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EXACT BOUNDARY CONTROLLABILITY FOR HIGHER ORDER NONLINEAR SCHRÖDINGER EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract. The exact boundary controllability of the higher order nonlinear Schrödinger equation with constant coefficients on a bounded domain with various boundary conditions is studied. We derive the exact boundary controllability for this equation for sufficiently small initial and final states.

1. Introduction

We consider the initial-value problem

$$\begin{gathered} iu_t + \alpha u_{xx} + i\beta u_{xxx} + |u|^2 u = 0, \quad x, t \in \mathbb{R} \\ u(x,0) = u_0(x) \end{gathered} \tag{1.1}$$

where α, $\beta \in \mathbb{R}$, $\beta \neq 0$ and u is a complex valued function. The above equation is a particular case of the equation

$$\begin{gathered} iu_t + \alpha u_{xx} + i\beta u_{xxx} + \gamma|u|^2 u + i\delta |u|^2 u_x + i\epsilon u^2 \overline{u}_x = 0, \quad x, t \in \mathbb{R} \\ u(x,0) = u_0(x) \end{gathered} \tag{1.2}$$

where α, β, γ, δ, with $\beta \neq 0$ and u is a complex valued function. This equation was first proposed by Hasegawa and Kodama [10] as a model for the propagation of a signal in a fiber optic (see also [13]). The equation (1.2) can be reduced to other well known equations. For instance, setting $\alpha = 1$, $\beta = \epsilon = \gamma = 0$ in (1.2) we have the semi linear Schrödinger equation, i. e.,

$$u_t - iu_{xx} - i\gamma|u|^2 u = 0. \tag{1.3}$$

If we let $\beta = \gamma = 0$ and $\alpha = 1$ in (1.2) we obtain the derivative nonlinear Schrödinger equation

$$u_t - iu_{xx} - \delta|u|^2 u_x - \epsilon u^2 \overline{u}_x = 0. \tag{1.4}$$

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Letting $\alpha = \gamma = \epsilon = 0$ in (1.2), the equation that arises is the complex modified Korteweg-de Vries equation,

$$u_t + \beta u_{xxx} + \delta |u|^2 u_x = 0. \tag{1.5}$$

The initial-value problem for the equations (1.3), (1.4) and (1.5) has been extensively studied, see for instance [1, 8, 14, 18, 20, 21, 24, 26] and references therein. In 1992, Laurey [17] considered the equation (1.2) and proved local well-posedness of the initial-value problem associated for data in $H^s(\mathbb{R})$ with $s > 3/4$, and global well-posedness in $H^s(\mathbb{R})$ where $s \geq 1$. In 1997, Staffilani [30] established local well-posedness for data in $H^s(\mathbb{R})$ with $s \geq 1/4$, improving Laurey's result. Similar results were given in [6, 7] for (1.2) where $w(t)$, $\beta(t)$ are real functions.

For the case of the (1.1) if we consider the Gauge transformation

$$u(x,t) = e^{i\frac{\alpha}{3}x + i2\frac{\alpha^3}{27}} v(x - \frac{\alpha^2}{3}t, t) \equiv e^{\theta} v(\eta, \xi)$$

where $\theta = i\frac{\alpha}{3}x + i2\frac{\alpha^3}{27}$, $\eta = x - \frac{\alpha^2}{3}t$ and $\xi = t$, then

$$\begin{gathered}
u_t = i2\frac{\alpha^3}{27} e^{\theta} v - \frac{\alpha^2}{3} e^{\theta} v_{\eta} + e^{\theta} v_{\xi} \\
u_{xx} = -\frac{\alpha^2}{9} e^{\theta} v + i\frac{2}{3}\alpha e^{\theta} v_{\eta} + e^{\theta} v_{\eta\eta} \\
u_{xxx} = -i\frac{\alpha^3}{27} e^{\theta} v - \frac{1}{3}\alpha^2 e^{\theta} v_{\eta} + i\alpha e^{\theta} v_{\eta\eta} + e^{\theta} v_{\eta\eta\eta}.
\end{gathered}$$

Replacing in (1.1) and considering $\beta = 1$(rescaling the equation) we obtain

$$\begin{gathered}
iv_{\xi} + iv_{\eta\eta\eta} + |v|^2 v - \frac{4}{27}\alpha^3 v = 0, \quad x, t \in \mathbb{R} \\
v(x,0) = v_0(x) \equiv u_0(x) e^{-i\frac{\alpha}{3}}
\end{gathered} \tag{1.6}$$

Thus (1.1) is reduced to a complex modified Korteweg-de Vries type equation. In this paper, we consider the boundary control of the Schrödinger equation

$$iu_t + \alpha u_{xx} + i\beta u_{xxx} + |u|^2 u + i\delta u_x = 0 \tag{1.7}$$

where α, β, $\delta \in \mathbb{R}$, $\beta \neq 0$ and u is a complex valued function on the domain (a, b), $t > 0$, and with the boundary condition

$$u(a,t) = h_0, \quad u(b,t) = h_1, \quad u_x(a,t) = h_2, \quad u_x(b,t) = h_3. \tag{1.8}$$

In this paper we want to study directly the exact boundary controllability problem for the higher order Schrödinger equation by adapting the method of [21] which combines the Hilbert Uniqueness Method (HUM) and multiplier techniques. This method has been successfully applied to study controllability of wave and plate equations, Schrödinger and KdV equations (see for instance [1, 8, 9, 11, 14, 15, 18, 20, 22, 24] and references therein). The first result of this paper concerns boundary controllability of the higher order linear Schrödinger equation.

Theorem 1.1. *Let $H_p^2 = \{w \in H^2(0, 2\pi) : w(0) = w(2\pi),\ w'(0) = w'(2\pi)\}$ and $T > 0$. Then, for any $y_0, y_T \in (H_p^2)'$ (the dual space of H_p^2), there exist $h_k \in L^2(0,T)$ $(k = 0, 1, 2)$ such that the solution $y \in C([0,T] : (H_p^2)')$ of the boundary initial-value higher order Schrödinger equation*

$$iy_t + i\beta y_{xxx} + \alpha y_{xx} = 0, \quad (x,t) \in (0, 2\pi) \times (0,T); \tag{1.9}$$

$$\partial_x^k y(2\pi,t) - \partial_x^k y(0,t) = h_k(t), \quad k = 0,1,2; \tag{1.10}$$
$$y(.,0) = y_0 \tag{1.11}$$

satisfies $y(.,T) = y_T$.

We see that explicit controls may be given. Unfortunately, the state y is only known to belong to $C([0,T] : (H_p^2)')$ so it seems quite difficult to deduce from Theorem 1.1 controllability results for higher order nonlinear Schrödinger equation (1.7).

The second result relates exact boundary controllability for the linear higher order Schrödinger equation with boundary control on y_x at $x = L$. In this part a condition on the coefficients α and β given by the second and the third order derivatives that appear in $(HSCHROD)$ is needed. A condition on the length L of the domain appears.

Theorem 1.2. *Let* $|\alpha| < 3\beta$, $\delta > 0$ *and*

$$\mathcal{N} = \Big\{ 2\pi\beta\sqrt{\frac{k^2+kl+l^2}{3\beta\delta+\alpha^2}} : k,l \in \mathbb{N}^* \Big\}.$$

Then for any $T > 0$ *and* $L \in (0,+\infty)\backslash\mathcal{N}$, *and for any* $y_0, y_T \in L^2(0,L)$, *there exists* $h \in L^2(0,T)$ *such that the mild solution* $y \in C([0,T] : L^2(0,L)) \cap L^2(0,T : H^1(0,L))$ *of the system*

$$iy_t + i\beta y_{xxx} + \alpha y_{xx} + i\delta y_x = 0 \tag{1.12}$$
$$y(0,t) = y(L,t) = 0 \tag{1.13}$$
$$y_x(L,t) = h(t) \tag{1.14}$$
$$y(x,0) = y_0(x) \tag{1.15}$$

satisfies $y(.,T) = y_T$.

To prove this we use the Hilbert uniqueness method and the multiplier method. It turns out that the study of (1.12)-(1.15) as a boundary initial-value problem is more delicate than the study of (1.9)-(1.11), and -because of the extra term y_x in (1.14)- the observability result holds true if and only if $L \notin \mathcal{N}$. On the other hand, the solution y belongs this time to a functional space in which we may give a sense to the nonlinear term $|y|^2y$ in (1.1). By means of the Banach Contraction Fixed Point Theorem and Theorem 1.2 we get the main result of the paper, that is the exact boundary controllability of the higher order nonlinear Schrödinger equation on a bounded domain.

Theorem 1.3. *Let* $|\alpha| < 3\beta$, $\delta > 0$, $T > 0$ *and* $L > 0$. *Then, there exists* $r_0 > 0$ *such that for any* y_0, $y_T \in L^2(0,L)$ *with* $\|y_0\|_{L^2(0,L)} < r_0$, $\|y_T\|_{L^2(0,L)} < r_0$, *there is function* y *in*

$$C([0,T] : L^2(0,L)) \cap L^2([0,T] : H^1(0,L)) \cap W^{1,1}([0,T] : H^{-2}(0,L)) \tag{1.16}$$

which is a solution of

$$iy_t = -(i\beta y_{xxx} + \alpha y_{xx} + |y|^2 y + i\delta y_x) \quad \text{in } \mathcal{D}'(0,T : H^{-2}(0,L)) \tag{1.17}$$
$$y(0,.) = 0 \quad \text{in } L^2(0,L) \tag{1.18}$$

and such that $y(.,0) = y_0$, $y(.,T) = y_T$. *If moreover* $L \notin \mathcal{N}$, *then in addition, it is possible to assume that* $y(L,.) = 0$ *in* $L^2(0,T)$ *and take* $y_x(L,.)$ *in* $L^2(0,T)$ *as a control function.*

In a forthcoming paper we study the case $|\alpha| \geq 3\beta$ for Theorems 1.2 and 1.3 using the Gauge transformation (KdVm described above) and following the same idea shown here.

This paper is organized as follows: Section 2 outlines briefly the notation and terminology to be used subsequently and some previous result. Section 3 we derive from the Hilbert uniqueness method a direct proof of the exact controllability result for the higher order linear Schrödinger equation. In section 4, we consider another boundary controllability problem for the higher order linear Schrödinger equation, in which only the value of the first spatial derivative (at $x = L$) of the state function is assumed to be controlled: this boundary initial-value problem is first shown to admit solutions, later on, an observability result is given and used to show using the Hilbert uniqueness method the exact boundary controllability for higher order linear Schrödinger equation with these boundary conditions. Finally, in section 5, we prove the main result of this paper, that is, the exact local boundary controllability of the higher order nonlinear Schrödinger equation on a bounded domain.

2. Preliminaries

For an arbitrary Banach space X, the associated norm will be denoted by $\|\cdot\|_X$. If $\Omega = (a,b)$ is a bounded open interval and k a non-negative integer, we denote by $C^k(\Omega) = C^k(a,b)$ the functions that, along with their first k ones, are continuous on $[a,b]$ with the norm

$$\|f\|_{C^k(\Omega)} = \sup_{x\in\Omega,\, 0\leq j\leq k} |f^{(j)}(x)|. \tag{2.1}$$

As usual, $\mathcal{D}(\Omega)$ is the subspace of $C^\infty(\overline{\Omega})$ consisting of functions with compact support in Ω. Its dual space $\mathcal{D}'$ is the space of Schwartz distributions on Ω. For $1 \leq p < \infty$, $L^p(\Omega)$ denotes those functions f which are p-power absolutely integrable on Ω with the usual modification n case $p = \infty$. If $s \geq 0$ is an integer and $1 \leq p \leq \infty$, $W^{s,p}(\Omega)$ is the Sobolev space consisting of those $L^p(\Omega)$-functions whose first s generalized derivatives lie in $L^p(\Omega)$, with the usual norm

$$\|f\|^p_{W^{s,p}(\Omega)} = \sum_{k=0}^{s} \|f^{(k)}\|^p_{L^p(\Omega)}. \tag{2.2}$$

If $p = 2$ we write $H^2(\Omega)$ for $W^{s,2}(\Omega)$. The notation $H^s(\Omega)$ is frequent where s is a positive integer.

$$\|\cdot\|_s = \|\cdot\|_{H^s(a,b)}. \tag{2.3}$$

For $s \geq 1$, $H^s_0((a,b))$ is the closed linear subspace of $H^s((a,b))$ of functions f such that $f(a) = f'(a) = \cdots = f^{s-1}(a) = 0$. $H^s_{\rm loc}(\Omega)$ is the set of real-valued functions f defined on Ω such that, for each $\varphi \in \mathcal{D}(\Omega)$, $\varphi f \in H^s(\Omega)$. This space is equipped with the weakest topology such that all of the mapping $f \mapsto \varphi f$, for $\varphi \in \mathcal{D}(\Omega)$, are continuous from $H^s(\Omega)$ into $H^s_{\rm loc}(\Omega)$. With this topology, $H^s_{\rm loc}(\Omega)$ is a Fréchet space. If X is a Banach space, T a positive real number and $1 \leq p \leq +\infty$, we will denote by $L^p(0,T;X)$ the Banach space of all measurable functions $u : (0,T) \mapsto X$, such that $t \mapsto \|u(t)\|_X$ is in $L^p(0,T)$, with the norm

$$\|u\|_{L^p(0,T;X)} = \Big(\int_0^T \|u(t)\|^p_X\, dx\Big)^{1/p} \quad \text{if } 1 \leq p < +\infty,$$

and if $p = \infty$, then

$$\|u\|_{L^\infty(0,T;\, X)} = \sup_{0<t<T} \|u\|_X.$$

Similarly, if k is a positive integer, then $C^k(0, T : X)$ denote the space of all continuous functions $u : [0, T] \mapsto X$, such that their derivatives up to the k order exist and are continuous.

For notation, we write $\partial = \partial/\partial x$, $\partial_t = \partial/\partial t$ and $u_j = \partial_x^j u = \partial^j u/\partial x^j$.

Definition. For $k = \{2, 3\}$, we define the space

$$H_p^k = \Big\{ u \in H^k(0, 2\pi) : \frac{d^j u}{dx^j}(0) = \frac{d^j u}{dx^j}(2\pi) \text{ for } 0 \le j \le k-1 \Big\}$$

We remark that $H^k(0, 2\pi)$ denotes the classical Sobolev space on the interval $(0, 2\pi)$.

