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OSCILLATION CRITERIA FOR FIRST-ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. Oscillation criteria are obtained for all solutions of first-order nonlinear neutral delay differential equations. Our results extend and improve some results well known in the literature. Some examples are considered to illustrate our main results.

1. Introduction

In recent years, the literature on the oscillation of neutral delay differential equations has grown very rapidly. It is a relatively new field with interesting applications in real world life problems. In fact, neutral delay differential equations appear in modelling of the networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, as the Euler equation in some variational problems, in the theory of automatic control and in neuro-mechanical systems in which inertia plays an important role. See Hale [17], Driver [8], Brayton and Willoughby [6], Popove [32], and Boe and Chang [5], and the references cited therein. Also this is evident by the number of references in the recent books by Ladde et al. [14] and by Ladas [16].

We consider a general first-order nonlinear neutral delay differential equation

$$(x(t) - q(t)x(t-r))' + f(t, x(\tau(t))) = 0, \tag{1.1}$$

where for $t \geq t_0$

$$q, \tau \in C([t_0, \infty), \mathbb{R}^+),\ q(t) \neq 1,\ r \in (0, \infty),\ \tau(t) < t,\ \lim_{t\to\infty} \tau(t) = \infty, \tag{1.2}$$

$$f \in C([t_0, \infty) \times R, R), \quad uf(t, u) \geq 0, \tag{1.3}$$

$$\sum_{i=1}^{n} \prod_{j=1}^{i} \frac{1}{q(t_j)} \to \infty \quad \text{as } n \to \infty. \tag{1.4}$$

we assume that the nonlinear function $f(t, u)$ in (1.1) satisfies the following conditions:

(H) There are a piecewise continuous function $p : [t_0, \infty) \to \mathbb{R}^+ = [0, \infty)$, a function $g \in C(R, \mathbb{R}^+)$, and a number $\varepsilon_0 > 0$ such that

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(i) g is nondecreasing on $\mathbb{R}^+$
(ii) $g(-u) = g(u)$, $\lim_{u\to 0} g(u) = 0$,
(iii) $\int_0^\infty g(e^{-u})du < \infty$,
(iv) $\frac{1}{|u|}|f(t,u) - p(t)u| \le p(t)g(u)$ for $t \ge t_0$ and $0 < |u| < \varepsilon_0$,
(v) For each $\varphi \in C([t_0,\infty), R)$ with $\lim_{t\to\infty} \varphi(t) > 0$,

$$\int_{t_0}^{\infty} \int f(t, \varphi(\tau(t)))dt = \infty, \quad \int_{t_0}^{\infty} f(t, -\varphi(\tau(t)))dt = -\infty.$$

As usual a solution $x(t)$ of equation (1.1) is said to be oscillatory if it has arbitrarily large zeros in $[t_0,\infty)$. Otherwise it is nonoscillatory and the equation (1.1) is called oscillatory if every solution of this equation is oscillatory.

When $q(t) = 0$, (1.1) reduces to

$$x'(t) + f(t, x(\tau(t))) = 0,$$

which was studied by Tang and Shen [36]. They obtained some infinite integral sufficient conditions for oscillations.

The oscillatory behavior of other neutral delay differential equations have been investigated by many authors, see [1, 2, 3, 4, 9, 10, 12, 13, 14, 15, 16, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 37, 38, 39, 40] and references therein.

In recent papers Elabbasy and Saker [10], Kubiaczyk and Saker [22] obtained an infinite integral conditions for oscillation of the linear neutral delay differential equation

$$(x(t) - q(t)x(t-r))' + p(t)x(t-\tau) = 0\,.$$

Let $\delta(t) = \max\{\tau(t) : t_0 \le s \le t\}$ and $\delta^{-1}(t) = \min\{s \ge t_0 : \delta(s) = t\}$. Clearly, δ and δ^{-1} are non-decreasing and satisfy

(A) $\delta(t) < t$ and $\delta^{-1}(t) > t$
(B) $\delta(\delta^{-1}(t)) = t$ and $\delta^{-1}(\delta(t)) \le t$.

Let $\delta^{-k}(t)$ be defined on $[t_0,\infty)$ by

$$\delta^{-(k+1)}(t) = \delta^{-1}(\delta^{-k}(t)), \quad k = 1, 2, \dots \tag{1.5}$$

Throughout this paper, we use the sequence $\{p_k\}$, of functions defined by

$$p_1(t) = \int_t^{\delta^{-1}(t)} p(s)ds, \quad t \ge t_0,$$
$$p_{k+1}(t) = \int_t^{\delta^{-1}(t)} p(s)p_k(s)ds, \quad t \ge t_0,\ k = 1, 2, \dots$$

Our main results are the following.

Theorem 1.1. *Assume that* (1.2), (1.4), (1.3), *and (H) hold, and there exist a bounded positive function* $\sigma(t)$ *such that*

$$\int_{\tau(t)}^{t} B(s)ds > \frac{1}{e}, \tag{1.6}$$

and

$$\int_{t_0}^{\infty} p(t)\sigma(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) - 1\Big]dt = \infty, \tag{1.7}$$

where $B(t) = p(t)/\sigma(t)$. *Then every solution of* (1.1) *oscillates.*

Theorem 1.2. *Assume that* (1.2), (1.4), (1.3) *and (H) hold, and that*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds \geq 0. \tag{1.8}$$

and suppose that there exists a positive integer n *such that*

$$\int_{t_0}^{\infty} p(t)\ln(e^{n-1}p_n(t)+1)dt = \infty. \tag{1.9}$$

Then every solution of (1.1) *oscillates.*

Corollary 1.3. *Assume that* (1.2) (1.3), (1.4), (1.8) *and (H) hold, and that*

$$\int_{t_0}^{\infty} p(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)dsBig) - 1\Big]dt = \infty. \tag{1.10}$$

Then every solution of (1.1) *oscillates.*

Corollary 1.4. *Assume that* (1.2), (1.3), (1.4), (1.8) *and (H) hold, and that*

$$\int_{t_0}^{\infty} p(t)\ln\Big(\int_{t}^{\delta^{-1}(t)} p(s)ds + 1\Big)dt = \infty.$$

Then every solution of (1.1) *oscillates.*

Corollary 1.5. *Assume that* (1.2), (1.3), (1.4), (1.8) *and (H) hold, and suppose that there exists a positive integer* n *such that*

$$\int_{t_0}^{\infty} p(t)\ln(e^{n}p_n(t))dt = \infty.$$

Then every solution of (1.1) *oscillates.*

Note that if

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > 2,$$

then by Lemma 2.4 every solution of (1.1) oscillates. Thus, we will consider the case

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds \leq 2.$$

This implies that for some $\epsilon > 0$ and large t,

$$\int_{\tau(t)}^{t} p(s)ds \leq 2+\epsilon\,.$$

Thus we have

$$\liminf_{t\to\infty} p_k(t) \leq (2+\epsilon)^{k-1}\liminf_{t\to\infty}\int_{t}^{\delta^{-1}(t)} p(s)ds \leq (2+\epsilon)^{k-1}\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds.$$

As a result, by Theorem 1.2 we have

Corollary 1.6. *Assume that* (1.2), (1.3), (1.4) *and (H) hold, and that there exists a positive integer* n *such that*

$$\lim_{t\to\infty}\inf p_n(t) > 0.$$

Then every solution of (1.1) *oscillates.*

The proofs of the above Theorems and also some Lemmas to be used in these proofs will be given in the next two sections. Some examples which illustrate and the advantage of our results will be given in section 4.

