that $\mathcal{D}(0,T;W_0^{1,2}(\Omega))$ is included in $D(L)$, then we deduce that $D(L)$ is dense in $L^2(0,T;W_0^{1,2}(\Omega))$. Concerning the monotonicity of L, we have for $u\in D(L)$,

$$\begin{aligned}
&\langle Lu,u\rangle\\
&=\int_0^T\langle\frac{\partial u}{\partial t},u\rangle dt+\int_Q(\operatorname{div}(Vu)u+(k+\mu)u^2)\,dP\,dt,\\
&=\frac12\int_0^T\frac{d}{dt}\|u(t)\|_*^2dt+\int_\Sigma(V,\eta)u^2d\sigma-\int_Q(V,\nabla u)u\,dP\,dt+\int_Q(k+\mu)u^2\,dP\,dt,\\
&=\frac12\|u(T)\|_*^2-\frac12\int_Q(V,\nabla u^2)\,dP\,dt+\int_Q(k+\mu)u^2\,dP\,dt,
\end{aligned}$$

with η is exterior normal, and $(,)$ is the scalar product. Hence by integration by parts we obtain that

$$\langle Lu, u\rangle = \frac{1}{2}\|u(T)\|_*^2 + \int_Q (k + \frac{1}{2}\operatorname{div}(V) + \mu)u^2\, dP\, dt,$$

choosing k large so that

$$k + \frac{1}{2}\operatorname{div}(V) + \mu \geq 0,$$

it follows that L is monotone for all $u \in D(L)$. In addition for $u \in D(L)$,

$$\begin{aligned}\langle Lu, v\rangle &= \int_0^T \langle \frac{\partial u}{\partial t}, v\rangle dt + \int_Q (\operatorname{div}(Vu)v + (k+\mu)uv)\, dP\, dt,\\ &= \int_0^T \langle u, -\frac{\partial v}{\partial t}\rangle dt + \langle u(T), v(T)\rangle + \int_Q (-(V, \nabla v) + (k+\mu)v)u\, dP\, dt,\end{aligned}$$

thus, the associated adjoint operator is

$$L^* v = -\frac{\partial v}{\partial t} - (V, \nabla v) + (k+\mu)v,$$

with

$$D(L^*) = \{v \in L^2(0,T; W_0^{1,2}(\Omega)); \frac{\partial v}{\partial t} \in L^2(0,T; W^{-1,2}(\Omega)), v(T) = 0\}.$$

The proof of monotonicity of L^* is similar to the one of L. Then L is a maximal monotone operator. It remain to see that A is coercive, indeed for $u \in L^2(0,T; W_0^{1,2}(\Omega))$ and applying the hypothesis on h, it holds

$$\langle Au, u\rangle = \sum_{i=1}^3 \int_Q a_i^\varepsilon |\frac{\partial u}{\partial x_i}|^2\, dP\, dt \geq M_\varepsilon \|u\|^2_{L^2(0,T; W_0^{1,2}(\Omega))}.$$

According to Theorem 2.5 and Remark 2.6, we get the existence of a unique solution $u_\varepsilon \in D(L)$ of the perturbed problem (4.3). Hence for all $v \in L^2(0,T; W_0^{1,2}(\Omega))$, we have

$$\int_0^T \langle \frac{\partial u_\varepsilon}{\partial t}, v\rangle dt + \int_Q (\operatorname{div}(Vu_\varepsilon) + (\mu+k)u_\varepsilon)v\, dP\, dt + \sum_{i=1}^3 \int_Q a_i^\varepsilon \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial v}{\partial x_i}\, dP\, dt = 0. \tag{4.4}$$