Definition. For $n \in \mathbb{Z}$, let the n-th Fourier coefficient of $u \in L^2(0, 2\pi)$,

$$\widehat{u}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} u(x)\, dx \tag{2.4}$$

Lemma 2.1. *For $n \in \mathbb{Z}$, we have*

$$\sum_{n\in\mathbb{Z}} |\widehat{u}(n)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |u(x)|^2\, dx \tag{2.5}$$

The proof of the above lemma is straightforward. We remark that for $k = 2$ (similarly for $k = 3$) we have

$$\widehat{u}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} u(x)\, dx = -\frac{1}{n^2} \partial^2 \widehat{u}(x)$$

then $-n^2\widehat{u}(n) = \partial^2\widehat{u}(n)$. Applying $|\cdot|$ and squaring we obtain $[n^2|\widehat{u}(n)|^2]^2 = |\partial^2\widehat{u}(n)|^2$ where by applying $\sum_{n\in\mathbb{Z}}$ and using (2.2) it follows that

$$\sum_{n\in\mathbb{Z}} [n^2|\widehat{u}(n)|^2]^2 = \sum_{n\in\mathbb{Z}} |\partial^2\widehat{u}(n)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\partial^2 u(x)|^2\, dx < \infty.$$

Hence, we have that for all $u \in L^2(0, 2\pi)$, $k \in \{2, 3\}$

$$u \in H_p^k \quad \text{if and only if} \quad \sum_{n\in\mathbb{Z}} [n^k|\widehat{u}(n)|^2]^2 < \infty, \tag{2.6}$$

and the Sobolev norm

$$\|u\|_{H^k(0,2\pi)} = \Big[\sum_{j=0}^k \int_0^{2\pi} |\partial^j u(x)|^2\, dx \Big]^{1/2} = \Big[\sum_{j=0}^k \|\partial^j u\|^2_{L^2(0,2\pi)} \Big]^{1/2}$$

reduces to

$$\|u\|_{H^k(0,2\pi)} = \Big[\sum_{n\in\mathbb{Z}} (1 + n^2 + \ldots + n^{2k}) |\widehat{u}(n)|^2 \Big]^{1/2} \quad \text{for } u \in H_p^k. \tag{2.7}$$

In what follows, the Hilbert space H_p^k is endowed with the norm $\|u\|_{H^k(0,2\pi)}$.

Lemma 2.2 (Ingham's Inequality [12]). *Assume the strictly increasing sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ of real numbers satisfies the "gap" condition $\lambda_{k+1}-\lambda_k \geq \gamma$, for all $k\in\mathbb{Z}$, for some $\gamma>0$. Then, for all $T>2\pi/\gamma$ there are two positive constants C_1, C_2 depending only on γ and T such that*

$$C_1(T,\gamma)\sum_{k=-\infty}^{\infty}|a_k|^2 \leq \int_0^T \big|\sum_{k=-\infty}^{\infty} a_k e^{it\lambda_k}\big|\,dx \leq C_2(T,\gamma)\sum_{k=-\infty}^{\infty}|a_k|^2 \tag{2.8}$$

for every complex sequence $(a_k)_{k\in\mathbb{Z}}\in l^2$, where

$$\begin{aligned} C_1(T,\gamma) &= \frac{2T}{\pi}\big(1-\frac{4\pi^2}{T^2\gamma^2}\big)>0,\\ C_2(T,\gamma) &= \frac{8T}{\pi}\big(1+\frac{4\pi^2}{T^2\gamma^2}\big)>0 \end{aligned} \tag{2.9}$$

and l^2 is the Hilbert space of square summable sequences, sequences $\{a_k\}$ such that $\sum_{k\in\mathbb{N}}|a_k|^2<\infty$.

Finally, we denote by c, a generic constant, not necessarily the same at each occasion, which depends in an increasing way on the indicated quantities.

3. Exact boundary controllability of the higher order linear Schrödinger equation by means of control on data $[\partial^k y(.,t)]_0^{2\pi}$ for $k=0,1,2$

For simplicity, in this section, we restrict ourselves to the case where the space domain $[0,L]$ is $[0,2\pi]$; although Theorem 1.1 holds for arbitrary $L>0$.

Lemma 3.1. *Let A denote the operator $Au=(-\beta\partial^3+i\alpha\partial^2)u$ on the domain $D(A)=H_p^3\subseteq L^2(0,2\pi)$. Then A generates a strongly continuous unitary group $(S(t))_{t\in\mathbb{R}}$ on $L^2(0,2\pi)$.*

Proof. Let $A: D(A)\subseteq L^2(0,2\pi)\mapsto L^2(0,2\pi)$ such that $u\mapsto Au=-\beta\partial^3u+i\alpha\partial^2u$. We have

$$\begin{aligned} \langle Au,v\rangle &= \langle -\beta\partial^3u+i\alpha\partial^2u, v\rangle\\ &= -\beta\langle\partial^3u,v\rangle+i\alpha\langle\partial^2u,v\rangle\\ &= \beta\langle u,\partial^3v\rangle+i\alpha\langle u,\partial^2v\rangle\\ &= \langle u,\beta\partial^3v\rangle+\langle u,-i\alpha\partial^2v\rangle\\ &= \langle u,-(-\beta\partial^3v+i\alpha\partial^2v)\rangle\\ &= \langle u,-Av\rangle \end{aligned}$$

then $A^*=-A$. Hence, by the Stone theorem [25], A is the infinitesimal generator of a unitary group of class C_0 (all groups of class C_0 are strongly continuous) on $L^2(0,2\pi)$. ☐

Definition. Let $T>0$. For $u_T=\sum_{n\in\mathbb{Z}}c_ne^{int}\in L^2(0,2\pi)$, the mild solution of the uncontrolled problem

$$\begin{gathered} \partial_tu+\beta\partial^3u-i\alpha\partial^2u=0,\quad x\in(0,2\pi),\ t\in\mathbb{R};\\ \partial^ku(0,t)=\partial^ku(2\pi,t),\quad k=0,1,2;\\ u(.,T)=u_T(.) \end{gathered} \tag{3.1}$$

is given by

$$u(x,t) = \sum_{n\in\mathbb{Z}} c_n e^{i(\beta n^3 - \alpha n^2)(t-T)+inx} \tag{3.2}$$

Remark 3.2. Let $u(x,t) = \sum_{n\in\mathbb{Z}} \widehat{u}(n,t)e^{inx}$, then

$$u(x,t) = \sum_{n\in\mathbb{Z}} c_n e^{i[(\beta n^3 - \alpha n^2)(t-T)+nx]}.$$

In fact,

$$\partial_t u(x,t) = \sum_{n\in\mathbb{Z}} \partial_t \widehat{u}(n,t)e^{inx}$$
$$\partial^2 u(x,t) = \sum_{n\in\mathbb{Z}} (in)^2 \widehat{u}(n,t)e^{inx} = -\sum_{n\in\mathbb{Z}} n^2 \widehat{u}(n,t)e^{inx}$$
$$\partial^3 u(x,t) = \sum_{n\in\mathbb{Z}} (in)^3 \widehat{u}(n,t)e^{inx} = -i\sum_{n\in\mathbb{Z}} n^3 \widehat{u}(n,t)e^{inx},$$

hence, if u is the solution of (3.1), we obtain

$$\sum_{n\in\mathbb{Z}} \partial_t \widehat{u}(n,t)e^{inx} - i\beta \sum_{n\in\mathbb{Z}} n^3 \widehat{u}(n,t)e^{inx} + i\alpha \sum_{n\in\mathbb{Z}} n^2 \widehat{u}(n,t)e^{inx} = 0.$$

Multiplying by $e^{-imx}(m \in \mathbb{Z})$ and integrating over $x \in (0,2\pi)$ we obtain

$$\sum_{n\in\mathbb{Z}} \partial_t \widehat{u}(n,t) - i(\beta n^3 - \alpha n^2)\widehat{u}(n,t) \int_0^{2\pi} e^{i(n-m)x} dx = 0.$$

Using that

$$\int_0^{2\pi} e^{i(n-m)x} dx = \begin{cases} 0, & \text{if } n \neq m \\ 2\pi, & \text{if } n = m \end{cases}$$

we have that $\sum_{n\in\mathbb{Z}} \partial_t \widehat{u}(n,t) - i(\beta n^3 - \alpha n^2)\widehat{u}(n,t) = 0$, then $\partial_t \widehat{u}(n,t) - i(\beta n^3 - \alpha n^2)\widehat{u}(n,t) = 0$ where

$$\partial_t \Big[e^{-i(\beta n^3 - \alpha n^2)t} \widehat{u}(n,t) \Big] = 0.$$

Integrating over $t \in [0,T]$ yields

$$\widehat{u}(n,t) = \widehat{u}(n,0)e^{i(\beta n^3 - \alpha n^2)t}$$

multiplying by e^{inx} and applying $\sum_{n\in\mathbb{Z}}$ we obtain

$$\begin{aligned} u(x,t) &= \sum_{n\in\mathbb{Z}} \widehat{u}(n,t)e^{inx} \\ &= \sum_{n\in\mathbb{Z}} \widehat{u}(n,0)e^{i[(\beta n^3 - \alpha n^2)t+nx]} \\ &= \sum_{n\in\mathbb{Z}} \widehat{u}(n,0)e^{i(\beta n^3 - \alpha n^2)T} e^{i[(\beta n^3 - \alpha n^2)(t-T)+nx]} \\ &= \sum_{n\in\mathbb{Z}} c_n e^{i(\beta n^3 - \alpha n^2)(t-T)+inx}. \end{aligned}$$

where $c_n = \widehat{u}(n,0)e^{i(\beta n^3 - \alpha n^2)T}$ and $u(x,T) = u_T = \sum_{n\in\mathbb{Z}} c_n e^{inx}$.

For the rest of this article, u will denote the solution of (3.1) associated with u_T. We show the following result for the non-homogeneous problem.

Theorem 3.3. *Let $H_p^2 = \{w \in H^2(0,2\pi) : w(0) = w(2\pi), w'(0) = w'(2\pi)\}$ and $T > 0$. Then for any y_0, $y_T \in (H_p^2)'$ (the dual space of H_p^2), there exist $h_k \in L^2(0,T)$ $(k = 0,1,2)$ such that the solution $y \in C([0,T] : (H_p^2)')$ of the boundary initial-value higher order Schrödinger equation*

$$\begin{gathered}\partial_t y + \beta\partial^3 y - i\alpha\partial^2 y = 0, \quad (x,t) \in (0,2\pi)\times(0,T);\\ \partial^k y(2\pi,t) - \partial^k y(0,t) = h_k(t), \quad k = 0,1,2;\\ y(.,0) = y_0\end{gathered} \tag{3.3}$$

satisfies $y(.,T) = y_T$.

Remark 3.4. Given $y_0 \in (H_p^2)'$, $h_k \in L^2(0,T)$ $(k = 0,1,2)$, we want to find y such that it satisfies (3.3). We first prove that (3.3) admits a unique solution $y \in C([0,T] : (H_p^2)')$ in a certain sense, and this solution is the classical one whenever $y \in D(A)$, and $h_k (k = 0,1,2)$ are smooth enough and vanish at 0.

Lemma 3.5. *(1) Assume that $h_k \in C_0^2([0,T]) = \{h \in C^2([0,T] : \mathbb{C}) : h(0) = 0\}$ and $y_0 \in H_p^3$. Then there exists a unique solution $y \in C([0,T] : H^3(0,2\pi)) \cap C^1([0,T] : L^2(0,2\pi))$ of (3.3). Moreover, for any $u_T \in H_p^3$ and any $t \in [0,T]$ we have*

$$\begin{aligned}&\int_0^{2\pi} u(x,t)\overline{y(x,t)}\,dx\\ &= \int_0^{2\pi} u(x,0)\overline{y_0(x)}\,dx - (\beta - i\alpha)\int_0^t \partial^2 u(0,s)\overline{h_0(s)}\,ds\\ &\quad + \beta\int_0^t \partial u(0,s)\overline{h_1(s)}\,ds + \int_0^t u(0,s)(\overline{\beta h_2(s) + i\alpha h_1(s)})\,ds.\end{aligned} \tag{3.4}$$

(2) For $u_T \in H_p^2$, $u \in C([0,T] : H_p^2)$ and $\partial^2 u(0,.)$ makes sense in $L^2(0,T)$.
(3) Assume now that $y_0 \in (H_p^2)'$ and $h_k \in L^2(0,T) (k = 0,1,2)$. Then, there exists a unique $y \in C([0,T] : (H_p^2)')$ such that for all $u_T \in H_p^2$ and for all $t \in [0,T]$,

$$\begin{aligned}&\langle u(.,t), y(t)\rangle_{H_p^2\times(H_p^2)'}\\ &= \langle u(.,0), y_0\rangle_{H_p^2\times(H_p^2)'} - (\beta - i\alpha)\int_0^t \partial^2 u(0,s)\overline{h_0(s)}\,ds\\ &\quad + \beta\int_0^t \partial u(0,s)\overline{h_1(s)}\,ds + \int_0^t u(0,s)(\overline{\beta h_2(s) + i\alpha h_1(s)})\,ds\end{aligned} \tag{3.5}$$

Proof. **(1)** Let $\phi_i \in C^\infty([0,2\pi]) (i = 0,1,2)$ be such that

$$\phi_i^{(k)}(0) = 0 \quad \text{and} \quad \phi_i^{(k)}(2\pi) = \begin{cases} -1, & i = k\\ 0, & i \neq k.\end{cases}$$

We consider the change of function $z(x,t) = \sum_{i=0}^2 [h_i(t)\phi_i(x) + (S(t)y_0)(x) + y(x,t)]$, then

$$\begin{aligned}z(2\pi,t) - z(0,t) &= \sum_{i=0}^2 h_i(t)\phi_i(2\pi) + (S(t)y_0)(2\pi) + y(2\pi,t)\\ &\quad - \sum_{i=0}^2 h_i(t)\phi_i(0) + (S(t)y_0)(0) + y(0,t)\\ &= -h_0(t) + (S(t)y_0)(2\pi) - (S(t)y_0)(0) + y(2\pi,t) - y(0,t)\end{aligned}$$

$$
\begin{aligned}
&= -h_0(t) + (S(t)y_0)(2\pi) - (S(t)y_0)(0) + h_0(t)\\
&= (S(t)y_0)(2\pi) - (S(t)y_0)(0)
\end{aligned}
$$

using that $y_0 \in H_p^3$ we obtain $z(2\pi, t) = z(0, t)$. The other initial conditions are calculated in a similar way. Hence, this change of the function yields an equivalent problem to (3.3): Find z such that

$$
\begin{gathered}
\partial_t z + \beta\partial^3 z - i\alpha\partial^2 z = f(x,t)\\
= \sum_{i=0}^{2}\Big[h_i'(t)\phi_i(x) + \beta h_i(t)\phi_i^{(3)}(x) - i\alpha h_i(t)\phi_i^{(2)}(x)\Big]\\
\partial^k z(2\pi,t) = \partial^k z(0,t), \quad k = 0,1,2\\
z(.,0) = 0
\end{gathered}
\tag{3.6}
$$

Since $f \in C^1([0,T] : L^2(0,2\pi))$, this non-homogeneous problem admits a unique solution (see [25]), $z \in C([0,T] : H_p^3) \cap C^1([0,T] : L^2(0,2\pi))$. This proves the first assertion in (1).