2. Preliminary Lemmas

Lemma 2.1. *Assume that* (1.2), (1.4) *and* (1.3) *hold. Let* $x(t)$ *be an eventually positive solution of 1.1 and set*

$$z(t) = x(t) - q(t)x(t-r). \tag{2.1}$$

Then $z(t)$ *is eventually nonincreasing and positive function.*

Proof. From (1.1)), (1.3), we have $z'(t) = -f(t, x(\tau(t))) \le 0$ eventually. We prove that $z(t)$ is a positive function. If not, then there exist $T \ge t_0$ and $\alpha < 0$ such that $z(t) < \alpha$ for $t \ge T$. Then from (2.1), we have $x(t) < \alpha + q(t)x(t-r)$ which implies

$$x(t+r) < \alpha + q(t+r)x(t).$$

Now we choose k such that $t_k = t^* + kr > T$. Then $x(t_{k+1}) < \alpha + q(t_{k+1})x(t_k)$. Applying this inequality by induction, it gives

$$x(t_n) < \alpha\Big[1 + \sum_{i=k+2}^{n} \prod_{j=0}^{n-i} q(t_{n-j})\Big] + \prod_{i=k+1}^{n} q(t_i)x(t_k).$$

Now define q_n and d_n by

$$q_n = 1 + \sum_{i=k+2}^{n} \prod_{j=0}^{n-i} q(t_{n-j}), \quad d_n = \prod_{i=k+1}^{n} q(t_i),$$

and let

$$s_n = \sum_{i=1}^{n} \prod_{j=1}^{i} \frac{1}{q(t_j)}.$$

Then

$$s_n^* = \frac{q_n}{d_n} = \Big(s_n - \sum_{i=1}^{k+1} \prod_{j=1}^{i} \frac{1}{q(t_j)}\Big) q(t_{k+1}) \dots q(t_1) \to \infty \quad \text{as } n \to \infty,$$

by condition (1.4). Using the above inequality,

$$x(t_n) < \big[s_n^* + \frac{x(t_k)}{\alpha}\big]\alpha d_n \to -\infty \quad \text{as } n \to \infty,$$

and this contradicts the assumption that $x(t) > 0$. Then $z(t)$ must be positive function. The proof is complete. □

Note that the proof of Lemma 2.1 is similar to that in [7, Lemma 1]; we state it here for the sake of completeness.

Lemma 2.2. *Assume that* (1.2), (1.3), (1.4) *and (H) hold. Then every non-oscillatory solution of* (1.1) *converges to zero monotonically for large* t *as* $t \to \infty$.

Proof. Suppose that $x(t)$ is a non-oscillatory solution of equation (1.1) which we shall assume to be eventually positive [If $x(t)$ is eventually negative the proof is similar]. From Lemma 2.1, we have $z(t)$ is eventually non-increasing and positive function.

Choose a $t_1 \geq t_0$ such that $x(t) > 0$, $z(t) > 0$ for $t \geq t_1$. It follows from equations (1.1)-(1.3) and (H) that there exists $t_2 > t_1$ such that $\tau(t) \geq t_1$ and $z'(t) \leq 0$ for $t > t_2$. Hence the following limits exist and

$$\lim_{t\to\infty} x(t) \geq \lim_{t\to\infty} z(t) = \alpha \geq 0\,.$$

If $\alpha > 0$, then from (1.1) we have

$$z(t) - z(t_0) = -\int_{t_0}^{t} f(t, x(\tau(s)))ds.$$

It follows from assumption (H)(v) that $\lim_{t\to\infty} z(t) = -\infty$, which contradicts that $z(t)$ being positive function, then $\alpha = 0$, from (1.2), we have $\lim_{t\to\infty} x(t) = 0$. The proof of Lemma 2.2 is complete. $\square$

Lemma 2.3. *Assume that* (1.2)*, 1.3,* (1.4) *and (H) hold. If $x(t)$ is a nonoscillatory solution of* (1.1)*, then there exist $A > 0$, $\varepsilon > 0$ and $T \in (0,\infty)$ such that for $t \geq T$,*

$$|x(t)| \leq A\exp\Big(-\frac{1}{2}\int_{T}^{t} p(s)ds\Big) + \varepsilon, \tag{2.2}$$

Proof. We shall assume $x(t)$ to be eventually positive [If $x(t)$ is eventually negative the proof is similar]. By Lemma 2.2, there exists $t_1 > 0$ such that

$$0 < x(t) \leq x(\tau(t)) < \varepsilon \quad \text{for } t \geq t_1.$$

From (H), we find that for $t \geq t_1$

$$f(t, x(\tau(t))) \geq p(t)[1 - g(x(\tau(t)))]x(\tau(t)),$$

and $\lim_{t\to\infty} x(t) = 0$. By assumption (H), there exists $T > t_1$ such that for $t \geq T$,

$$f(t, x(\tau(t))) \geq \frac{1}{2}p(t)x(\tau(t)) \geq \frac{1}{2}p(t)x(t),$$

and it follows from (1.1) that for $t \geq T$,

$$(x(t) - q(t)x(t-r))' + \frac{1}{2}p(t)x(t) \leq 0, z'(t) + \frac{1}{2}p(t)z(t) \leq 0,$$

where $z(t) = x(t) - q(t)x(t-r)$. This yields, for $t \geq T$,

$$z(t) \leq A\exp\Big[-\frac{1}{2}\int_{T}^{t} p(s)ds\Big],$$

$$|x(t)| \leq A\exp\Big(-\frac{1}{2}\int_{T}^{t} p(s)ds\Big) + \varepsilon,$$

where $A = x(T) - q(T)x(T-r)$. $\square$

Lemma 2.4. *Assume that* (1.2)*, 1.3,* (1.4) *and (H) hold. If equation* (1.1)*) has a nonoscillatory solution, then*

$$\int_{\tau(t)}^{t} p(s)ds \leq 2 \quad \textit{and} \quad p_k(t) \leq 2^k, \quad k = 1, 2, \dots \tag{2.3}$$

eventually.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (1.1) which we shall assume to be eventually positive [if $x(t)$ is eventually negative the proof is similar]. By Lemma 2.2, there exists $T \geq 0$ such that

$$\begin{gathered} x(\tau(t)) \geq x(t) > 0 \quad \text{for } t \geq T, \\ (x(t) - q(t)x(t-r))' + \frac{1}{2}p(t)x(\tau(t)) \leq 0, \\ z'(t) + \frac{1}{2}p(t)z(\tau(t)) \leq 0 \quad \text{for } t \geq T. \end{gathered} \tag{2.4}$$