We prove now the positivity of the solution. We set $u = u^+ - u^-$, where u^+ and u^- are respectively the positive and negative part of u. Using u_ε^- as a test function in (4.4) and integrating on $(0,t)$, we get

$$\begin{aligned}&-\int_0^t \langle \frac{\partial u_\varepsilon^-}{\partial t}, u_\varepsilon^-\rangle dt - \int_0^t \int_\Omega (\operatorname{div}(Vu_\varepsilon^-) + (\mu+k)u_\varepsilon^-)u_\varepsilon^-\, dP\, dt\\ &-\sum_{i=1}^3 \int_0^t \int_\Omega a_i^\varepsilon |\frac{\partial u_\varepsilon^-}{\partial x_i}|^2\, dP\, dt = 0,\end{aligned}$$

integrating by parts two times, we obtain

$$-\frac{1}{2}\|u_\varepsilon^-(t)\|_*^2 = \int_0^t \int_\Omega (\frac{1}{2}\operatorname{div}(V) + \mu + k)(u_\varepsilon^-)^2\, dP\, dt + \sum_{i=1}^3 \int_0^t \int_\Omega a_i^\varepsilon |\frac{\partial u_\varepsilon^-}{\partial x_i}|^2\, dP\, dt;$$

hence, $-\frac{1}{2}\|u_\varepsilon^-(t)\|_*^2 \geq 0$. Then $u_\varepsilon^-(t) = 0$ for all $t \in (0,T)$. The proof is complete. $\square$

5. The exact solution

In this section we show that the perturbed solution defined in (4.4) tends to the desired solution of problem (4.2) in $L^2(Q)$ as ε tends to 0. Our main result is the following Theorem.

Theorem 5.1. *Let $l_0 \in L^2(\Omega)$ and consider u_ε the solution to problem (4.3), then u_ε converges weakly to u in $L^2(Q)$ where u is a distributional solution of the problem (4.2). In addition we have $\frac{\partial u}{\partial x_3} \in L^2(Q)$ and u satisfies*

$$\begin{aligned} &-\int_\Omega \int_0^T u \frac{\partial \phi}{\partial t} dx dt + \int_Q (-(V, \nabla\phi) + (\mu + k)\phi) u \, dP \, dt + \int_Q h \frac{\partial u}{\partial x_3} \frac{\partial \phi}{\partial x_3} \, dP \, dt \\ &= \int_\Omega l_0(P)\phi(0, P) dP, \end{aligned} \tag{5.1}$$

for all $\phi \in K$ where

$$K \equiv \{\phi \in L^2(0, T; W_0^{1,2}(\Omega)) : \frac{\partial \phi}{\partial t} \in L^2(0, T; W^{-1,2}(\Omega)) \cap L^2(Q),\ \phi(T) = 0\}. \tag{5.2}$$

Proof. By taking u_ε as a test function in (4.4), we obtain

$$\int_0^T \langle \frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon \rangle dt + \int_Q (\operatorname{div}(V u_\varepsilon) + (\mu + k) u_\varepsilon) u_\varepsilon \, dP \, dt + \sum_{i=1}^3 \int_Q a_i^\varepsilon |\frac{\partial u_\varepsilon}{\partial x_i}|^2 \, dP \, dt = 0, \tag{5.3}$$

integrating by parts and using the definition of u_ε, we deduce

$$\frac{1}{2}\|u_\varepsilon(T)\|_*^2 + \int_Q (k + \frac{1}{2}\operatorname{div}(V) + \mu) u_\varepsilon^2 \, dP \, dt + \sum_{i=1}^3 \int_Q a_i^\varepsilon \Big|\frac{\partial u_\varepsilon}{\partial x_i}\Big|^2 \, dP \, dt = \frac{1}{2}\|l_0\|_*^2.$$

Since $\operatorname{div}(V)$ and μ are bounded functions we conclude that $\|u_\varepsilon\|^2_{L^2(Q)} \le C$ and then there exists a subsequence called also u_ε such that $u_\varepsilon \rightharpoonup u$ weakly in $L^2(Q)$. Notice that, in the same way, we obtain that

$$\sum_{i=1}^3 \int_Q a_i^\varepsilon |\frac{\partial u_\varepsilon}{\partial x_i}|^2 \, dP \, dt \le C_1.$$