Let $u_T \in H_p^3$, then $u \in C([0,T] : H_p^3) \cap C^2([0,T] : L^2(0,2\pi))$. Multiplying the equation (3.1) by $\overline{y}$ and integrating in $x \in [0,2\pi]$ and $t \in [0,T]$ we have

$$
\int_0^t\int_0^{2\pi}\overline{y}[\partial_s u]\,dx\,ds + \beta\int_0^t\int_0^{2\pi}\overline{y}[\partial^3 u]\,dx\,ds - i\alpha\int_0^t\int_0^{2\pi}\overline{y}[\partial^2 u]\,dx\,ds = 0.
$$

Each term is treated separately. Integrating by parts,

$$
\begin{aligned}
&\int_0^t\int_0^{2\pi}\overline{y}[\partial_s u]\,dx\,ds\\
&= \int_0^{2\pi}\overline{y(x,t)}u(x,t)\,dx - \int_0^{2\pi}\overline{y(x,0)}u(x,0)\,dx - \int_0^t\int_0^{2\pi}[\partial_s\overline{y}]u\,dx\,ds\,,
\end{aligned}
$$

$$
\begin{aligned}
\beta\int_0^t\int_0^{2\pi}\overline{y}[\partial^5 u]\,dx\,ds &= \beta\int_0^t\overline{h_0(s)}[\partial^2 u(0,s)]\,ds - \beta\int_0^t\overline{h_1(s)}[\partial u(0,s)]\,ds\\
&\quad + \beta\int_0^t\overline{h_2(s)}u(0,s)\,ds - \int_0^t\int_0^{2\pi}[\partial^3\overline{y}]u\,dx\,ds\,,
\end{aligned}
$$

$$
\begin{aligned}
&- i\alpha\int_0^t\int_0^{2\pi}\overline{y}[\partial^2 u]\,dx\,ds\\
&= -i\alpha\int_0^t\overline{h_0(s)}[\partial^2 u(0,s)]\,ds + i\alpha\int_0^t\overline{h_1(s)}u(0,s)\,ds - i\alpha\int_0^t\int_0^{2\pi}[\partial^2\overline{y}]u\,dx\,ds\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_0^{2\pi}\overline{y(x,t)}\;u(x,t)\,dx - \int_0^{2\pi}\overline{y(x,0)}u(x,0)\,dx - \int_0^t\int_0^{2\pi}[\partial_s\overline{y}]u\,dx\,ds\\
&+ \beta\int_0^t\overline{h_0(s)}[\partial^2 u(0,s)]\,ds - \beta\int_0^t\overline{h_1(s)}[\partial u(0,s)]\,ds + \beta\int_0^t\overline{h_2(s)}\;u(0,s)\,ds\\
&- \int_0^t\int_0^{2\pi}[\partial^3\overline{y}]u\,dx\,ds - i\alpha\int_0^t\overline{h_0(s)}[\partial^2 u(0,s)]\,ds + i\alpha\int_0^t\overline{h_1(s)}u(0,s)\,ds\\
&- i\alpha\int_0^t\int_0^{2\pi}[\partial^2\overline{y}]u\,dx\,ds = 0\,,
\end{aligned}
$$

where

$$\begin{aligned}&\int_0^{2\pi} u(x,t)\overline{y(x,t)}\,dx\\&= \int_0^{2\pi} u(x,0)\overline{y_0(x)}\,dx - (\beta - i\alpha)\int_0^t [\partial^2 u(0,s)]\overline{h_0(s)}\,ds\\&\quad + \beta\int_0^t [\partial u(0,s)]\overline{h_1(s)}\,ds + \int_0^t u(0,s)(\overline{\beta h_2(s) + i\alpha h_1(s)})\,ds.\end{aligned}$$

Result (1) follows.

Now, we proof (2). By (3.2), for $t_1, t_2 \in [0,T]$

$$\begin{aligned}u(x,t_1) &= \sum_{n\in\mathbb{Z}} c_n e^{i(\beta n^3 - \alpha n^2)(t_1 - T) + inx},\\u(x,t_2) &= \sum_{n\in\mathbb{Z}} c_n e^{i(\beta n^3 - \alpha n^2)(t_2 - T) + inx};\end{aligned}$$

hence

$$\begin{aligned}&u(x,t_1) - u(x,t_2)\\&= \sum_{n\in\mathbb{Z}} c_n e^{i(\beta n^3 - \alpha n^2)T}\big(e^{i(\beta n^3-\alpha n^2)t_1} - e^{i(\beta n^3 - \alpha n^2)t_2}\big)e^{inx}.\end{aligned}$$

From (2.3), if $u_T \in H_p^2$ then $\sum_{n\in\mathbb{Z}} |n^2 c_n|^2 < \infty$ and $\sum_{n\in\mathbb{Z}} |nc_n|^2 < \infty$. Using Lebesgue's Theorem [27],

$$|u(x,t_1) - u(x,t_2)| = \sum_{n\in\mathbb{Z}} \big|(n^2+n)c_n\big(e^{i(\beta n^3-\alpha n^2)t_1} - e^{i(\beta n^3-\alpha n^2)t_2}\big)\big|^2$$

which approaches 0 as $t_1 \to t_2$. We conclude that $u \in C([0,T] : H_p^2)$. Hence $u(0,.)$, $\partial u(0,.)$ exist in $C([0,T]) \subseteq L^2(0,T)$. The same argument shows that if $u_T \in H_p^3$, $u \in C([0,T] : H_p^3)$ and

$$\partial^2 u(0,t) = \sum_{n\in\mathbb{Z}} \big(-n^2 c_n e^{-i(\beta n^3 - \alpha n^2)T}\big)e^{i(\beta n^3-\alpha n^2)t}. \tag{3.7}$$

The sum in (3.7) makes sense in $L^2(0,T)$ wherever $\sum_{n\in\mathbb{Z}}(n^2|c_n|)^2 < \infty$, that is, $u_T \in H_p^2$. From now on, $\partial^2 u(0,.)$ denotes for $u_T \in H_p^2$, the sum in (3.7). □

Remark 3.6. The linear map $u_T \mapsto \partial^2 u(0,.)$ is continuous since

$$\big\|\sum_{n\in\mathbb{Z}} \big(n^2 c_n e^{-i(\beta n^3-\alpha n^2)T}\big)e^{i(\beta n^3-\alpha n^2)t}\big\| \le \big([\frac{T}{2\pi}]+1\big)\sum_{n\in\mathbb{Z}}[n^2|c_n|]^2 \tag{3.8}$$

where $[x]$ denotes the integral part of a real number x. Identifying $L^2(0,2\pi)$ with its dual by means of the conjugate linear map $y \mapsto \langle ., y\rangle_{L^2(0,2\pi)}$, we have the following dense and compact embedding (see [23])

$$H_p^2 \hookrightarrow L^2(0,2\pi) \hookrightarrow (L^2(0,2\pi))' \hookrightarrow (H_p^2)'. \tag{3.9}$$

Moreover,

$$\langle u, y\rangle_{H_p^2\times(H_p^2)'} = \langle u,y\rangle_{L^2(0,2\pi)} = \int_0^{2\pi} u\overline{y}\,dx \tag{3.10}$$

for $u \in H_p^2$ and $y \in L^2(0,2\pi)$. Then

$$\langle u(.,t), y(t)\rangle_{H_p^2\times(H_p^2)'}$$

$$\begin{aligned}&= \langle u(.,0), y_0\rangle_{H_p^2\times(H_p^2)'} - (\beta - i\alpha)\int_0^t [\partial^2 u(0,s)]\overline{h_0(s)}\,ds\\&\quad + \beta\int_0^t \partial u(0,s)\overline{h_1(s)}\,ds + \int_0^t u(0,s)(\overline{\beta h_2(s) + i\alpha h_1(s)})\,ds\end{aligned}$$

for $h_k \in C_0^2([0,T])$ $(k = 0,1,2)$ and $y_0, u_T \in H_p^3$. Since H_p^3 is dense in H_p^2, using (2), we see that (3.5) also is true for $u_T \in H_p^2$.

Definition. For $y_0 \in (H_p^2)'$ and $h_k \in L^2(0,T)$ $(k = 0,1,2)$, we define a weak solution of (3.3) as a function $y \in C([0,T] : (H_p^2)')$ such that (3.5) holds for all $u_T \in H_p^2$ and all $t \in [0,T]$.

Claim. For t fixed in $[0,T]$, (3.5) defines $y(t) \in (H_p^2)'$ in a unique manner.

In fact, from the proof of (2) the map $\Xi : H_p^2 \to \mathbb{C}$, $u_T \mapsto \Xi(u_T)$, given by

$$\begin{aligned}\Xi(u_T) &= -(\beta - i\alpha)\int_0^t \overline{h_0(s)}[\partial^2 u(0,s)]\,ds + \beta\int_0^t \overline{h_1(s)}[\partial u(0,s)]\,ds\\&\quad + \int_0^t (\beta\overline{h_2(s)} - i\alpha\overline{h_1(s)})u(0,s)\,ds\end{aligned}$$

is a continuous linear form. On the other hand, the map $\Phi : H_p^2 \mapsto H_p^2$ with $u_T \to \Phi(u_T) = u(.,t)$ is an automorphism of the Hilbert space, hence, for each $t \in [0,T]$, $y(t)$ is uniquely defined in $(H_p^2)'$. Moreover, for $t \in [0,T]$,

$$\begin{aligned}\|y(t)\|_{(H_p^2)'} &= \sup_{\|u(.,t)\|_{H_p^2}\le 1} |\langle u(.,t), y(t)\rangle|\\
&= \sup_{\|u(.,t)\|_{H_p^2}\le 1} |\langle u(.,0), y_0\rangle_{H_p^4\times(H_p^2)'} - (\beta - i\alpha)\int_0^t [\partial^2 u(0,s)]\overline{h_0(s)}\,ds\\
&\quad + \beta\int_0^t [\partial u(0,s)]\overline{h_1(s)}\,ds + \int_0^t u(0,s)(\overline{\beta h_2(s) + i\alpha h_1(s)})\,ds|\\
&\le \sup_{\|u(.,t)\|_{H_p^2}\le 1} |\langle u(.,0), y_0\rangle_{H_p^2\times(H_p^2)'}|\\
&\quad + (|\beta| + |\alpha|)\sup_{\|u(.,t)\|_{H_p^2}\le 1}\int_0^t |\overline{h_0(s)}[\partial^2 u(0,s)]|\,ds\\
&\quad + |\beta|\sup_{\|u(.,t)\|_{H_p^2}\le 1}\int_0^t |\overline{h_1(s)}[\partial u(0,s)]|\,ds\\
&\quad + \sup_{\|u(.,t)\|_{H_p^2}\le 1}\int_0^t |(\beta\overline{h_2(s)} - i\alpha\overline{h_1(s)})u(0,s)|\,ds\\
&\le \sup_{\|u(.,t)\|_{H_p^2}\le 1} \|u(.,0)\|_{(H_p^2)'}\|y_0\|_{H_p^2}\\
&\quad + (|\beta| + |\alpha|)\sup_{\|u(.,t)\|_{H_p^2}\le 1} \|\overline{h_0(t)}\|_{L^2(0,T)}\|\partial^2 u(0,t)\|_{L^2(0,T)}\\
&\quad + |\beta|\sup_{\|u(.,t)\|_{H_p^2}\le 1} \|\overline{h_1(t)}\|_{L^2(0,T)}\|\partial u(0,t)\|_{L^2(0,T)}\\
&\quad + \sup_{\|u(.,t)\|_{H_p^2}\le 1} \|(\beta\overline{h_2(s)} - i\alpha\overline{h_1(s)})\|_{L^2(0,T)}\|u(0,t)\|_{L^2(0,T)}\end{aligned}$$

$$\le c\big(\|y_0\|_{(H_p^2)'} + \|h_0\|_{L^2(0,T)} + \|h_1\|_{L^2(0,T)} + \|h_2\|_{L^2(0,T)}\big)$$

where c is a positive constant which does not depend on t or on y_0, h_0, h_1, h_2. Since

$$y \in C([0,T] : L^2(0,2\pi)) \subseteq C([0,T] : (H_p^2)') \tag{3.11}$$

for $y \in H_p^3$ and $(h_0,h_1,h_2) \in [C_0^2([0,T])]^3$, and since H_p^3 is dense in $L^2(0,T)$ and $C_0^2([0,T])$ is dense in $L^2(0,L)$, it follows from (3.11) that $y \in C([0,T] : (H_p^2)')$.