Integrating both sides from $\tau(t)$ to t yields

$$z(t) - z(\tau(t)) + \frac{1}{2}\int_{\tau(t)}^{t} p(s)z(\tau(s))ds \leq 0 \quad \text{for } t \geq T.$$

By the decreasing nature of $z(t)$ for large t and the increasing nature of $\tau(t)$, there exists $T_1 \geq T$ such that

$$z(t) - z(\tau(t)) + \frac{1}{2}z(\tau(t))\int_{\tau(t)}^{t} p(s)ds \leq 0 \quad \text{for } t \geq T_1.$$

Then, $\int_{\tau(t)}^{t} p(s)ds \leq 2$.

Also, integrating both sides of equation (2.4) from t to $\delta^{-1}(t)$ yields

$$z(\delta^{-1}(t)) - z(t) + \frac{1}{2}\int_{t} \delta^{-1}(t)p(s)z(\tau(s))ds \leq 0 \quad \text{for } t \geq T.$$

By the decreasing nature of $z(t)$ for large t and the increasing nature of $\tau(t)$, there exists $T_1 \geq T$ such that

$$z(\delta^{-1}(t)) - z(t) + \frac{1}{2}\Big(\int_{t}^{\delta^{-1}(t)} p(s)ds\Big)z(\tau(\delta^{-1}(t))) \leq 0 \quad \text{for } t \geq T_1.$$

or

$$z(\delta^{-1}(t)) - z(t) + \frac{1}{2}\Big(\int_{t}^{\delta^{-1}(t)} p(s)ds\Big)z(t) \leq 0 \quad \text{for } t \geq T_1.$$

Then, we have

$$p_1(t) = \int_{t}^{\delta^{-1}(t)} p(s)ds \leq 2.$$

By iteration we deduce, from this, that $p_k(t) \leq 2^k$ which shows that (2.3) holds for $t \geq T_1$. The proof of Lemma 2.4 is complete. $\square$

Lemma 2.5. *Assume that* (1.2), (1.3), (1.8), (1.4) *and (H) hold. If $x(t)$ is a nonoscillatory solution of equation* (1.1)*), then $\frac{z(\tau(t))}{z(t)}$ is well defined for large t and is bounded.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (1.1) which we shall assume to be eventually positive [if $x(t)$ is eventually negative the proof is similar]. By the same argument as in the proof of Lemma 2.3, there exists $T > 0$, such that

$$\begin{gathered} x(\tau(t)) \geq x(t) > 0 \quad \text{for } t \geq T, \\ (x(t) - q(t)x(t-\sigma))' + \frac{1}{2}p(t)x(\tau(t)) \leq 0, \end{gathered}$$

$$z'(t) + \frac{1}{2}p(t)z(\tau(t)) \le 0 \quad \text{for } t \ge T.$$

The rest of the proof is similar to in [28, Lemma 5], and thus it is omitted. $\square$

3. Proofs of Theorems

Proof of Theorem 1.1. Assume that (1.1) has a nonoscillatory solution $x(t)$ which will be assumed to be eventually positive (if $x(t)$ is eventually negative the proof is similar). By Lemma 2.2, there exists $t_1 \ge t_0$ such that

$$0 < x(t) \le x(\tau(t)) < \varepsilon_0, \quad g(x(\tau(t))) < 1, \quad t \ge t_1, \tag{3.1}$$

where ε_0 is given by assumption (H). From (3.1) and (H), we have

$$f(t, x(\tau(t))) \ge p(t)[1 - g(x(\tau(t)))]x(\tau(t)), \quad t \ge t_1. \tag{3.2}$$

Set

$$\omega(t) = \frac{\sigma(t)z(\tau(t))}{z(t)} \quad \text{for } t \ge t_1.$$

From Lemmas 2.1 and 2.2, $\omega(t) \ge \sigma(t)$ for $t \ge t_1$. From (1.1) and (3.2), we have

$$\frac{z'(t)}{z(t)} + B(t)\omega(t)[1 - g(x(\tau(t)))] \le 0, \quad t \ge t_1. \tag{3.3}$$

Let $t_2 > t_1$ be such that $\tau(t) \ge t_1$ for $t \ge t_2$. Integrating both sides of (3.3) from $\tau(t)$ to t, we obtain

$$\omega(t) \ge \sigma(t) \exp\Big(\int_{\tau(t)}^{t} B(s)\omega(s)[1 - g(x(\tau(s)))]ds\Big), \quad t \ge t_2. \tag{3.4}$$

By (1.6), for $t \ge t_2$, we have

$$\int_{\delta(t)}^{t} p(s)ds = \int_{\tau(t^*)}^{t} p(s)ds \ge \int_{\tau(t^*)}^{t^*} p(s)ds \ge e^{-1}, \tag{3.5}$$

where $t^* \in [t_0, t]$ with $\tau(t^*) = \delta(t)$. From (1.6) and (3.4)), we find that for $t \ge t_2$,

$$\begin{aligned}
\omega(t) &\ge \sigma(t) \exp\Big(\int_{\tau(t)}^{t} B(s)(\omega(s) - \sigma(t))ds + \frac{\sigma(t)}{e}\Big) \\
&\quad \times \exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) \exp\exp\Big(-\int_{\tau(t)}^{t} B(s)\omega(s)g(x(\tau(s)))ds\Big) \\
&\ge \sigma(t)\Big(e\int_{\delta(t)}^{t} B(s)(\omega(s) - \sigma(t))ds + \sigma(t)\Big) \exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) \\
&\quad \times \exp\Big(-\int_{\tau(t)}^{t} B(s)\omega(s)g(x(\tau(s)))ds\Big).
\end{aligned}$$