By letting $\varepsilon \to 0$, we obtain

$$\limsup_{\varepsilon \to 0} \int_Q h |\frac{\partial u_\varepsilon}{\partial x_3}|^2 \, dP \, dt \le C_1. \tag{5.4}$$

We claim that u is a solution of (4.2) in the sense of distribution. To proof the claim we consider $\phi \in \mathcal{C}_0^\infty(\Omega \times (0, T))$, then using ϕ as a test function in (4.3) we obtain

$$\begin{aligned} &-\int_\Omega \int_0^T u_\varepsilon \phi_t \, dP \, dt + \int_Q (-(V, \nabla\phi) + (\mu + k)\phi) u_\varepsilon \, dP \, dt \\ &+ \sum_{i=1}^3 \int_Q u_\varepsilon \frac{\partial}{\partial x_i} \big(a_i^\varepsilon \frac{\partial \phi}{\partial x_i}\big) \, dP \, dt \\ &= \int_\Omega l_0(P)\phi(0, P) dP. \end{aligned}$$

Since $\nabla h \in (L^2(Q))^3$ and $u_\varepsilon \rightharpoonup u$ weakly in $L^2(Q)$, passing to the limit in the above equality we obtain

$$
\begin{aligned}
&-\int_\Omega \int_0^T u\phi_t \, dP\, dt + \int_Q (-(V,\nabla\phi) + (\mu+k)\phi)u \, dP\, dt + \int_Q u\frac{\partial}{\partial x_3}\big(h\frac{\partial \phi}{\partial x_3}\big)\, dP\, dt \\
&= \int_\Omega l_0(P)\phi(0,P)dP.
\end{aligned}
$$

Hence u is a distributional solution to problem (4.2) and the claim follows.

To get more regularity on u we set

$$
\Psi_\varepsilon(t,x_3) = \int_D hu_\varepsilon \, dx\, dy,
$$

where u_ε is the solution of (4.3). Using the hypothesis on h and V and by the classical result on the theory of regularity we obtain that $u_\varepsilon \in C^1([0,T]\times\bar\Omega)$. Thus

$$
\frac{\partial \Psi_\varepsilon}{\partial x_3} = \int_D (h\frac{\partial u_\varepsilon}{\partial x_3} + u_\varepsilon \frac{\partial h}{\partial x_3})\, dx\, dy,
$$

by integrating over $(0,T)\times(0,z^*)$ we get

$$
\int_0^T \int_0^{z^*} |\frac{\partial \Psi_\varepsilon}{\partial x_3}|^2 dx_3 dt \le \int_Q h^2 |\frac{\partial u_\varepsilon}{\partial x_3}|^2 \, dP\, dt + C\int_Q |u_\varepsilon|^2 \, dP\, dt \le C_2.
$$

Since Ψ_ε is bounded in $L^2((0,T)\times(0,z^*))$, which can be proved easily, we conclude that Ψ_ε is bounded in $L^2(0,T;W_0^{1,2}(0,z^*))$, hence up to a subsequence, called also Ψ_ε, we obtain that Ψ_ε converges weakly in $L^2(0,T;W_0^{1,2}(0,z^*))$ to Ψ where

$$
\Psi = \int_D h\, u\ dx\, dy.
$$

Note that the last identification follows by the fact that $u_\varepsilon \rightharpoonup u$ in weak topology of $L^2(Q)$ and by the uniqueness of the weak limit.

We claim that $\frac{\partial u}{\partial x_3} \in L^2(Q)$. To show this claim we prove that $\frac{\partial u}{\partial x_3} \in (L^2(Q))' \equiv L^2(Q)$. Note that $\frac{\partial u}{\partial x_3}$ is well defined as a distribution. Let $\phi \in \mathcal{C}_0^\infty(Q)$, then we have

$$
\begin{aligned}
\int_Q \frac{\partial u}{\partial x_3}\phi \, dP\, dt &= -\int_Q \frac{\partial \phi}{\partial x_3} u\, dP\, dt \\
&= -\lim_{\varepsilon\to 0}\int_Q \frac{\partial \phi}{\partial x_3} u_\varepsilon \, dP\, dt \\
&= \lim_{\varepsilon\to 0}\int_Q \frac{\partial u_\varepsilon}{\partial x_3}\phi \, dP\, dt \\
&\le \lim_{\varepsilon\to 0}\Big(\int_Q |\frac{\partial u_\varepsilon}{\partial x_3}|^2 \, dP\, dt\Big)^{1/2}\Big(\int_Q |\phi|^2 \, dP\, dt\Big)^{1/2}.
\end{aligned}
$$