Lemma 3.7 (Observability result). *Let $T > 0$. There exist positive numbers C_1^T, C_2^T such that for every $u_T \in H_p^2$*

$$\begin{aligned} C_1^T\|u_T\|^2_{H_p^2(0,2\pi)} &\le \|u(0,.)\|^2_{L^2(0,T)} + \|\partial u(0,.)\|^2_{L^2(0,T)} + \|\partial^2 u(0,.)\|^2_{L^2(0,T)} \\ &\le C_2^T\|u_T\|^2_{H_p^2(0,2\pi)} \end{aligned} \tag{3.12}$$

Proof. In $L^2(0,T)$ we have that

$$\begin{aligned} u(0,t) &= \sum_{n\in\mathbb{Z}} c_n e^{i(\beta n^3-\alpha n^2)(t-T)} \\ \partial u(0,t) &= \sum_{n\in\mathbb{Z}} inc_n e^{i(\beta n^3-\alpha n^2)(t-T)} \\ \partial^2 u(0,t) &= \sum_{n\in\mathbb{Z}} -n^2 c_n e^{i(\beta n^3-\alpha n^2)(t-T)}. \end{aligned}$$

Hence

$$\begin{aligned} &\|u(0,t)\|^2_{L^2(0,T)} + \|\partial u(0,t)\|^2_{L^2(0,T)} + \|\partial^2 u(0,t)\|^2_{L^2(0,T)} \\ &\le \big([\frac{T}{2\pi}]+1\big)\sum_{n\in\mathbb{Z}}(1+n^2+n^4)|c_n|^2 \\ &\le C_2^T\ \|u_T\|^2_{H_p^2(0,2\pi)} \quad \text{for } u_T \in H_p^2 \end{aligned} \tag{3.13}$$

where $C_2^T = ([\frac{T}{2\pi}]+1)$. To prove the left inequality we first take $T' \in (0,T)$ and $\gamma > 2\pi/T'$. Let $N \in \mathbb{N}^*$ be such that

$$n \in \mathbb{Z}, \quad |n| \ge N \Rightarrow [\beta(n+1)^5 - \alpha(n+1)^3] - [\beta n^5 - \alpha n^3] \ge \gamma.$$

By Ingham's inequality [12] there exists $c^{T'} > 0$ such that for all sequences $(a_n)_{n\in\mathbb{Z}}$ in $l^2(\mathbb{Z})$,

$$\sum_{|n|\ge N}|a_n|^2 \le c^{T'}\int_0^{T'}\Big|\sum_{|n|\ge N} a_n e^{i(\beta n^3-\alpha n^2)(t-T)}\Big|^2\,dt. \tag{3.14}$$

Let $\mathcal{Z}_n = \operatorname{Span}(e^{inx})$ for $n \in \mathbb{Z}$ and $\mathcal{Z} = \oplus_{n\in\mathbb{Z}}\mathcal{Z}_n \subseteq H_p^2$. We define a semi-norm p in $\mathcal{Z}$ by: $\forall u \in \mathcal{Z}$,

$$\begin{aligned} p(u) &= (|u(0)|^2 + |\partial u(0)|^2 + |\partial^2 u(0)|^2)^{1/2} \\ &= \Big(\big|\sum_{n\in\mathbb{Z}}\widehat{u}(n)\big|^2 + \big|\sum_{n\in\mathbb{Z}} in\widehat{u}(n)\big|^2 + \big|\sum_{n\in\mathbb{Z}} -n^2\widehat{u}(n)\big|^2\Big)^{1/2} \end{aligned} \tag{3.15}$$

(For $u \in \mathcal{Z}$, $\widehat{u}(n) = 0$ for $|n|$ large enough).

Let $u_T \in \mathcal{Z} \cap (\oplus_{|n|<N}\mathcal{Z}_n)^\perp$, that is, $c_n = 0$ for $|n| < N$ or for $|n|$ large enough. Using (3.2) and (3.14) we have

$$\|u_T\|^2_{H_p^2((0,2\pi))} = \sum_{n\ge N}(1+n^2+n^4)|c_n|^2 \le c^{T'}\int_0^T [p(u(.,t))]^2\,dt. \tag{3.16}$$

Since $T > T'$, it follows from (3.13), (3.16) and a result by Komornik (see [14]) that there exists a constant $C_1^T > 0$ such that for all u_T in $\mathcal{Z}$,

$$\begin{aligned} C_1^T \|u_T\|^2_{H^2_p(0,2\pi)} &\le \int_0^T [p(u(.,t))]^2\,dt \\ &= \|u(0,.)\|^2_{L^2(0,T)} + \|\partial u(0,.)\|^2_{L^2(0,T)} + \|\partial^2 u(0,.)\|^2_{L^2(0,T)} \end{aligned} \tag{3.17}$$

and the result follows. □

We remark that by a density argument we obtain the left inequality in (3.12) in the general case ($u_T \in H^2_p$).

Proof of Theorem 3.3. Without loss of generality we may assume that $y_0 = 0$. In fact, if y_0, $y_T \in (H^2_p)'$, if there exist $h_k \in L^2(0,T)$ $(k = 0,1,2)$ such that the weak solution $\widetilde{y}$ of (3.3) and $\widetilde{y}(.,0) = 0$ satisfies $\widetilde{y}(.,T) = y_T - S(T)y_0$, then $y_T = S(T)y_0 + \widetilde{y}(.,t)$ is the weak solution of (3.3) with the same control functions and its such that $y(.,T) = y_T$. In what follows we assume that $y_0 = 0$. For $u_T \in H^2_p$ we let $\Lambda : H^2_p \mapsto (H^2_p)'$,

$$u_T \mapsto \Lambda(u_T) = y_T.$$

where y is the weak solution of (3.3) and $h_k (k = 0,1,2)$ are chosen the following way:

$$\begin{gathered} \overline{h_0(t)} = \frac{-1}{(\beta + i\alpha)}\partial^2 u(0,t), \quad \overline{h_1(t)} = \frac{1}{\beta}\partial u(0,t), \\ \overline{h_2(t)} = i\frac{1}{\beta}u(0,t) + i\frac{\alpha}{\beta^2}\partial u(0,t) \end{gathered}$$

As above u stands for the solutions of (3.1) associated with u_T. Clearly $\Lambda : H^2_p \mapsto (H^2_p)'$ is a conjugate linear continuous map. Moreover

$$\begin{aligned} \langle u_T, \Lambda(u_T)\rangle_{H^2_p \times (H^2_p)'} &= \int_0^T (|u(0,t)|^2 + |\partial u(0,t)|^2 + |\partial^2 u(0,t)|^2)\,dt \\ &\ge C_1^T \|u_T\|^2_{H^2_p(0,2\pi)}. \end{aligned}$$

By Lemmas 3.5 and 3.7 it follows from Lax-Milgram's Theorem (see [34]) that Λ is invertible. Then the theorem follows. □

Remark 3.8. If $T = 2\pi$, Lemma 3.7 is trivial. Indeed, for any $u_T \in H^2_p$,

$$\|u_T\|^2_{H^4_p(0,2\pi)} = \|u(0,.)\|^2_{L^2(0,2\pi)} + \|\partial u(0,.)\|^2_{L^2(0,2\pi)} + \|\partial^2 u(0,.)\|^2_{L^2(0,2\pi)}.$$

4. Exact boundary controllability of the higher order linear Schrödinger equation by means of the control $\partial y(L,t)$

We consider now, the scalar space $\mathbb{R}$. In this section, L stands for some positive number. We shall prove the controllability in $L^2(0,L)$ of

$$\begin{gathered} \partial_t y + \beta\partial^3 y - i\alpha\partial^2 y + \delta\partial y = 0 \\ y(0,t) = y(L,t) = 0 \\ \partial y(L,t) = h(t) \\ y(.,0) = y_0 \end{gathered} \tag{4.1}$$

where $h \in L^2(0,T)$ stands for the control function. More precisely we shall prove that, for any $L > 0$, $T > 0$, y_0, $y_T \in L^2(0,L)$ there exists $h \in L^2(0,T)$ such that a mild solution

$$y \in C([0,T] : L^2(0,L)) \cap L^2(0,T : H^1(0,L)) \cap H^1(0,T : H^{-2}(0,L)) \tag{4.2}$$

of (4.1) which verifies the equation (4.1) in $\mathcal{D}'(0,T : H^{-2}(0,L))$ and y_0 in $L^2(0,L)$ may be found such that $y(.,T) = y_T$.

We begin by showing the well-posedness of the initial-value homogeneous problem with $|\alpha| < 3\beta$

$$\begin{gathered}\partial_t y + \beta\partial^3 y - i\alpha\partial^2 y + \delta\partial y = 0 \\ y(0,t) = y(L,t) = 0 \\ \partial y(L,t) = 0 \\ y(.,0) = y_0\,.\end{gathered} \tag{4.3}$$

Let A denote the operator $Aw = -\beta w''' + i\alpha w'' - \delta w'$ on the (dense) domain $D(A) \subseteq L^2(0,L)$, defined by

$$D(A) = \{w \in H^3(0,L) : w(0) = w(L) = w'(L) = 0\}$$

Lemma 4.1. *Operator A generates a strongly continuous semigroup of contractions on $L^2(0,L)$.*

Proof. A is closed. Let $w \in D(A)$. Then

$$\begin{aligned}&\mathrm{Re}\langle w, Aw\rangle_{L^2(0,L)} \\ &= \mathrm{Re}\int_0^L [-\beta w''' + i\alpha w'' - \delta w']w(x)\,dx \\ &= \mathrm{Re}\Big[-\beta\int_0^L w'''(x)w(x)\,dx + i\alpha\int_0^L w''w(x)\,dx - \delta\int_0^L w'(x)w(x)\,dx\Big].\end{aligned}$$

Each term is treated separately. Integrating by parts,

$$\begin{gathered}\int_0^L w'''(x)w(x)\,dx = \frac{1}{2}[w'(0)]^2, \\ \int_0^L w''(x)w(x)\,dx = -\int_0^L [w'(x)]^2\,dx\end{gathered}$$

Then

$$\mathrm{Re}\langle w, Aw\rangle_{L^2(0,L)} = -\frac{\beta}{2}[w'(0)]^2 \le 0 \quad \text{if} \quad \beta > \frac{1}{3}|\alpha|$$

hence, A is dissipative. It can be seen that $A^*(w) = \beta w''' - i\alpha w'' + \delta w'$ with domain $D(A^*) = \{w \in H^5(0,L) : w(0) = w(L) = w'(0) = 0\}$, so that

$$\mathrm{Re}\langle w, A^*w\rangle_{L^2(0,L)} = -\frac{\beta}{2}[w'(L)]^2 \le 0, \quad \text{if } \beta > \frac{1}{3}|\alpha|$$

and A^* is dissipative. Hence, by the Lumer-Phillips Theorem, A is the infinitesimal generator of a C_0 semigroup of contractions on $L^2(0,L)$. The result follows. $\square$

We denote by $(S(t))_{t\ge 0}$ the semi-group of contractions associated with A, and we let $\mathbb{H}$ denote the Banach space $C([0,T] : L^2(0,L)) \cap L^2([0,T] : H^1(0,L))$ endowed with the norm

$$\begin{aligned}\|y\|_{\mathbb{H}} &= \sup_{t\in[0,T]} \|y(.,t)\|_{L^2(0,L)} + \Big(\int_0^T \|y(.,t)\|^2_{H^1(0,L)}\,dt\Big)^{1/2}\\ &= \sup_{t\in[0,T]} \|y(.,t)\|_{L^2(0,L)} + \|y(.,t)\|_{L^2(0,T:H^1(0,L))}.\end{aligned} \tag{4.4}$$

Using the multiplier method, we get useful estimates for the mild solutions of (4.3).

Lemma 4.2. *Let $|\alpha| < 3\beta$. Then*

1. *The map $y_0 \in L^2(0,L) \mapsto S(\cdot)y_0 \in \mathbb{H}$ is continuous.*
2. *For $y_0 \in L^2(0,L)$, $\partial y(0,.)$ makes sense in $L^2(0,L)$, and for all $y_0 \in L^2(0,L)$,*

$$\|\partial y(.,t)\|_{L^2(0,T)} \le \|y_0\|_{L^2(0,L)} \tag{4.5}$$

$$\|y_0\|^2_{L^2(0,L)} \le \frac{1}{T}\|S(\cdot)y_0\|^2_{L^2((0,T)\times(0,L))} + \|\partial y(0,.)\|^2_{L^2(0,T)} \tag{4.6}$$

Proof. (1) For $y_0 \in L^2(0,L)$ we write y the mild solution $S(\cdot)y_0$ of (R_2). By Lemma 4.1, $y \in C([0,T]:L^2(0,L))$ and

$$\|y\|_{C([0,T]:L^2(0,L))} \le \|y_0\|_{L^2(0,L)} \tag{4.7}$$

To see that $y \in L^2(0,T:H^2(0,L))$ we first assume that $y \in D(A)$. Let $\xi = \xi(x,t) \in C^\infty([0,T]\times[0,L])$. Then, multiplying the equation (4.3) by $i\xi y$ we have

$$\begin{gathered} i\xi\overline{y}\partial_t y + i\xi\overline{y}\partial^3 y + \alpha\xi\overline{y}\partial^2 y + i\delta\xi\overline{y}\partial y = 0\\ -i\xi y\partial_t\overline{y} - i\xi y\partial^3\overline{y} + \alpha\xi y\partial^2\overline{y} - i\delta\xi y\partial\overline{y} = 0\end{gathered}$$

(applying conjugates). Subtracting, integrating over $x \in (0,L)$ and using straightforward calculus, we obtain

$$\begin{gathered} i\partial_t\int_0^L \xi|y|^2\,dx - i\int_0^L \partial_t\xi|y|^2\,dx + i\beta\int_0^L \xi\overline{y}\partial^3 y\,dx + i\beta\int_0^L \xi y\partial^3\overline{y}\,dx\\ +\alpha\int_0^L \xi\overline{y}\partial^2 y\,dx - \alpha\int_0^L \xi y\partial^2\overline{y}\,dx - i\delta\int_0^L \partial\xi|y|^2\,dx = 0.\end{gathered}$$