Let $\upsilon(t) = \omega(t) - \sigma(t)$ for $t \ge t_1$. Then $\upsilon(t) \ge 0$ for $t \ge t_1$, and so for $t \ge t_2$,

$$\begin{aligned}
&\upsilon(t) - e\int_{\delta(t)}^{t} B(s)\upsilon(s)ds \\
&\ge \Big(e\int_{\delta(t)}^{t} B(s)\upsilon(s)ds + \sigma(t)\Big)\Big[\sigma(t)\exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) \\
&\quad \times \exp\Big(-\int_{\tau(t)}^{t} B(s)\omega(s)g(x(\tau(s)))ds\Big) - 1\Big],
\end{aligned}$$

that is, for $t \geq t_2$,

$$\begin{aligned} &B(t)\upsilon(t) - B(t)e\int_{\delta(t)}^{t} B(s)\upsilon(s)ds \\ &\geq B(t)\Big(e\int_{\delta(t)}^{t} B(s)\upsilon(s)ds + \sigma(t)\Big)\Big[\sigma(t)\exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) \\ &\quad\times \exp\Big(-\int_{\tau(t)}^{t} B(s)\omega(s)g(x(\tau(s)))ds\Big) - 1\Big]. \end{aligned} \tag{3.6}$$

By Lemmas 2.1–2.5, there exist $T > t_2$, $A > 0$, $\varepsilon > 0$ and $M > 0$ such that for $t \geq T$,

$$x(\tau(t)) \leq A\exp\Big(-\frac{1}{2}\int_{T}^{\tau(t)} p(s)ds\Big) + \varepsilon, \tag{3.7}$$

$$\int_{\tau(t)}^{t} p(s)ds \leq 2, \tag{3.8}$$

$$\omega(t) \leq \sigma(t)M, \quad \sigma(t) \leq \eta. \tag{3.9}$$

Let

$$\alpha(t) = \frac{1}{2}\int_{T}^{t} p(s)ds, \quad t \geq T.$$

Clearly, (1.6) implies that $\alpha(t) \to \infty$ as $t \to \infty$. For $t \geq t_2$, set

$$\begin{aligned} D(t) &= p(t)\Big(e\int_{\delta(t)}^{t} B(s)\upsilon(s)ds + \sigma(t)\Big)\exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}|Big) \\ &\quad\times \Big[1 - \exp\Big(-\int_{\tau(t)}^{t} B(s)\omega(s)g(x(\tau(s)))ds\Big)\Big]. \end{aligned} \tag{3.10}$$

One can easily see that

$$0 \leq 1 - e^{-c} \leq c \quad \text{for } c \geq 0. \tag{3.11}$$

It follows from (3.10) that for $t \geq t_2$,

$$\begin{aligned} D(t) &\leq p(t)\Big(e\int_{\delta(t)}^{t} B(s)\upsilon(s)ds + \sigma(t)\Big)\exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) \\ &\quad\times \int_{\tau(t)}^{t} B(s)\omega(s)g(x(\tau(s)))ds. \end{aligned} \tag{3.12}$$

Therefore,

$$\begin{aligned} &D(t) \\ &\leq p(t)\Big(e\int_{\delta(t)}^{t} B(s)\upsilon(s)ds + \sigma(t)\Big)\exp\Big(\int_{\tau(t)}^{t} p(s)ds\Big)\int_{\tau(t)}^{t} B(s)\omega(s)g(x(\tau(s)))ds. \end{aligned}$$

Let $T^* > T$ be such that $\tau(\tau(t)) \geq T$ for $t \geq T^*$ and $\alpha(T^*) > 2 + \ln A$. Set $M_1 = e^2\eta M[2e(M-1)+\eta]$ and $A_1 = eA$. Noting that

$$e\int_{\delta(t)}^{t} B(s)\upsilon(s)ds + \sigma(t) \leq 2e(M-1) + \eta \quad \text{for } t \geq T.$$

from (3.7)–(3.9), (3.12), and assumption (H), we obtain $N \geq T^*$,

$$\begin{aligned}
&\int_{T^ast}^{N} D(t)dt \\
&\leq M_1 \int_{T^ast}^{N} p(t) \int_{\tau(t)}^{t} p(s) g\Big(A \exp\Big(\frac{1}{2}\int_{T}^{\tau(t)} p(s)ds\Big) + \varepsilon\Big)\, ds\, dt \\
&= M_1 \int_{T^ast}^{N} p(t) \int_{\tau(t)}^{t} p(s) g\Big(A \exp\Big(-\frac{1}{2}\int_{T}^{s} p(\mu)d\mu + \frac{1}{2}\int_{\tau(s)}^{s} p(\mu)d\mu\Big) + \varepsilon\Big)\, ds\, dt \\
&\leq M_1 \int_{T^ast}^{N} p(t) \int_{\tau(t)}^{t} p(s) g\Big(A_1 e^{-\alpha(s)} + \varepsilon\Big)\, ds\, dt \\
&= 2M_1 \int_{T^ast}^{N} p(t) \int_{\alpha(\tau(t))}^{\alpha(t)} g(A_1 e^{-u} + \varepsilon) du\, dt \\
&= 2M_1 \int_{T^ast}^{N} p(t) \int_{\alpha(t)-\beta(t)}^{\alpha(t)} g(A_1 e^{-u} + \varepsilon)\, du\, dt, \quad \beta(t) = \frac{1}{2}\int_{\tau(t)}^{t} p(s)ds \\
&\leq 4M_1 \int_{\alpha(T^*)}^{\alpha(N)} p(t) \int_{v-1}^{v} g(A_1 e^{-u} + \varepsilon) du\, dv \\
&\leq 4M_1 \int_{\alpha(T^*)-1}^{\alpha(N)} g(A_1 e^{-u} + \varepsilon) du \\
&= 4M_1 \int_{\ln(A_1 e^{1-\alpha(T^*)}+\varepsilon)^{-1}}^{\ln(A_1 e^{-\alpha(N)}+\varepsilon)^{-1}} g(e^{-u}) \frac{e^{-u}}{e^{-u} - \varepsilon} du \\
&\leq 4M_1 \int_{0}^{\alpha(N)} g(e^{-u}) \frac{e^{-u}}{e^{-u} - \varepsilon} du \\
&\leq 4M_1 \int_{0}^{\infty} g(e^{-u}) du < \infty.
\end{aligned}$$

and

$$\int_{T}^{\infty} D(t)dt < \infty. \tag{3.13}$$

Substituting (3.10) into (3.6), for $t \geq t_2$, we obtain

$$\begin{aligned}
&B(t)\upsilon(t) - eB(t) \int_{\delta(t)}^{t} B(s)\upsilon(s)ds \\
&\geq p(t)\Big(e \int_{\delta(t)}^{t} B(s)\upsilon(s)ds + \sigma(t)\Big)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) - 1\Big] - D(t),
\end{aligned}$$

$$\begin{aligned}
&B(t)\upsilon(t) - eB(t) \int_{\delta(t)}^{t} B(s)\upsilon(s)ds \\
&\geq p(t)\sigma(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) - 1\Big] - D(t).
\end{aligned}$$