Then we conclude that

$$
|\int_Q \frac{\partial u}{\partial x_3}\phi \, dP\, dt| \le C\Big(\int_Q |\phi|^2 \, dP\, dt\Big)^{1/2}
$$

for all $\phi \in \mathcal{C}_0^\infty(Q)$. Hence by density we conclude that $\frac{\partial u}{\partial x_3} \in (L^2(Q))' \equiv L^2(Q)$ and then the claim follows.

Therefore we conclude that

$$\int_Q h \frac{\partial u_\varepsilon}{\partial x_3} \frac{\partial v}{\partial x_3}\, dP\, dt \to \int_Q h \frac{\partial u}{\partial x_3} \frac{\partial v}{\partial x_3}\, dP\, dt,$$

for all $v \in L^2(0,T;W_0^{1,2}(0,z^*))$. Let $\phi \in \mathcal{C}_0^\infty(Q)$, using a density result, see for example [9], [2], we get that $\{\eta(t,x_1,x_2) \times \psi(t,x_3)\}$ is a total family in $\mathcal{C}_0^\infty(Q)$. Then by the above computation we obtain that

$$\int_Q h\eta \frac{\partial u_\varepsilon}{\partial x_3} \frac{\partial \psi}{\partial x_3}\, dP\, dt \to \int_Q h\eta \frac{\partial u}{\partial x_3} \frac{\partial \phi}{\partial x_3}\, dP\, dt.$$

Hence by the density result obtained in [9] and in [2] we get the same conclusion for all $\phi \in \mathcal{C}_0^\infty(Q)$; hence

$$\int_Q h \frac{\partial u_\varepsilon}{\partial x_3} \frac{\partial \phi}{\partial x_3}\, dP\, dt \to \int_Q h \frac{\partial u}{\partial x_3} \frac{\partial \phi}{\partial x_3}\, dP\, dt.$$

Since $\phi \in \mathcal{C}_0^\infty(Q)$ is dense in $L^2(0,T;W_0^{1,2}(D \times (0,z^*)))$ and by the fact that $\frac{\partial u}{\partial x_3} \in L^2(Q)$ we get that (5.1) holds for all $\phi \in L^2(0,T;W_0^{1,2}(D \times (0,z^*)))$. By letting ε tend to 0 in (4.4), we obtain

$$\begin{aligned} &\int_Q u \frac{\partial \phi}{\partial t} dxdt + \int_Q (-(V,\nabla\phi) + (\mu+k)\phi)u\, dP\, dt + \int_Q h \frac{\partial u}{\partial x_3} \frac{\partial \phi}{\partial x_3}\, dP\, dt \\ &= \int_\Omega l_0(P)\phi(0,P) dP, \end{aligned}$$

for all $\phi \in K$, where K is defined in (5.2). The proof is complete. □

Now, we give a remark on the uniqueness of solution. In [4], the authors consider the Cauchy problem

$$u_t - \Delta_x u = \partial_y (f(u)), \;\; (x,y) \in \mathbb{R}^N, t > 0\,. \tag{5.5}$$

Using the vanishing viscosity argument and the notion of Entropy solution, they obtain existence and uniqueness of solution to problem (5.5). The argument used depends on the presence of the linear operator Δ_x and the estimate obtained in [5].

The extension of the above uniqueness result to a non-autonomous problem seems to be a more difficult technical problem and it will be treated in forthcoming work.

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