Each term is treated separately. Integrating by parts

$$\begin{aligned}\int_0^L \xi\overline{y}\partial^3 y\,dx &= \int_0^L \partial^2\xi\overline{y}\partial y\,dx + 2\int_0^L \partial\xi|\partial y|^2\,dx - \xi(0,t)|\partial y(0,t)|^2\\ &\quad+\int_0^L \xi\partial y\partial^2\overline{y}\,dx\\ \int_0^L \xi y\partial^3\overline{y}\,dx &= \int_0^L \partial^2\xi y\partial\overline{y}\,dx + \int_0^L \partial\xi|\partial y|^2\,dx - \int_0^L \xi\partial y\partial^2\overline{y}\,dx,\\ \int_0^L \xi\overline{y}\partial^2 y\,dx &= -\int_0^L \partial\xi\overline{y}\partial y\,dx - \int_0^L \xi|\partial y|^2\,dx,\\ \int_0^L \xi y\partial^2\overline{y}\,dx &= -\int_0^L \partial\xi y\partial\overline{y}\,dx - \int_0^L \xi|\partial y|^2\,dx\,.\end{aligned}$$

Then

$$i\partial_t\int_0^L \xi|y|^2\,dx - i\int_0^L \partial_t\xi|y|^2\,dx + i\beta\int_0^L \partial^2\xi\overline{y}\partial y\,dx + 2i\beta\int_0^L \partial\xi|\partial y|^2\,dx$$

$$
\begin{aligned}
&- i\beta\xi(0,t)|\partial y(0,t)|^2 + i\beta\int_0^L \xi\partial y\partial^2\overline{y}\,dx + i\beta\int_0^L \partial^2\xi y\partial\overline{y}\,dx + i\beta\int_0^L \partial\xi|\partial y|^2\,dx\\
&- i\beta\int_0^L \xi\partial y\partial^2\overline{y}\,dx - \alpha\int_0^L \partial\xi\overline{y}\partial y\,dx - \alpha\int_0^L \xi|\partial y|^2\,dx + \int_0^L \partial\xi y\partial\overline{y}\,dx\\
&+ \int_0^L \xi|\partial y|^2\,dx - i\delta\int_0^L \partial\xi|y|^2\,dx = 0\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&i\partial_t\int_0^L \xi|y|^2\,dx - i\int_0^L \partial_t\xi|y|^2\,dx + i\beta\int_0^L \partial^2\xi\partial(|y|^2)\,dx + 3i\beta\int_0^L \partial\xi|\partial y|^2\,dx\\
&- i\beta\xi(0,t)|\partial y(0,t)|^2 - 2i\alpha\operatorname{Im}\int_0^L \partial\xi\overline{y}\partial y\,dx - i\delta\int_0^L \partial\xi|y|^2\,dx = 0\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\partial_t\int_0^L \xi|y|^2\,dx - \int_0^L \partial_t\xi|y|^2\,dx - \beta\int_0^L \partial^3\xi|y|^2\,dx + 3\beta\int_0^L \partial\xi|\partial y|^2\,dx\\
&- \beta\xi(0,t)|\partial y(0,t)|^2 - \delta\int_0^L \partial\xi|y|^2\,dx\\
&= 2\alpha\operatorname{Im}\int_0^L \partial\xi\overline{y}\partial y\,dx\\
&\le |\alpha|\int_0^L \partial\xi|y|^2\,dx + |\alpha|\int_0^L \partial\xi|\partial y|^2\,dx,
\end{aligned}
$$

where

$$
\begin{aligned}
&\partial_t\int_0^L \xi|y|^2\,dx + \int_0^L [3\beta-|\alpha|]\partial\xi|\partial y|^2\,dx - \int_0^L \partial_t\xi|y|^2\,dx - \beta\int_0^L \partial^3\xi|y|^2\,dx\\
&- \beta\xi(0,t)|\partial y(0,t)|^2 - \delta\int_0^L \partial\xi|y|^2\,dx - |\alpha|\int_0^L \partial\xi|y|^2\,dx \le 0.
\end{aligned}
\tag{4.8}
$$

Choosing $\xi(x,t) = x$ leads to

$$
\partial_t\int_0^L x|y|^2\,dx + \int_0^L [3\beta-|\alpha|]|\partial y|^2\,dx - (\delta+|\alpha|)\int_0^L |y|^2\,dx \le 0.
$$

Integrating over $t \in [0,T]$ we obtain

$$
\begin{aligned}
&\int_0^L x|y|^2\,dx + [3\beta-|\alpha|]\int_0^T\int_0^L |\partial y|^2\,dx\,dt\\
&\le (\delta+|\alpha|)\int_0^T\int_0^L |y|^2\,dx\,dt + \int_0^L x|y_0|^2\,dx\\
&\le (\delta+|\alpha|)\int_0^T\int_0^L |y|^2\,dx\,dt + L\int_0^L |y_0|^2\,dx.
\end{aligned}
$$

Using that $|\alpha| < 3\beta$, the second and the third terms in the left hand on the above equation are positive, thus we obtain

$$
\begin{aligned}
&[3\beta-|\alpha|]\|\partial y\|^2_{L^2(0,T:L^2(0,L))}\\
&\le \big[(|\delta|+|\alpha|)\|y\|^2_{L^2(0,T:L^2(0,L))} + L\|y_0\|^2_{L^2(0,L)}\big],
\end{aligned}
$$

where

$$\begin{aligned} &\|\partial y\|^2_{L^2(0,T:L^2(0,L))}\\ &\le \frac{1}{(3\beta-|\alpha|)}\left[(|\delta|+|\alpha|)\|y\|^2_{L^2(0,T:L^2(0,L))}+L\|y_0\|^2_{L^2(0,L)}\right]. \end{aligned}\tag{4.9}$$

Then, using (4.7),

$$\begin{aligned} \|y\|_{L^2(0,T:H^1(0,L))} &\le \Big[T+\frac{1}{(3\beta-|\alpha|)}[(|\delta|+|\alpha|)T+L]\Big]^{1/2}\|y_0\|_{L^2(0,L)}\\ &\le \big[\frac{1}{(3\beta-|\alpha|)}[(|\delta|+3\beta)T+L]\big]^{1/2}\|y_0\|_{L^2(0,L)} \end{aligned}\tag{4.10}$$

By the density of $D(A)$ in $L^2(0,L)$ the result extends to arbitrary $y_0 \in L^2(0,L)$.
(2) We also assume $y_0 \in D(A)$ and taking $\xi(x,t)=1$ in (4.8), we get

$$\beta|\partial y(0,t)|^2 \le \int_0^L |y_0|^2\,dx - \int_0^L |y|^2\,dx \le \int_0^L |y_0|^2\,dx. \tag{4.11}$$

On the other hand the choice $\xi(x,t)=T-t$ yields

$$\partial_t\int_0^L (T-t)|y|^2\,dx + \int_0^L |y|^2\,dx - \beta(T-t)|\partial y(0,t)|^2 \le 0. \tag{4.12}$$

Integrating over $t\in[0,T]$ we have

$$-T\int_0^L |y_0|^2\,dx + \int_0^L\int_0^L |y|^2\,dx\,dt - \beta\int_0^L (T-t)|\partial y(0,t)|^2\,dt \le 0.$$

Hence

$$\begin{aligned} \int_0^L |y_0|^2\,dx &\le \frac{1}{T}\int_0^L\int_0^L |y|^2\,dx\,dt - \frac{\beta}{T}\int_0^L (T-t)|\partial y(0,t)|^2\,dt\\ &\le \frac{1}{T}\int_0^L\int_0^L |y|^2\,dx\,dt + \beta\int_0^L |\partial y(0,t)|^2\,dt. \end{aligned}\tag{4.13}$$

By (4.12) there exists a unique continuous (linear) extension of the map $y_0 \in D(A) \mapsto \partial y(0,.) \in L^2(0,T)$ to the whole space $L^2(0,L)$. In what follows we also will denote by $\partial y(0,.)$ the value of this map at any $y_0 \in L^2(0,L)$. It is trivial to see that (4.12) and (4.13) are true for any $y_0 \in L^2(0,L)$. □

Lemma 4.3 (Observability result). *Let $|\alpha|<3\beta$, $\delta>0$ and*

$$\mathcal{N} = \Big\{2\pi\beta\sqrt{\frac{k^2+kl+l^2)}{3\beta\delta+\alpha^2}} : k,l\in\mathbb{N}^*\Big\}. \tag{4.14}$$

Then, for all $L\in(0,+\infty)\backslash\mathcal{N}$, for all $T>0$, there exists $C=C(L,T)>0$ such that for all $y_0\in L^2(0,L)$,

$$\|y_0\|_{L^2(0,L)} \le C\,\|\partial y(0,.)\|_{L^2(0,T)}. \tag{4.15}$$

Proof. (By contradiction) If the statement is false, there exists a sequence $(y_0^n)_{n\ge0} \in L^2(0,L)$ such that $\|y_0^n\|_{L^2(0,L)}=1$ for any n, but $\|\partial y^n(0,.)\|_{L^2(0,T)} \to 0$ as $n\to\infty$, where $y^n = S(\cdot)y_0^n$. Using (4.11) have that $\{y^n\}$ is bounded in $L^2(0,T : H^2(0,L))$ ($\hookrightarrow L^2(0,T:H^1(0,L)$). On the other hand,

$$\partial_t y^n = -(\beta\partial^3 y^n - i\alpha\partial^2 y^n + \delta\partial y^n) \quad \text{is bounded in } L^2(0,T:H^{-2}(0,L)). \tag{4.16}$$

But $H^1(0,L) \overset{c}{\hookrightarrow} L^2(0,L) \hookrightarrow H^{-2}(0,L)$, then from Lions-Aubin's Theorem (see [23]), the set $\{y^n\}$ is relatively compact in $L^2(0,T : L^2(0,L))$. Without loss of generality, we may assume that the sequence $\{y^n\}$ is convergent in $L^2(0,T : L^2(0,L))$. We infer from (4.6) that $\{y_0^n\}$ is a Cauchy sequence in $L^2(0,T)$. Let $y_0 = \lim_{n\to 0} y_0^n$ and $y = S(\cdot)y_0$. By Lemma 4.2, $\partial y^n(0,.) \to \partial y(0,.)$ in $L^2(0,T)$. Thus, $\|y_0\|_{L^2(0,L)} = 1$ and $\partial y(0,.) = 0$, but such function does not exist because of the following lemma. □

Lemma 4.4. *For $T > 0$ let $\mathcal{F}_T$ denote the space of the initial states $y_0 \in L^2(0,L)$ such that the mild solution $y = S(\cdot)y_0$ of (4.3) satisfies $\partial y(0,.) = 0$ in $L^2(0,T)$. Then, for $L \in (0,\infty)\backslash\mathcal{N}$, $\mathcal{F}_T = \{0\}$, for all $T > 0$.*

Proof. It is obvious that if $T < T'$ then $\mathcal{F}_{T'} \subseteq \mathcal{F}_T$.

Claim. For any $T > 0$, $\mathcal{F}_T$ is a finite-dimensional vector space. In fact, if $\{y_0^n\}$ is a sequence in the unit ball $\mathbb{B}_{\mathcal{F}_T} = \{y \in \mathcal{F}_T : \|y\|_{L^2(0,L)} \leq 1\}$ the same argument as above shows that there exist a convergent subsequence. Since the unit ball is compact, by the Riesz Theorem (see [27]) $\mathcal{F}_T$ is finite dimensional and the claim follows.

Let $T' > 0$ be given. To prove that $\mathcal{F}_{T'} = \{0\}$, it is sufficient to find $0 < T < T'$ such that $\mathcal{F}_T = \{0\}$. Since the map $T \mapsto \dim(\mathcal{F}_T)_{n\in\mathbb{N}}$ is non-increasing, there exist $T, \epsilon > 0$ such that $T < T + \epsilon < T'$ and $\dim \mathcal{F}_T = \dim \mathcal{F}_{T+\epsilon}$, where we obtain that $\mathcal{F}_t = \mathcal{F}_T$ for $T \leq t \leq T + \epsilon$. Let $y_0 \in \mathcal{F}_T$, $y = S(\cdot)y_0$ and $0 < t < \epsilon$. Since $S(\tau)(S(t)y_0) = S(\tau + t)y_0$ for $0 \leq \tau \leq T$ and $y_0 \in \mathcal{F}_{T+\epsilon}$, we see that

$$\frac{S(t)y_0 - y_0}{t} \in \mathcal{F}_T \tag{4.17}$$

Let $\mathcal{M}_T = \{\widetilde{y} = S(\tau)\widetilde{y}_0 : 0 \leq \tau \leq T, \widetilde{y}_0 \in \mathcal{F}_T\} \subseteq C([0,T] : L^2(0,L))$. Since $y \in H^1(0,T+\epsilon : H^{-2}(0,L))$,

$$\lim_{t\to 0^+} \frac{y(t+\cdot) - y}{t} = y' \quad \text{in } L^2(0,T : H^{-2}(0,L)). \tag{4.18}$$

On the other hand, by (4.17), $\frac{y(t+\cdot)-y}{t} \in \mathcal{M}_T$ for $0 < t < \epsilon$ and $\mathcal{M}_T$ is closed in $L^2(0,T : H^{-2}(0,L))$, since $\dim \mathcal{M}_T < \infty$. It follows that $y' \in C([0,T] : L^2(0,L))$ and $y \in C^1([0,T] : L^2(0,L))$. Hence, we may write

$$y'(0) = \lim_{t\to 0^+} \frac{S(t)y_0 - y_0}{t} \quad \text{in } L^2(0,L).$$

Then

$$y_0 \in D(A), \quad A(y_0) = y'(0) \in \mathcal{F}_T \quad \text{and} \quad \partial y(0,.) \in C([0,T]). \tag{4.19}$$

Hence,

$$\Big(\frac{dy_0}{dx}\Big)_{x=0} = \partial y(0,0) = 0.$$

If $\mathcal{F}_T \neq \{0\}$, the map $y_0 \in \mathbb{C}\mathcal{F}_T \mapsto A(y_0) \in \mathbb{C}\mathcal{F}_T$ (where $\mathbb{C}\mathcal{F}_T$ denote the complexification of $\mathcal{F}_T$) has at least one eigenvalue, thus there exist $\lambda \in \mathbb{C}$, $y_0 \in H^3(0,L)\backslash\{0\}$ such that

$$\lambda y_0 = -\beta y_0''' + i\alpha y_0'' - i\delta y_0' y_0(0) = y_0(L) = y_0'(0) = y_0'(L) = 0. \tag{4.20}$$

We prove in the following Lemma that this does not hold if $L \in \mathcal{N}$. □