Integrating both sides from T^* to $N > \tau^{-1}(T^*)$, we have

$$\begin{aligned} &\int_{T^ast}^{N} B(t)\upsilon(t)dt - e\int_{T^ast}^{N} B(t)\int_{\delta(t)}^{t} B(s)\upsilon(s)\,ds\,dt \\ &\geq \int_{T^ast}^{N} p(t)\sigma(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) - 1\Big]dt - \int_{T^ast}^{N} D(t)dt. \end{aligned} \tag{3.14}$$

By interchanging the order of integrations and by (3.5), we have

$$\begin{aligned} e\int_{T^ast}^{N} B(t)\int_{\delta(t)}^{t} B(s)\upsilon(s)\,ds\,dt &\geq e\int_{T^*}^{\delta(N)} B(t)\upsilon(t)\int_{t}^{\delta^{-1}(t)} B(s)\,ds\,dt \\ &\geq \int_{T^*}^{\delta(N)} B(t)\upsilon(t)dt. \end{aligned} \tag{3.15}$$

From this and (3.14), it follows that

$$\int_{\delta(N)}^{N} B(t)\upsilon(t)dt \geq \int_{T^ast}^{N} p(t)\sigma(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) - 1\Big]dt - \int_{T^ast}^{N} D(t)dt. \tag{3.16}$$

By (3.8) and (3.9),

$$\int_{\delta(N)}^{N} B(t)\upsilon(t)dt \leq (M-1)\int_{\delta(N)}^{N} p(t)dt \leq (M-1)\int_{\tau(N)}^{N} p(t)dt \leq 2(M-1),$$

and so by (3.16),

$$2(M-1) \geq \int_{T^ast}^{N} p(t)\sigma(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) - 1\Big]dt - \int_{T^ast}^{N} D(t)dt.$$

This implies that

$$2(M-1) \geq \int_{T^*}^{\infty} p(t)\sigma(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) - 1\Big]dt - \int_{T^*}^{\infty} D(t)dt,$$

which together with (3.13) yields

$$\int_{T^*}^{\infty} p(t)\sigma(t)\Big[\exp\Big(\int_{\tau(t)}^{t} p(s)ds - \frac{\sigma(t)}{e}\Big) - 1\Big]dt < \infty.$$

This contradicts (1.7) and so the proof is complete.

Proof of Theorem 1.2. Assume that (1.1) has a nonoscillatory solution $x(t)$ which will be assumed to be eventually positive (if $x(t)$ is eventually negative the proof is similar). By Lemma 2.1 and assumption (H), there exists $t_0^* \geq t_0$ such that

$$0 < x(t) \leq x(\delta(t)) \leq x(\tau(t)) < \varepsilon_0, \quad g(x(\tau(t))) < 1, \quad t \geq t_0^*, \tag{3.17}$$

where ε_0 is given by assumption (H). (3.17) and (H) yield that for $t \geq t_0^*$,

$$\begin{aligned} f(t, x(\tau(t))) &\geq p(t)[1 - g(x(\tau(t)))]x(\tau(t)) \\ &\geq p(t)[1 - g(x(\tau(t)))]z(\delta(t)), \end{aligned} \tag{3.18}$$

and it follows from (1.1) that

$$\frac{z'(t)}{z(t)} + p(t)\frac{z(\delta(t))}{z(t)}[1 - g(x(\tau(t)))] \leq 0, \quad t \geq t_0^*. \tag{3.19}$$

By Lemmas 2.1–2.5, there exist $T > t_2$, $A > 0$, $\varepsilon > 0$ and $M > 0$ such that for $t \geq T$,

$$x(\tau(t)) \leq A \exp\Big(-\frac{1}{2}\int_T^{\tau(t)} p(s)ds) + \varepsilon, \tag{3.20}$$

$$\int_{\delta(t)}^t p(s)ds \leq \int_{\tau(t)}^t p(s)ds \leq 2, \quad p_k(t) \leq 2^k, \quad k = 1, 2, \dots, \tag{3.21}$$

$$\frac{z(\delta(t))}{z(t)} \leq \frac{z(\tau(t))}{z(t)} \leq M. \tag{3.22}$$

Let $t_k = \delta^{-k}(T)$, $k = 1, 2, \dots$ Clearly $t_k \to \infty$ as $k \to \infty$. Set $\lambda(t) = -z'(t)/z(t)$, for $t \geq T$. Then

$$\frac{z(\delta(t))}{z(t)} = \exp \int_{\delta(t)}^t \lambda(s)ds, \quad t \geq t_1,$$

and from (3.19), for $t \geq t_1$, we have

$$\lambda(t) \geq p(t) \exp \int_{\delta(t)}^t \lambda(s)ds - p(t)g(x(\tau(t)))\frac{z(\delta(t))}{z(t)}. \tag{3.23}$$

It follows from (3.20)–(3.23) that for $t \geq t_1$,

$$\begin{aligned} &\lambda(t) \\ &\geq p(t) \exp \int_{\delta(t)}^t \lambda(s)ds - Mp(t)g\Big(A \exp\Big(-\frac{1}{2}\int_T^{\tau(t)} p(s)ds\Big) + \varepsilon\Big) \\ &\geq p(t) \exp \int_{\delta(t)}^t \lambda(s)ds - Mp(t)g\Big(A_1 \exp\Big(-\frac{1}{2}\int T^t p(s)ds\Big) + \varepsilon\Big), \end{aligned} \tag{3.24}$$

where $A_1 = eA$. By the inequality $e^c \geq ec$ for $c \geq 0$, we have for $t \geq t_1$,

$$\lambda(t) \geq ep(t) \int_{\delta(t)}^t \lambda(s)ds - Mp(t)g\Big(A_1 \exp\Big(-\frac{1}{2}\int_T^t p(s)ds\Big) + \varepsilon\Big). \tag{3.25}$$

Set

$$\alpha(t) = \frac{1}{2}\int_T^t p(s)ds, \quad t \geq T, \tag{3.26}$$

and

$$\begin{gathered} \lambda_0(t) = \lambda(t), \quad t \geq T, \\ \lambda_k(t) = p(t) \int_{\delta(t)}^t \lambda_{k-1}(s)ds, \quad t \geq t_k, \ k = 1, 2, \dots, n, \end{gathered} \tag{3.27}$$

and

$$\begin{gathered} G_0(t) = 0, \quad t \geq T, \\ G_k(t) = ep(t) \int_{\delta(t)}^t G_{k-1}(s)ds \quad + Mp(t)g(A_1 \exp(-\alpha(t)) + \varepsilon), \end{gathered} \tag{3.28}$$

for $t \geq t_k$, $k = 1, 2, \dots, n$. Clearly (1.8) implies that $\alpha(t)$ is nondecreasing on $[T, \infty)$ and $\alpha(t) \to \infty$ as $t \to \infty$. By iteration we deduce from (3.25) that

$$\lambda(t) \geq e^k \lambda_k(t) - G_k(t), \quad t \geq t_k, \ k = 1, 2, \dots n - 1, \tag{3.29}$$

and so by (3.24),

$$\lambda(t) \geq p(t) \exp\Big(e^{n-1} \int_{\delta(t)}^{t} \lambda_{n-1}(s)ds\Big) \exp\Big(-\int_{\delta(t)}^{t} G_{n-1}(s)ds\Big) - G_1(t), \quad (3.30)$$

for $t \geq t_n$. From (3.28), one can easily obtain

$$G_{k+1}(t) - G_k(t) = ep(t) \int_{\delta(t)}^{t} [G_k(s) - G_{k-1}(s)]ds, \quad (3.31)$$

for $t \geq t_{k+1}$, $k = 1, 2, \dots, n-1$. By (3.21), (3.26) and (3.28), for $t \geq t_2$, we have

$$\begin{aligned} \int_{\delta(t)}^{t} G_1(s)ds &= M \int_{\delta(t)}^{t} p(s) g(A_1 \exp(-\alpha(s)) + \varepsilon)ds \\ &= 2M \int_{\alpha(\delta(t))}^{\alpha(t)} g(A_1 e^{-u} + \varepsilon)du \\ &\leq 2M \int_{\alpha(t)-1}^{\alpha(t)} g(A_1 e^{-u} + \varepsilon)du. \end{aligned} \quad (3.32)$$

Thus, from (3.31), we get

$$[G_2(t) - G_1(t) = ep(t) \int_{\delta(t)}^{t} G_1(s)ds \leq 2eMp(t) \int_{\alpha(t)-1}^{\alpha(t)} g(A_1 e^{-u} + \varepsilon)du, \quad t \geq t_2,$$

$$\begin{aligned} G_3(t) - G_2(t) &= ep(t) \int_{\delta(t)}^{t} [G_2(s) - G_1(s)]ds \\ &\leq 2e^2 Mp(t) \int_{\delta(t)}^{t} p(s) \int_{\alpha(s)-1}^{\alpha(s)} g(A_1 e^{-u} + \varepsilon)du\, ds \\ &= 4e^2 Mp(t) \int_{\alpha(\delta(t))}^{\alpha(t)} \int_{v-1}^{v} g(A_1 e^{-u} + \varepsilon)du\, dv \\ &\leq 4e^2 Mp(t) \int_{\alpha(t)-1}^{\alpha(t)} \int_{v-1}^{v} g(A_1 e^{-u} + \varepsilon)du\, dv \\ &\leq 4e^2 Mp(t) \int_{\alpha(t)-2}^{\alpha(t)} g(A_1 e^{-u} + \varepsilon)du, \quad t \geq t_3. \end{aligned}$$

By induction, one can prove in general that for $k = 2, 3, \dots, n-1$,

$$G_k(t) - G_{k-1}(t) \leq (2e)^{k-1}(k-2)! Mp(t) \int_{\alpha(t)-(k-1)}^{\alpha(t)} g(A_1 e^{-u} + \varepsilon)du, \quad t \geq t_k,$$

and so

$$\begin{aligned} G_{n-1}(t) &= \sum_{k=1}^{n-1} [G_k(t) - G_{k-1}(t)] \\ &\leq G_1(t) + Mp(t) \sum_{k=2}^{n-1} (2e)^{k-1}(k-2)! \int_{\alpha(t)-(k-1)}^{\alpha(t)} g(A_1 e^{-u} + \varepsilon)du, \end{aligned} \quad (3.33)$$

for $t \geq t_{n-1}$. By (3.21), (3.22) and (3.27), we obtain

$$
\begin{aligned}
\lambda_1(t) &= p(t)\int_{\delta(t)}^{t}\lambda(s)ds = p(t)\ln\big[\frac{z(\delta(t))}{z(t)}\big] \\
&\leq p(t)\ln M, \quad t \geq t_1, \\
\lambda_2(t) &= p(t)\int_{\delta(t)}^{t}\lambda_1(s)ds \leq p(t)\ln M\int_{\delta(t)}^{t}p(s)ds \\
&\leq 2p(t)\ln M, \quad t\geq t_2, \\
&\dots \\
\lambda_{n-1}(t) &\leq 2^{n-2}p(t)\ln M, \quad t \geq t_{n-1}.
\end{aligned}
\tag{3.34}
$$

For $t \geq t_n$, set

$$
D(t) \quad = p(t)\exp\Big(e^{n-1}\int_{\delta(t)}^{t}\lambda_{n-1}(s)ds\Big)\Big[1-\exp\Big(-\int_{\delta(t)}^{t}G_{n-1}(s)ds\Big)\Big] + G_1(t).
$$