Lemma 4.5. *Let $|\alpha| < 3\beta$, $L \in (0, +\infty)$ and*

$$\begin{gathered} \exists\lambda \in \mathbb{C},\ \exists y_0 \in H^3(0,L)\backslash\{0\} \quad \textit{such that} \\ \lambda y_0 + \beta y_0''' - i\alpha y_0'' + \delta y_0' = 0, \\ y_0(0) = y_0(L) = y_0'(0) = 0, \end{gathered} \tag{4.21}$$

Then (4.21) *is satisfied if and only if $L \in \mathcal{N}$.*

Proof. Let $y_0 \in (\mathbb{K})$, we denote by $u \in H^2(\mathbb{R})$ its prolongation by 0; i. e.,

$$u(x) = \begin{cases} y_0, & \text{if } x \in (0,L) \\ 0, & \text{if } x \in (0,L)^c. \end{cases}$$

Then

$$\lambda u + \beta u''' - i\alpha u'' + \delta u' = \beta y_0''(0)\delta_0 - \eta y_0''(L)\delta_L \quad \text{in } \mathcal{D}'(\mathbb{R}), \tag{4.22}$$

where δ_{x_0} denotes the Dirac measure at x_0. Is easy to see that (4.21) is equivalent to the existence of complex numbers μ, η, λ (with $(\mu, \eta) \neq (0,0)$) and of a function $u \in H^2(\mathbb{R})$ with compact support in $[-L, L]$ such that

$$\lambda u + \beta u''' - i\alpha u'' + \delta u' = \eta\delta_0 - \mu\delta_L \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{4.23}$$

Taking Fourier transform we have

$$(\lambda + \beta(i\xi)^3 - i\alpha(i\xi)^2 + \delta(i\xi))\widehat{u}(\xi) = \eta - \mu e^{-iL\xi};$$

hence setting $\lambda = -ip$, we obtain

$$\widehat{u}(\xi) = i\,\Big[\frac{\eta - \mu e^{-iL\xi}}{\beta\xi^3 - \alpha\xi^2 - \delta\xi + p}\Big].$$

Using Paley-Wiener's theorem (see [27]) and the usual characterization of $H^2(\mathbb{R})$ functions by means of their Fourier transform, we see that (4.21) is equivalent to the existence of $p \in \mathbb{C}$ and $(\eta, \mu) \in \mathbb{C}^2\backslash\{(0,0)\}$ such that the map

$$f(\xi) = \Big[\frac{\eta - \mu e^{-iL\xi}}{\beta\xi^3 - \alpha\xi^2 - \delta\xi + p}\Big]$$

satisfies

(1) f is an entire function in $\mathbb{C}$
(2) $\int_{\mathbb{R}} |f(\xi)|^2(1 + |\xi|^2)^2\,d\xi < \infty$
(3) For all $\xi \in \mathbb{C}$, $|f(\xi)| \le C(1+|\xi|)^N e^{L|\operatorname{Im}\xi|}$ for some positive constants C, N.

Since the roots of $\eta - \mu e^{-iL\xi}$ are simple unless $\eta = \mu = 0$, (1) holds provided that the roots of $\beta\xi^3 - \alpha\xi^2 - \delta\xi + p$ are simple, and the roots of $\eta - \mu e^{-iL\xi}$. We have that if (1) holds, then (2) and (3) are satisfied. It follows that (4.21) is equivalent to the existence of complex number p, μ_0 and of positive integers k, l, m, and n such that, if we set

$$\mu_1 = \mu_0 + k\frac{2\pi}{L}, \quad \mu_2 = \mu_1 + l\frac{2\pi}{L} = \mu_0 + (k+l)\frac{2\pi}{L}$$

we have

$$\xi^3 - \frac{\alpha}{\beta}\xi^2 - \frac{\delta}{\beta}\xi + \frac{1}{\beta}p = (\xi - \mu_0)(\xi - \mu_1)(\xi - \mu_2)$$

that is

$$\mu_0 + \mu_1 + \mu_2 = \frac{\alpha}{\beta},$$

$$\mu_0\mu_1 + \mu_0\mu_2 + \mu_1\mu_2 = -\frac{\delta}{\beta},$$
$$\mu_0\mu_1\mu_2 = \frac{1}{\beta}p\,.$$

Straightforward calculus leads to

$$L = 2\pi\beta\sqrt{\frac{k^2+kl+l^2}{3\beta\delta+\alpha^2}},$$
$$\mu_0 = \frac{1}{3}\Big[\frac{\alpha}{\beta} - (2k+l)\frac{2\pi}{L}\Big],$$
$$p = \beta\mu_0\big(\mu_0 + k\frac{2\pi}{L}\big)\big(\mu_0 + (k+l)\frac{2\pi}{L}\big).$$

Hence, (4.21) is satisfied if and only if $L \in \mathcal{N}$. This complete the proof of Lemmas 4.3, 4.4, and 4.5. $\square$

Remark 4.6. *For $L \in \mathcal{N}$, if p is given as above and y_0 (with $\operatorname{Re} y_0 \neq 0$) is as in (4.21) with $\lambda = -ip$, then $y(x,t) = \operatorname{Re}(e^{-ipt}y_0(x))$ is a nontrivial smooth solution of (4.3) such that $\partial y(0,.) \equiv 0$. Thus, the result in Lemma 4.3. holds if and only if $L \notin \mathcal{N}$.*

The goal of the following lemma is to define in a certain weak sense a solution of the non-homogeneous problem (R_1).

Lemma 4.7. *Let $|\alpha| < 3\beta$. There exists a unique linear continuous map $\Pi : L^2(0,L)\times L^2(0,T) \mapsto \mathbb{H}$ such that, for $y_0 \in D(A)$ and $h \in C^2([0,T])$ with $h(0) = 0$, $\Pi(y_0,h)$ is the unique classical solution of (4.1).*

Proof. We assume here that $y_0 \in D(A)$ and $h \in C_0^2([0,T]) = \{h \in C^2([0,T] : \mathbb{R}) : h(0) = 0\}$. Let $\phi \in C^\infty([0,L])$ be such that $\phi(0) = \phi(L) = 0$ and $\phi'(L) = -1$. Then the change of function $z(x,t) = y(x,t) - (S(t)y_0)(x) + h(t)\phi(x)$ transforms (4.1) into

$$\begin{gathered}
\partial_t z + \beta\partial^3 z - i\alpha\partial^2 z + \delta\partial z = h'(t)\phi(x) + h(t)[\beta\phi''' - i\alpha\phi'' + \delta\phi'] = f(x,t)\\
z(0,t) = z(L,t) = 0\\
\partial z(L,t) = 0\\
z(.,0) = 0
\end{gathered} \tag{4.24}$$

Using Lemma 4.1. and that $f \in C^1([0,T] : L^2(0,L))$, we obtain that there exists a unique solution (see [25]) for the non-homogeneous problem $z \in C([0,T] : D(A)) \cap C^1([0,T] : L^2(0,L))$ of (4.24). Hence, for smooth data $y_0 \in D(A)$, $h \in C_0^2([0,1])$, (4.1) admits a unique classical solution

$$y \in C([0,T] : H^3(0,L)) \cap C^1([0,T] : L^2(0,L)).$$

On the other hand, we assume that $y_0 \in D(A)$, $h \in C_0^2([0,T])$. Let $\xi = \xi(x,t) \in C^\infty([0,T]\times[0,L])$. From equation (4.1) we have (multiplying by i)

$$i\partial_t y + i\beta\partial^3 y + \alpha\partial^2 y + i\delta\partial y = 0. \tag{4.25}$$

Multiplying by $\xi\overline{y}$ we obtain

$$i\xi\overline{y}\partial_t y + i\beta\xi\overline{y}\partial^3 y + \alpha\xi\overline{y}\partial^2 y + i\delta\xi\overline{y}\partial y = 0,$$
$$-i\xi y\partial_t\overline{y} - i\beta\xi y\partial^3\overline{y} + \alpha\xi y\partial^2\overline{y} - i\delta\xi y\partial\overline{y} = 0,$$

(applying conjugate). Subtracting and integrating over $x \in [0, L]$ we obtain

$$\begin{aligned} &i\partial_t \int_0^L \xi|y|^2\,dx - i\int_0^L \partial_t\xi|y|^2\,dx + i\beta\int_0^L \xi\overline{y}\partial^3 y\,dx + i\beta\int_0^L \xi y\partial^3\overline{y}\,dx \\ &+ \alpha\int_0^L \xi\overline{y}\partial^2 y\,dx - \alpha\int_0^L \xi y\partial^2\overline{y}\,dx - i\delta\int_0^L \partial\xi|y|^2\,dx = 0. \end{aligned} \tag{4.26}$$

Each term is treated separately. Integrating by parts

$$\begin{gathered} \int_0^L \xi\overline{y}\partial^3 y\,dx = \int_0^L \partial^2\xi\overline{y}\partial y\,dx + 2\int_0^L \partial\xi|\partial y|^2\,dx \\ + \xi(L,t)|h(t)|^2 - \xi(0,t)|\partial y(0,t)|^2 + \int_0^L \xi\partial y\partial^2\overline{y}\,dx, \\ \int_0^L \xi y\partial^3\overline{y}\,dx = \int_0^L \partial^2\xi y\partial\overline{y}\,dx + \int_0^L \partial\xi|\partial y|^2\,dx - \int_0^L \xi\partial y\partial^2\overline{y}\,dx, \\ \int_0^L \xi\overline{y}\partial^2 y\,dx = -\int_0^L \partial\xi\overline{y}\partial y\,dx - \int_0^L \xi|\partial y|^2\,dx, \\ \int_0^L \xi y\partial^2\overline{y}\,dx = -\int_0^L \partial\xi y\partial\overline{y}\,dx - \int_0^L \xi|\partial y|^2\,dx\,. \end{gathered}$$

Then

$$\begin{aligned} &i\partial_t \int_0^L \xi|y|^2\,dx - i\int_0^L \partial_t\xi|y|^2\,dx + i\beta\int_0^L \partial^2\xi\overline{y}\partial y\,dx + 2i\beta\int_0^L \partial\xi|\partial y|^2\,dx \\ &+ \beta\xi(L,t)|h(t)|^2 - i\beta\xi(0,t)|\partial y(0,t)|^2 + i\beta\int_0^L \xi\partial y\partial^2\overline{y}\,dx + i\beta\int_0^L \partial^2\xi y\partial\overline{y}\,dx \\ &+ i\beta\int_0^L \partial\xi|\partial y|^2\,dx - i\beta\int_0^L \xi\partial y\partial^2\overline{y}\,dx - \alpha\int_0^L \partial\xi\overline{y}\partial y\,dx - \alpha\int_0^L \xi|\partial y|^2\,dx \\ &+ \int_0^L \partial\xi y\partial\overline{y}\,dx + \int_0^L \xi|\partial y|^2\,dx - i\delta\int_0^L \partial\xi|y|^2\,dx = 0\,. \end{aligned}$$

Hence,

$$\begin{aligned} &i\partial_t \int_0^L \xi|y|^2\,dx - i\int_0^L \partial_t\xi|y|^2\,dx + i\beta\int_0^L \partial^2\xi\partial(|y|^2)\,dx + 3i\beta\int_0^L \partial\xi|\partial y|^2\,dx \\ &+ \beta\xi(L,t)|h(t)|^2 - i\beta\xi(0,t)|\partial y(0,t)|^2 - 2i\alpha\,\mathrm{Im}\int_0^L \partial\xi\overline{y}\partial y\,dx - i\delta\int_0^L \partial\xi|y|^2\,dx \\ &= 0\,. \end{aligned}$$

Thus

$$\begin{aligned} &\partial_t \int_0^L \xi|y|^2\,dx - \int_0^L \partial_t\xi|y|^2\,dx - \beta\int_0^L \partial^3\xi|y|^2\,dx + 3\beta\int_0^L \partial\xi|\partial y|^2\,dx \\ &+ \beta\xi(L,t)|h(t)|^2 - \beta\xi(0,t)|\partial y(0,t)|^2 - \delta\int_0^L \partial\xi|y|^2\,dx \\ &= 2\alpha\,\mathrm{Im}\int_0^L \partial\xi\overline{y}\partial y\,dx \le |\alpha|\int_0^L \partial\xi|y|^2\,dx + |\alpha|\int_0^L \partial\xi|\partial y|^2\,dx\,, \end{aligned}$$

where

$$\begin{aligned}&\partial_t \int_0^L \xi|y|^2\,dx + \int_0^L [3\beta-|\alpha|]\partial\xi|\partial y|^2\,dx - \int_0^L \partial_t\xi|y|^2\,dx - \beta\int_0^L \partial^3\xi|y|^2\,dx\\ &+\beta\xi(L,t)|h(t)|^2 - \beta\xi(0,t)|\partial y(0,t)|^2 - \delta\int_0^L \partial\xi|y|^2\,dx - |\alpha|\int_0^L \partial\xi|y|^2\,dx \le 0.\end{aligned}$$

Integrating over $t\in[0,T]$ we have

$$\begin{aligned}&\int_0^L \xi|y|^2\,dx + \int_0^t\int_0^L [3\beta-|\alpha|]\partial\xi|\partial y|^2\,dx\,ds - \int_0^t\int_0^L \partial_t\xi|y|^2\,dx\,ds\\ &-\beta\int_0^t\int_0^L \partial^3\xi|y|^2\,dx\,ds + \beta\int_0^t \xi(L,s)|h(s)|^2\,ds\\ &-\beta\int_0^t \xi(0,s)|\partial y(0,s)|^2\,ds\\ &-\delta\int_0^t\int_0^L \partial\xi|y|^2\,dx\,ds - |\alpha|\int_0^t\int_0^L \partial\xi|y|^2\,dx\,ds\\ &\le \int_0^L \xi(x,0)|y_0|^2\,dx.\end{aligned}\tag{4.27}$$

Choosing $\xi(x,t)=-1$ leads to

$$\int_0^L |y|^2\,dx - \beta\int_0^t |h(s)|^2\,ds + \beta\int_0^t |\partial y(0,s)|^2\,ds \le \int_0^L |y_0|^2\,dx,$$

where

$$\|y\|^2_{L^2(0,L)} + \beta\|\partial y(0,.)\|^2_{L^2(0,T)} \le \|y_0\|^2_{L^2(0,L)} + \beta\|h\|^2_{L^2(0,T)}.$$