From (3.11), (3.21), (3.32), (3.33) and (3.34), we have

$$
\begin{aligned}
D(t) &\leq p(t)\exp\Big(e^{n-1}\int_{\delta(t)}^{t}\lambda_{n-1}(s)ds\Big)\int_{\delta(t)}^{t}G_{n-1}(s)ds + G_1(t) \\
&\leq G_1(t) + p(t)\exp\Big(2^{n-2}e^{n-1}\ln M\int_{\delta(t)}^{t}p(s)ds\Big) \\
&\quad \times\int_{\delta(t)}^{t}\Big[G_1(s) + Mp(s)\sum_{k=2}^{n-1}(2e)^{k-1}(k-2)!\int_{\alpha(s)-(k-1)}^{\alpha(s)}g(A_1e^{-u}+\varepsilon)du\Big]ds \\
&\leq G_1(t) + 2Mp(t)\exp((2e)^{n-1}\ln M)\int_{\alpha(t)-1}^{\alpha(t)}g(A_1e^{-u}+\varepsilon)du \\
&\quad + Mp(t)\exp((2e)^{n-1}\ln M) \\
&\quad \times\sum_{k=2}^{n-1}(2e)^{k-1}(k-2)!\int_{\delta(t)}^{t}p(s)\int_{\alpha(s)-(k-1)}^{\alpha(s)}g(A_1e^{-u}+\varepsilon)du\,ds \\
&\leq G_1(t) + M_1p(t)\sum_{k=1}^{n-1}(2e)^{k-1}(k-1)!\int_{\alpha(t)-k}^{\alpha(t)}g(A_1e^{-u}+\varepsilon)du, \quad t\geq t_n,
\end{aligned}
\tag{3.35}
$$

where $M_1 = 2M\exp((2e)^{n-1}\ln M)$. Let $T^* > t_n$ be such that $\alpha(T^*) > n + \ln A_1$. It follows from (3.35) and (H) that

$$
\begin{aligned}
&\int_{T^*}^{\infty}D(t)dt \\
&\leq \int_{T^*}^{\infty}G_1(t)dt + M_1\sum_{k=1}^{n-1}(2e)^{k-1}(k-1)!\int_{T^*}^{\infty}p(t)\int_{\alpha(t)-k}^{\alpha(t)}g(A_1e^{-u})\,du\,dt \\
&\leq 2M\int_{\alpha(T^*)}^{\infty}g(A_1e^{-u})du + 2M_1\sum_{k=1}^{n-1}(2e)^{k-1}(k-1)!\int_{\alpha(T^*)}^{\infty}\int_{v-k}^{v}g(A_1e^{-u})du\,dv \\
&\leq 2M\int_{\alpha(T^*)}^{\infty}g(A_1e^{-u})du + 2M_1\sum_{k=1}^{n-1}(2e)^{k-1}k!\int_{\alpha(T^*)-(k+1)}^{\infty}g(A_1e^{-u})du
\end{aligned}
$$

$$\int_{T^*}^{\infty} D(t)dt \le 2M\int_0^{\infty} g(e^{-u})du + 2M_1\sum_{k=1}^{n-1}(2e)^{k-1}k!\int_0^{\infty} g(e^{-u})du < \infty.$$

Since

$$\begin{aligned}
&p(t)\exp\Big(e^{n-1}\int_{\delta(t)}^{t}\lambda_{n-1}(s)ds\Big)\exp\Big(-\int_{\delta(t)}^{t}G_{n-1}(s)ds\Big) - G_1(t)\\
&= p(t)\exp\Big(e^{n-1}\int_{\delta(t)}^{t}\lambda_{n-1}(s)ds\Big) - D(t), \quad t \ge t_n,
\end{aligned}$$

it follows from (3.30) that

$$\lambda(t) \ge p(t)\exp\Big(e^{n-1}\int_{\delta(t)}^{t}\lambda_{n-1}(s)ds\Big) - D(t), \quad t \ge t_n. \tag{3.36}$$

One can easily show that $\gamma e^x \ge x + \ln(\gamma+1)$ for $\gamma > 0$, and so for $t \ge t_n$,

$$\begin{aligned}
p_n(t)\lambda(t) &\ge p(t)e^{1-n}(e^{n-1}p_n(t))\exp\Big(e^{n-1}\int_{\delta(t)}^{t}\lambda_{n-1}(s)ds\Big) - p_n(t)D(t)\\
&\ge p(t)\int_{\delta(t)}^{t}\lambda_{n-1}(s)ds + e^{1-n}p(t)\ln(e^{n-1}p_n(t)+1) - p_n(t)D(t),
\end{aligned}$$

that is, for $t \ge t_n$,

$$p_n(t)\lambda(t) - p(t)\int_{\delta(t)}^{t}\lambda_{n-1}(s)ds \ge e^{1-n}p(t)\ln(e^{n-1}p_n(t)+1) - p_n(t)D(t). \tag{3.37}$$

For $N > \delta^{-n}(T^*)$, we have

$$\begin{aligned}
&\int_{T^ast}^{N} p_n(t)\lambda(t)dt - \int_{T^ast}^{N} p(t)\int_{\delta(t)}^{t}\lambda_{n-1}(s)\,ds\,dt\\
&\ge e^{1-n}\int_{T^ast}^{N} p(t)\ln(e^{n-1}p_n(t)+1)dt - \int_{T^*}^{N} p_n(t)D(t)dt.
\end{aligned} \tag{3.38}$$

Let $\delta^1(t) = \delta(t)$, $\delta^{k+1}(t) = \delta(\delta^k(t))$, $k = 1, 2, \dots, n$. Then by interchanging the order of integration, we have

$$\begin{aligned}
\int_{T^ast}^{N} p(t)\int_{\delta(t)}^{t}\lambda_{n-1}(s)\,ds\,dt &\ge \int_{T^*}^{\delta(N)}\lambda_{n-1}(t)\int_{t}^{\delta^{-1}(t)} p(s)\,ds\,dt\\
&= \int_{T^*}^{\delta(N)} p(t)p_1(t)\int_{\delta(t)}^{t}\lambda_{n-2}(s)\,ds\,dt\\
&\ge \int_{T^*}^{\delta^2(N)}\lambda_{n-2}(t)\int_{t}^{\delta^{-1}(t)} p(s)p_1(s)\,ds\,dt\\
&= \int_{T^*}^{\delta^2(N)} p(t)p_2(t)\int_{\delta(t)}^{t}\lambda_{n-3}(s)\,ds\,dt\\
&\dots\\
&\ge \int_{T^*}^{\delta^n(N)}\lambda(t)p_n(t)dt.
\end{aligned}$$

From this and (3.38), we have

$$\int_{\delta^n(N)}^{N} p_n(t)\lambda(t)dt \geq e^{1-n}\int_{T^*}^{N} p(t)\ln(e^{n-1}p_n(t)+1)dt - \int_{T^ast}^{N} p_n(t)D(t)dt, \quad (3.39)$$

which together with (3.21) yields

$$2^n\int_{\delta^n(N)}^{N} \lambda(t)dt \geq e^{1-n}\int_{T^ast}^{N} p(t)\ln(e^{n-1}p_n(t)+1)dt - 2^n\int_{T^ast}^{N} D(t)dt,$$

or

$$\ln\frac{x(\delta^n(N))}{x(N)} \geq 2^{-n}e^{1-n}\int_{T^ast}^{N} p(t)\ln(e^{n-1}p_n(t)+1)dt - \int_{T^*}^{N} D(t)dt. \quad (3.40)$$

In view of (1.9) and (3), we have

$$\lim_{N\to\infty}\frac{x(\delta^n(N))}{x(N)} = \infty. \quad (3.41)$$

On the other hand, (3.22) implies that

$$\frac{x(\delta^n(N))}{x(N)} = \frac{x(\delta^1(N))}{x(N)}.\frac{x(\delta^2(N))}{x(\delta^1(N))}\cdots\frac{x(\delta^n(N))}{x(\delta^{n-1}(N))} \leq M^n.$$

This contradicts (3.41) and completes the proof.