Setting $\|(y_0,h)\| = \big[\|y_0\|^2_{L^2(0,L)} + \beta\|h\|^2_{L^2(0,T)}\big]^{1/2}$, we get

$$\|y\|_{C([0,T]:L^2(0,L))} \le \|(y_0,h)\| \tag{4.28}$$

which yields

$$\|y\|_{L^2([0,T]\times(0,L))} \le \sqrt{T}\,\|(y_0,h)\|. \tag{4.29}$$

Now, we take $\xi(x,t)=x$, and $t=T$ in (4.27)

$$\begin{aligned}&\int_0^L x|y(x,T)|^2\,dx + \int_0^T\int_0^L [3\beta-|\alpha|]|\partial y|^2\,dx\,dt\\ &+\beta L\int_0^T |h(s)|^2\,ds - \delta\int_0^T\int_0^L |y|^2\,dx\,dt - |\alpha|\int_0^T\int_0^L |y|^2\,dx\,dt\\ &\le \int_0^L x|y_0|^2\,dx\,,\end{aligned}$$

where

$$\begin{aligned}&\int_0^L x|y(x,T)|^2\,dx + \int_0^T\int_0^L [3\beta-|\alpha|]|\partial y|^2\,dx\,dt\\ &\le \int_0^L x|y_0|^2\,dx - \beta L\int_0^T |h(s)|^2\,ds + (\delta+|\alpha|)\int_0^T\int_0^L |y|^2\,dx\,dt\\ &\le (\delta+|\alpha|)\int_0^T\int_0^L |y|^2\,dx\,dt + L\Big(\int_0^L |y_0|^2\,dx + \beta\int_0^T |h(s)|^2\,ds\Big)\\ &= (\delta+|\alpha|)\|y\|^2_{L^2([0,T]\times(0,L))} + L\big(\|y_0\|^2_{L^2(0,L)} + \beta\|h\|^2_{L^2(0,T)}\big)\end{aligned}$$

$$
\begin{aligned}
&= (\delta + |\alpha|)T\|(y_0,h)\|^2 + L\big(\|(y_0,h)\|^2\big)\\
&= [(\delta + |\alpha|)T + L]\|(y_0,h)\|^2 .
\end{aligned}
$$

Then

$$[3\beta - |\alpha|] \int_0^T \int_0^L |\partial y|^2\, dx\, dt \le [(\delta + |\alpha|)T + L]\|(y_0,h)\|^2,$$

where

$$\|\partial y\|^2_{L^2(0,T:L^2(0,L))} \big[\frac{(\delta + |\alpha|)T + L}{3\beta - |\alpha|}\big]\|(y_0,h)\|^2 . \tag{4.30}$$

Adding (4.29) and (4.30) and using the fact that $|\alpha| < 3\beta$ we obtain

$$\|y\|_{L^2(0,T:H^1(0,L))} \le \big[\frac{(\delta + 3\beta)T + L}{3\beta - |\alpha|}\big]^{1/2}\|(y_0,h)\|. \tag{4.31}$$

Using (4.28) and (4.31), and the density of $D(A)$ in $L^2(0,L)$ and of $C_0^2([0,T])$ in $L^2(0,T)$, we see that the linear map $(y_0,h) \in D(A) \times C_0^2([0,T]) \mapsto y \in \mathbb{H}$ may be extended in a unique manner to the whole space $L^2(0,T) \times L^2(0,L)$ to give a linear map $\Pi : L^2(0,T) \times L^2(0,L) \mapsto \mathbb{H}$. □

Remark 4.8. *(a) For $y_0 \in L^2(0,L)$ and $h \in L^2(0,T)$, the weak solution $\Pi(y_0,h)$ is solution of (4.1) in $\mathcal{D}'(0,T : H^{-2}(0,L))$. Moreover, $\Pi(y_0,h)(.,0) = y_0$ and $\Pi(y_0,h)(.,T)$ are well-defined in $L^2(0,L)$, since $\Pi(y_0,h) \in C([0,T] : L^2(0,L))$.*
(b) $\Pi(y_0,0) = S(\cdot)y_0$, hence, $\Pi(y_0,0) = S(\cdot)y_0 + \Pi(0,h)$.

To apply the Hilbert uniqueness method, we need some observability result concerning the following backward well-posed homogeneous problem: For $|\alpha| < 3\beta$ and $\delta > 0$

$$
\begin{gathered}
\partial_t u + \beta\partial^3 u - i\alpha\partial^2 u + \delta\partial u = 0,\\
u(0,t) = u(L,t) = 0,\\
\partial u(0,t) = 0,\\
u(T,0) = u_T(x)\, .
\end{gathered}
\tag{4.32}
$$

The change of variables $\tau T - t$ and $\zeta = L - x$ transform (4.32) into (4.3) and vice-versa. Using Lemmas 4.1, 4.2 and 4.3, we readily get the following result.

Lemma 4.9. *]Observability result] Let $L, T > 0$, $|\alpha| < 3\beta$ and $\delta > 0$. For any $u_T \in L^2(0,L)$ the mild solution of (4.32) belongs to $\mathbb{H}$, the function $\partial u(L,.)$ makes sense in $L^2(0,T)$. If moreover, $L \notin \mathcal{N}$, there exists a constant $C = C(L,T) > 0$ such that for any $u_T \in L^2(0,L)$ we have that*

$$\|\partial u(L,.)\|_{L^2(0,T)} \le \|u_T\|_{L^2(0,T)} \le C\ \|\partial u(L,.)\|_{L^2(0,T)}. \tag{4.33}$$

It remains to apply the Hilbert uniqueness method.

Theorem 4.10. *Let $|\alpha| < 3\beta$, $\delta > 0$ and*

$$\mathcal{N} = \big\{2\pi\beta\sqrt{\frac{k^2 + kl + l^2)}{3\beta\delta + \alpha^2}} : k, l \in \mathbb{N}^*\big\}.$$

Then, for any $T > 0$ and $L \in (0,+\infty)\backslash\mathcal{N}$, and for any y_0, $y_T \in L^2(0,L)$, there exists $h \in L^2(0,T)$ such that the mild solution $y \in C([0,T] : L^2(0,L)) \cap L^2(0,T : H^1(0,L))$ of

$$\partial_t y + \beta\partial^3 y - i\alpha\partial^2 y + \delta\partial y = 0 \tag{4.34}$$

$$y(0,t) = y(L,T) = 0 \tag{4.35}$$

$$\partial y(L,t) = h(t) \tag{4.36}$$

$$y(x,0) = y_0(x) \tag{4.37}$$

satisfies $y(.,T) = y_T$.

Proof. By Remark 4.8 (b) we may assume, without loss of generality, that $y_0 = 0$. (see the proof of Theorem 3.3) Let $(u_T, h) \in C_c^\infty(0,L) \times C_c^\infty(0,L)$, let u (resp. y) be the classical solution of (4.34)-(4.37) (resp. (R_1)). Multiplying (4.34) by u and integrating over $x \in [0,L]$ we have

$$\int_0^L u\partial_t y\,dx + \beta\int_0^L u\partial^3 y\,dx - i\alpha\int_0^L u\partial^2 y\,dx + \delta\int_0^L u\partial y\,dx = 0 \tag{4.38}$$

Each term is treated separately. Integrating by parts,

$$\begin{gathered}
\int_0^L u\partial_t y\,dx = \partial_t\int_0^L uy\,dx - \int_0^L \partial_t uy\,dx,\\
\int_0^L u\partial^3 y\,dx = -\partial u(L,t)h(t) - \int_0^L \partial^3 uy\,dx,\\
\int_0^L u\partial^2 y\,dx = \int_0^L \partial^2 uy\,dx,\\
\int_0^L u\partial y\,dx = -\int_0^L \partial uy\,dx\,.
\end{gathered}$$

Then in (4.38) we obtain

$$\begin{aligned}
&\partial_t\int_0^L uy\,dx - \int_0^L \partial_t uy\,dx - \beta\partial u(L,t)h(t) - \beta\int_0^L \partial^3 uy\,dx\\
&- i\alpha\int_0^L \partial^2 uy\,dx - \delta\int_0^L \partial uy\,dx = 0
\end{aligned}$$

where

$$\begin{aligned}
\partial_t\int_0^L uy\,dx - \beta\partial u(L,t)h(t) &= -2i\alpha\int_0^L \partial u\partial y\,dx\\
&\le |\alpha|\int_0^L |\partial u|^2\,dx + |\alpha|\int_0^L |\partial y|^2\,dx\,.
\end{aligned} \tag{4.39}$$

Integrating over $t \in [0,T]$ and using that $y_0 = 0$ we obtain

$$\begin{aligned}
&\int_0^L u_T(x)y(x,T)\,dx\\
&\le \beta\int_0^T \partial u(L,t)h(t)\,dt + |\alpha|\int_0^T\int_0^L |\partial u|^2\,dx\,dt + |\alpha|\int_0^T\int_0^L |\partial y|^2\,dx\,dt
\end{aligned} \tag{4.40}$$

By a density argument we see that (4.40) holds for $u_T \in L^2(0,L)$ and $h \in L^2(0,T)$. Let Λ denote the linear continuous map $\Lambda : L^2(0,L) \mapsto L^2(0,L)$ with $u_T \mapsto \Lambda(u_T) = y(.,T)$ and y standing for the solution of (4.1) associated with the data $h(.) = \partial u(L,.) \in L^2(0,T)$. It follows (4.40) and by Lemma 4.9 that

$$\langle\Lambda(u_T), u_T\rangle_{L^2(0,L)} = \|\partial u(L,.)\|^2_{L^2(0,T)} \ge C^{-2}\,\|u_T\|^2_{L^2(0,L)}.$$

Therefore, by Lax-Milgram's theorem (see [34]), Λ is invertible. The proof is complete. $\square$

Remark 4.11. *When $y_0 = 0$, the Hilbert uniqueness method yields u, a linear continuous selection of the control, namely the map $\Gamma_0 : L^2(0,L) \mapsto L^2(0,T)$ with $y_T \mapsto \Gamma_0(y_T) = \partial u(L,.)$ where u denotes the solution of (4.32) associated with $u_T = \Lambda^{-1}(y_T)$.*

5. Exact boundary controllability for a higher order nonlinear Schrödinger equation with constant coefficients on a bounded domain

In this section we prove that the following boundary-control system (for $|\alpha| < 3\beta$ and $\delta > 0$)

$$\begin{gathered} \partial_t y + \beta\partial^3 y - i\alpha\partial^2 y - i|y|^2 y + \delta\partial y = 0 \\ y(0,t) = y(L,t) = 0 \\ \partial y(L,t) = h(t), \quad h \in L^2(0,T) \\ y(.,0) = y_0 \end{gathered} \tag{5.1}$$

is exactly controllable in a neighborhood of the null state. More precisely we show that for any $L > 0$ and $T > 0$ there exists a radius $r_0 > 0$ such that for every $y_0, y_T \in L^2(0,L)$ with $\|y_0\|_{L^2(0,L)} < r_0$, $\|y_T\|_{L^2(0,L)} < r_0$ we may find $y \in \mathbb{H} = C([0,T] : L^2(0,L)) \cap L^2(0,T : H^1(0,L))$ such that

1. $\partial_t y = -(\beta\partial^3 y - i\alpha\partial^2 y - i|y|^2 y + \partial y)$ in $\mathcal{D}'(0,T : H^{-2}(0,L))$.
2. $y(.,0) = y_0$, $y(.,T) = y_T$.

Remark 5.1. For $y \in \mathbb{H}$, $\partial y \in L^2(0,T : L^2(0,L))$, $\partial^3 y \in L^2(0,T : H^{-2}(0,L))$, and $|y|^2 y \in L^1(0,T : L^2(0,L))$. Hence,

$$\partial_t y = -(\beta\partial^3 y - i\alpha\partial^2 y - i|y|^2 y + \delta\partial y) \in L^1(0,T : H^{-2}(0,L));$$

i. e., $y \in W^{1,1}(0,T : H^{-2}(0,L))$.

To solve (5.1), we write $y = S(t)y_0 + y_1 + y_2$ where $(S(t))_{t\geq 0}$ denotes the semigroup associated with the operator A of section 4, y_1 and y_2 are respectively solutions of the two nonhomogeneous problems:

$$\begin{gathered} \partial_t y_1 + \beta\partial^3 y_1 - i\alpha\partial^2 y_1 + \delta\partial y_1 = 0 \\ y_1(0,t) = y_1(L,t) = 0 \\ \partial y_1(L,t) = h(t) \\ y_1(.,0) = y_0 \end{gathered} \tag{5.2}$$

and

$$\begin{gathered} \partial_t y_2 + \beta\partial^3 y_2 - i\alpha\partial^2 y_2 + \delta\partial y_2 = f \\ y_2(0,t) = y_2(L,t) = 0 \\ \partial y_2(L,t) = 0 \\ y_2(.,0) = 0\,. \end{gathered} \tag{5.3}$$

In (5.3) we have the set $f = -|y|^2 y$. Let $\Gamma_1 : h \in L^2(0,T) \mapsto y_1 \in \mathbb{H}$ be the map which associates the weak solution of (5.2) with h. By Lemma 4.7, Γ_1 is a linear continuous map.

Lemma 5.2. *For $|\alpha| < 3\beta$, we have*

(1) *If* $y \in L^2(0,T : H^1(0,L))$, $|y|^2 y \in L^1(0,T : L^2(0,L))$ *and the map* $y \mapsto |y|^2 y$ *is continuous.*
(2) *For* $f \in L^1(0,T : L^2(0,L))$ *the mild solution* y_2 *of* (5.3) *belong to* $\mathbb{H}$. *Moreover the linear map* $\Gamma_2 : f \mapsto y_2$ *is continuous.*

We remark that for $f \in L^1(0,T : L^2(0,L))$ the mild solution y_2 of (5.3) is given by

$$y_2(.,t) = \int_0^t S(t-s) f(.,s)\, ds$$

Proof of Lemma 5.2. (1) Let $y, z \in L^2(0,T : H^1(0,L))$. Let $\mathcal{H}_1$ be the norm of the Sobolev embedding $H^1(0,L) \hookrightarrow L^2(0,L)$. We have

$$|y|^2 y - |z|^2 z = (|y|^2 - |z|^2) y + |z|^2 (y - z) = (|y| - |z|)(|y| + |z|) y + |z|^2 (y - z);$$

hence

$$\begin{aligned} \big||y|^2 y - |z|^2 z\big| &= \big||y| - |z|\big|(|y| + |z|)|y| + |z|^2 |y - z| \\ &\le |y - z|(|y| + |z|)|y| + |z|^2 |y - z|\,. \end{aligned}$$

Applying the triangular inequality and Holder's inequality,

$$\begin{aligned} \||y|^2 y - |z|^2 z\|_{L^1(0,T:L^2(0,L))} &\le \int_0^T \||(y-z)(.,t)|(|y| + |z|)|y|\|_{L^2(0,L)}\, dt \\ &\quad + \int_0^T \|\, |z|^2 |(y-z)(.,t)|\, \|_{L^2(0,L)}\, dt \\ &\le \int_0^T \|y\|^2_{L^\infty(0,L)} \|z\|_{L^\infty(0,L)} \|(y-z)(.,t)\|_{L^2(0,L)}\, dt \\ &\quad + \int_0^T \|z\|^2_{L^\infty(0,L)} \|(y-z)(.,t)\|_{L^2(0,L)}\, dt \\ &\le \mathcal{H}_1 \int_0^T \|(y-z)(.,t)\|_{L^2(0,L)}\, dt \\ &\le \mathcal{H}_1 \|(y-z)(.,t)\|_{L^2(0,T:H^1(0,L))}. \end{aligned} \tag{5.4}$$

Choosing $z = 0$ yields $|y|^2 y \in L^1(0,T : L^2(0,L))$, and (5.4) with z tending to y gives the continuity of the map $|y|^2 y$.
(2) Since

$$\|1_{[0,t]}(s) S(t-s) f(.,s)\|_{L^2(0,L)} \le \|f(.,s)\|_{L^2(0,L)},$$

using Lebesgue's Theorem, the mild solution $y_2(.,t) = \int_0^t S(t-s) f(.,t)\, ds$ belongs to $C([0,T] : L^2(0,L))$. Moreover, for every $t \in [0,T]$,

$$\|y_2(.,t)\|_{L^2(0,L)} \le \int_0^t \|f(.,s)\|_{L^2(0,L)}\, ds \le \|f\|_{L^1(0,T:L^2(0,L))} \tag{5.5}$$

so the linear map $f \in L^1(0,T : L^2(0,L)) \mapsto y_2 \in C([0,T] : L^2(0,L))$ is continuous. To show that this map is well-defined and continuous from $L^1(0,T : L^2(0,L))$ into $L^2(0,T : H^1(0,L))$, it is clearly sufficient to prove that there exists $c_2 > 0$ such that for for all $f \in C^1([0,T] : L^2(0,L))$,

$$\|\partial y_2\|_{L^2((0,T)\times(0,L))} \le c_2 \|f\|_{L^1(0,T:L^2(0,L))}.$$

In fact, multiplying (5.3) by $ix\overline{y}_2$,

$$
\begin{gathered}
ix\overline{y}_2\partial_t y_2 + i\beta x\overline{y}_2\partial^3 y_2 + \alpha x\overline{y}_2\partial^2 y_2 + i\delta x\overline{y}_2\partial y_2 = ix\overline{y}_2 f, \\
-ixy_2\partial_t\overline{y}_2 - i\beta xy_2\partial^3\overline{y}_2 + \alpha xy_2\partial^2\overline{y}_2 - i\delta xy_2\partial\overline{y}_2 = -ixy_2 f
\end{gathered}
$$

(applying conjugate). Subtracting and integrating over $x \in [0, L]$ we have

$$
\begin{aligned}
& i\partial_t \int_0^L x|y_2|^2\,dx + i\beta \int_0^L x\overline{y}_2\partial^3 y_2\,dx + i\beta \int_0^L xy_2\partial^3\overline{y}_2\,dx \\
& + \alpha \int_0^L x\overline{y}_2\partial^2 y_2\,dx - \alpha \int_0^L xy_2\partial^2\overline{y}_2\,dx - i\delta \int_0^L |y_2|^2\,dx \\
& = 2i\,\mathrm{Re} \int_0^L x\overline{y}_2 f\,dx\,.
\end{aligned}
$$

Each term is treated separately. Integrating by parts,

$$
\begin{gathered}
\int_0^L x\overline{y}_2\partial^3 y_2\,dx = 2\int_0^L |\partial y_2|^2\,dx + \int_0^L x\partial y_2\partial\overline{y}_2\,dx\,, \\
\int_0^L xy_2\partial^3\overline{y}_2\,dx = \int_0^L |\partial y_2|^2\,dx - \int_0^L x\partial y_2\partial\overline{y}_2\,dx\,, \\
\int_0^L x\overline{y}_2\partial^2 y_2\,dx = -\int_0^L \overline{y}_2\partial y_2\,dx - \int_0^L x|\partial y_2|^2\,dx\,, \\
\int_0^L xy_2\partial^2\overline{y}_2\,dx = -\int_0^L y_2\partial\overline{y}_2\,dx - \int_0^L x|\partial y_2|^2\,dx\,.
\end{gathered}
$$

Then

$$
\begin{aligned}
& i\partial_t \int_0^L x|y_2|^2\,dx + 3i\beta \int_0^L |\partial y_2|^2\,dx - 2i\alpha \int_0^L \overline{y}_2\partial y_2\,dx - i\delta \int_0^L |y_2|^2\,dx \\
& = 2i\,\mathrm{Re} \int_0^L x\overline{y}_2 f\,dx\,,
\end{aligned}
$$

or

$$
\begin{aligned}
& \partial_t \int_0^L x|y_2|^2\,dx + 3\beta \int_0^L |\partial y_2|^2\,dx - 2\alpha \int_0^L \overline{y}_2[\partial y_2]\,dx - \delta \int_0^L |y_2|^2\,dx \\
& = 2iRe \int_0^L x\overline{y}_2 f\,dx\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \partial_t \int_0^L x|y_2|^2\,dx + 3\beta \int_0^L |\partial y_2|^2\,dx \\
& = 2iRe \int_0^L x\overline{y}_2 f\,dx + \int_0^L |y_2|^2\,dx + 2\alpha \int_0^L \overline{y}_2[\partial y_2]\,dx \\
& \le 2\int_0^L x|y_2||f|\,dx + \delta \int_0^L |y_2|^2\,dx + |\alpha| \int_0^L |y_2|^2\,dx + |\alpha| \int_0^L |\partial y_2|^2\,dx\,.
\end{aligned}
$$

Thus

$$
\partial_t \int_0^L x|y_2|^2\,dx + \int_0^L (3\beta - |\alpha|)|\partial y_2|^2\,dx
$$

$$\leq 2\int_0^L x|y_2||f|\,dx + (|\delta|+|\alpha|)\int_0^L |y_2|^2\,dx\,.$$

Integrating over $t \in [0,T]$ we obtain

$$\begin{aligned}&\int_0^L x|y_2|^2\,dx + \int_0^T\int_0^L (3\beta-|\alpha|)|\partial y_2|^2\,dx\,dt\\ &\leq 2\int_0^T\int_0^L x|y_2||f|\,dx\,dt + (|\delta|+|\alpha|)\int_0^T\int_0^L |y_2|^2\,dx\,dt + \int_0^L x|y_{02}|^2\,dx\,.\end{aligned}$$

Using (5.5) the result follows. □

Theorem 5.3. *Let $|\alpha| < 3\beta$, $\delta > 0$, $T > 0$ and $L > 0$. Then, there exists $r_0 > 0$ such that for any y_0, $y_T \in L^2(0,L)$ with $\|y_0\|_{L^2(0,L)} < r_0$, $\|y_T\|_{L^2(0,L)} < r_0$, there exists*

$$y \in C([0,T]:L^2(0,L)) \cap L^2(0,T:H^1(0,L)) \cap W^{1,1}(0,T:H^{-2}(0,L)) \tag{5.6}$$

solution of

$$i\partial_t y_t = -(i\beta\partial^3 y + \alpha\partial^2 y + |y|^2 y + i\delta\partial y) \quad \text{in } \mathcal{D}'(0,T:H^{-2}(0,L)) \tag{5.7}$$

$$y(0,.) = 0 \quad \text{in } L^2(0,T) \tag{5.8}$$

such that $y(.,0) = y_0$, $y(.,T) = y_T$. Moreover, if $L \notin \mathcal{N}$, then in addition it can be assumed that $y(L,.) = 0$ in $L^2(0,T)$ and take $\partial y(L,.)$ in $L^2(0,T)$ as control function.

Proof. We first assume that $L \notin \mathcal{N}$. We show that for $T > 0$ there exists $r_0 > 0$ small enough such that if $\|y_0\|_{L^2(0,L)} < r_0$, $\|y_T\|_{L^2(0,L)} < r_0$, the state y_T may be reached from y_0 for a higher order nonlinear Schrödinger equation. Let y_0, y_T be states in $L^2(0,L)$ such that $\|y_0\|_{L^2(0,L)} < r$, $\|y_T\|_{L^2(0,L)} < r$, $r > 0$ to be chosen later. Let $\Theta : L^2(0,T:H^1(0,L)) \mapsto \mathbb{H}$, defined by

$$\Theta(y) = S(\cdot)y_0 + (\Gamma_1\circ\Gamma_0)(y_T - S(T)y_0 + \Gamma_2(|y|^2y)(.,T)) + \Gamma_2(-|y|^2y)$$

where Γ_0 is well-defined in Remark 4.11, Γ_1 and Γ_2 are defined in this section. Θ is well-defined and continuous by Lemmas 4.2, 4.7, and Remark 4.11. We have that each fixed point of Θ verifies (5.1) in $\mathcal{D}'(0,T:H^{-2}(0,L))$ and $u(.,T) = y_T$. To prove the existence of a fixed-point for Θ we apply the Banach contraction fixed-point theorem to the restriction of Θ to some closed ball $\overline{\mathbb{B}}(0,R)$ in $L^2(0,T:H^1(0,L))$ (R will be chosen later). We need that

$$\Theta(\overline{\mathbb{B}}(0,R)) \subseteq \overline{\mathbb{B}}(0,R), \tag{5.9}$$

$$\exists\, C_3 \in]0,1[\, \forall y,z \in \overline{\mathbb{B}}(0,R): \quad \|\Theta(y)-\Theta(z)\| \leq C_3\|y-z\|, \tag{5.10}$$

where $\|\cdot\|$ stands for the norm $L^2(0,T:H^1(0,L))$. Let κ_1 (resp. κ_2,κ_2') denotes the norm of Γ_1 (resp. Γ_2,Γ_2) as a map from $L^2(0,T)$(resp. $L^1(0,T:L^2(0,L))$ into $L^2(0,T:H^1(0,L))$ (resp. $L^2(0,T:H^1(0,L))$, $C([0,T]:L^2(0,L))$), and κ denote the norm of Γ_0 as a map from $L^2(0,L)$ into $L^2(0,L)$. Set $\kappa_3 = \sqrt{\frac{(\delta+3\beta)T+L}{3\beta-|\alpha|}}$. Let $y,z \in L^2(0,T:H^1(0,L))$. Assume that $\|y\| \leq R$, $\|z\| \leq R$. Then by (4.10) and

(5.4),

$$\begin{aligned}\|\Theta(y)\| &\le \sqrt{\frac{(\delta+3\beta)T+L}{3\beta-|\alpha|}}\|y_0\|_{L^2(0,L)} \\ &\quad + \kappa_1\kappa(\|y_T\|_{L^2(0,L)} + \|y_0\|_{L^2(0,L)} + \kappa_2' C_1\|y\|^2) + \kappa_2 C_1\|y\|^2 \\ &\le C_1(\kappa_2+\kappa\kappa_1\kappa_2')R^2 + (2\kappa\kappa_1+\kappa_3)r.\end{aligned} \tag{5.11}$$

Hence, we have the first condition on R and r :

$$C_1(\kappa_2+\kappa\kappa_1\kappa_2')R^2 + (2\kappa\kappa_1+\kappa_3)r \le R. \tag{5.12}$$

Now write

$$\Theta(y)-\Theta(z) = \Gamma_2(|z|^2z-|y|^2y) + (\Gamma_1\circ\Gamma_0)(\Gamma_2(|y|^2y-|z|^2z)(.,T)). \tag{5.13}$$

Therefore, by (5.4),

$$\|\Theta(y)-\Theta(z)\| = 2C_1(\kappa_2+\kappa\kappa_1\kappa_2')R\|y-z\| \tag{5.14}$$

Condition (5.9) will hold provided that

$$2C_1(\kappa_2+\kappa\kappa_1\kappa_2')R < 1. \tag{5.15}$$

Let R be some positive number verifying (5.14). Then (5.11) holds true if we take $r = R/\big(2(2\kappa\kappa_1+\kappa_3)\big)$. Setting

$$r_0 = \frac{1}{4C_1(2\kappa\kappa_1+\kappa_3)(\kappa_2+\kappa\kappa_1\kappa_2')}\,,$$

we see that $r \to r_0$ as $R \to 1/(2C_1(\kappa_2+\kappa\kappa_1\kappa_2'))$. It follows that if $\|y_0\|_{L^2(0,L)} < r_0$ every y_T with $\|y_T\|_{L^2(0,L)} < r_0$ may be reached by a solution of the higher order nonlinear Schrödinger equation coming from y_0. The proof of the theorem is completed when $L \notin \mathcal{N}$. If now $L \in \mathcal{N}$, it is sufficient to consider some $\widetilde{L} > L$ such that $\widetilde{L} \notin \mathcal{N}$ and to apply the theorem to the functions $\widetilde{y}_0, \widetilde{y}_T \in L^2(0,\widetilde{L})$, where $\widetilde{y}_0, \widetilde{y}_T$ denote the prolongations by zero of the given states $\widetilde{y}_0, \widetilde{y}_T \in L^2(0,L)$, and then to restrict the solution $\widetilde{y}$ to the domain $(0,T)\times(0,L)$. The proof follows. $\square$

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