4. Examples

In this section we introduce some examples to illustrate our main results.

Example 4.1. Consider the neutral delay differential equation

$$\big(x(t) - (\frac{5}{2} + \sin t)x(t-\pi)\big)' + f(t, x(\tau(t))) = 0, \quad t \geq 3. \quad (4.1)$$

For $f(t,u) = p(t)f(u)$, with

$$f(u) = \begin{cases} u[1 + (1+\ln^2|u|)^{-1}], & u \neq 0,\\ 0, & u = 0,\end{cases} \quad (4.2)$$

$$g(u) = \begin{cases} 1, & |u| > 1,\\ (1+\ln^2|u|)^{-1}, & 0 < |u| \leq 1,\\ 0, & u = 0,\end{cases} \quad (4.3)$$

$$p(t) = \frac{1}{et\ln 2} + \frac{1}{t\ln t}, \quad \tau(t) = \frac{t}{2}, \quad (4.4)$$

with $\int_3^\infty p(t)dt = \infty$. It is easily seen that condition (H) holds. We check that the conditions (1.8) and (1.10) in Corollary 1.3 hold. In fact, for $t \geq 3$,

$$\int_{\frac{t}{2}}^{t} p(s)ds = \int_{\frac{t}{2}}^{t}\big[\frac{1}{es\ln 2} + \frac{1}{s\ln s}\big]ds = \frac{1}{e} - \ln\big[1 - \frac{\ln 2}{\ln t}\big] \geq \frac{1}{e},$$

$\liminf_{t\to\infty}\int_{t/2}^{t} p(s)ds = 1/e$, and

$$\begin{aligned}\int_3^\infty p(t)\big[\exp\big[\int_{t/2}^{t} p(s)ds - \frac{1}{e}\big] - 1\big]dt &\geq \int_3 \infty p(t)\big[\int_{t/2}^{t} p(s)ds - \frac{1}{e}\big]dt\\ &\geq -\frac{1}{e\ln 2}\int_3^\infty \frac{1}{t}\ln[1 - \frac{\ln 2}{\ln t}]dt = \infty,\end{aligned}$$

because
$$\int_3^\infty \frac{1}{t\ln t}dt = \infty \quad \text{and} \quad \lim_{t\to\infty}(\ln t)\ln[1-\frac{\ln 2}{\ln t}] = -\ln 2.$$
By Theorem 1.1 every solution of (4.1) oscillates.

Example 4.2. Consider the neutral delay differential equation
$$\big(x(t)-(\frac{5}{2}+\sin t)x(t-\frac{\pi}{2})\big)' + f(t,x(\tau(t))) = 0, \quad t\geq 3, \tag{4.5}$$
For
$$p(t)=\frac{\delta}{t}, \quad \tau(t)=\frac{t}{\lambda}, \quad \delta<\frac{1}{e\ln\lambda}, \quad \lambda>1, \quad f(t,u)=p(t)f(u),$$
where $f(u)$ and $g(u)$ are defined by (4.2) and (4.3) and with $\int_3^\infty p(t)dt=\infty$, and
$$\int_{t/\lambda}^t p(s)ds = \int_{t/\lambda}^t \frac{\delta}{s}ds = \delta\big(\ln t - \ln\frac{t}{\lambda}\big) = \delta\ln\lambda < \frac{1}{e}$$
It is easily seen that condition (H) holds. We check that the conditions (1.6) and (1.10) in Theorem 1.2 hold. In fact, for $t\geq 3$,
$$\lim_{t\to\infty}\inf\int_{t/\lambda}^t p(s)ds = \lim_{t\to\infty}\inf\int_{t/\lambda}^t \frac{\delta}{s}ds = \delta(\ln t-\ln\frac{t}{\lambda}) = \delta\ln\lambda > 0$$
and
$$\begin{aligned}\int_3^\infty p(t)[\exp(\int_{t/\lambda}^t p(s)ds)-1]dt &\geq \int_3^\infty p(t)(\int_{t/\lambda}^t p(s)ds)dt \\ &\geq \int_3^\infty \frac{\delta^2\ln\lambda}{t}dt = \delta^2\ln\lambda(\infty)=\infty\end{aligned}$$
By Corollary 1.3 every solution of (4.5) oscillates.

Example 4.3. Consider the neutral delay differential equation
$$\big(x(t)-(\frac{5}{2}+\sin t)x(t-\pi)\big)' + f(t,x(\tau(t))) = 0, \quad t\geq 3, \tag{4.6}$$
where
$$\tau(t)=t-1 \quad \text{and} \quad f(t,u)=[\exp 3(\sin t-1)+|u|]^{1/3}u.$$
Let $p(t)=\exp(\sin t)-0.1$ and $g(u)=e^2|u|^{1/3}$. It is easy to see that assumption (H) holds. Clearly
$$\liminf_{t\to\infty}\int_{t-1}^t p(s)ds < \frac{1}{e},$$
$$\int_0^\infty p(t)\ln\Big(\int_t^{t+1}p(s)ds+1\Big)dt \geq \int_0^\infty \exp(\sin t-1)\ln\Big(\int_t^{t+1}\exp(\sin s)ds\Big)dt$$
By Jensen's inequality,
$$\begin{aligned}\int_0^\infty p(t)\ln\Big(\int_t^{t+1}p(s)ds+1\Big)dt &\geq \int_0^\infty \exp(\sin t-1)\int_t^{t+1}\sin s\,ds\,dt \\ &= \frac{2\sin 2^{-1}}{e}\int_0^\infty \exp(\sin t)\sin(t+\frac{1}{2})dt.\end{aligned}$$

On the other hand, it is easy to see that $\int_0^t \exp(\sin s)\cos s\, ds$ is bounded and

$$\int_0^{2\pi} \exp(\sin t)\sin t dt > 0.$$

Thus

$$\int_0^{\infty} p(t)\ln\Big(\int_t^{t+1} p(s)ds + 1\Big)dt = \infty.$$

By Corollary 1.4, every solution of (4.6) oscillates.

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