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POSITIVE SOLUTIONS TO A GENERALIZED SECOND-ORDER THREE-POINT BOUNDARY-VALUE PROBLEM ON TIME SCALES

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Abstract. Let $\mathbb{T}$ be a time scale with $0, T \in \mathbb{T}$. We investigate the existence and multiplicity of positive solutions to the nonlinear second-order three-point boundary-value problem

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \quad t \in [0, T] \subset \mathbb{T},$$
$$u(0) = \beta u(\eta), \quad u(T) = \alpha u(\eta)$$

on time scales $\mathbb{T}$, where $0 < \eta < T$, $0 < \alpha < \frac{T}{\eta}$, $0 < \beta < \frac{T-\alpha\eta}{T-\eta}$ are given constants.

1. Introduction

In recent years, many authors have begun to pay attention to the study of boundary-value problems on time scales. Here two-point boundary-value problems have been extensively studied; see [1, 2, 3, 4, 5] and the references therein. However, the research for three-point boundary-value problems is still a fairly new subject, even though it is growing rapidly; see [6, 7, 8, 9].

In 2002, inspired by the study of the existence of positive solutions in [10] for the three-point boundary-value problem of differential equations, Anderson [9] considered the following three-point boundary-value problem on a time scale $\mathbb{T}$,

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \quad t \in [0, T] \subset \mathbb{T}, \tag{1.1}$$
$$u(0) = 0, \quad u(T) = \alpha u(\eta). \tag{1.2}$$

He investigated the existence of at least one positive solution and of at least three positive solutions for the problem (1.1)-(1.2) by using Guo-Krasnoselskii's fixed-point theorem and Leggett-Williams fixed-point theorem, respectively.

In this paper, we extend Anderson's results to the more general boundary-value problem on time scale $\mathbb{T}$,

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \quad t \in [0, T] \subset \mathbb{T}, \tag{1.3}$$
$$u(0) = \beta u(\eta), \quad u(T) = \alpha u(\eta), \tag{1.4}$$

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where $\alpha > 0$, $\beta \geq 0$, $\eta \in (0,T) \subset \mathbb{T}$ are given constants. Clearly if $\beta = 0$, then (1.4) reduces to (1.2). We also point out that when $\mathbb{T} = \mathbb{R}$, $\beta = 0$, (1.3)-(1.4) becomes a boundary-value problem of differential equations and just is the problem considered in [10]; when $\mathbb{T} = \mathbb{Z}$, $\beta = 0$, (1.3)-(1.4) becomes a boundary-value problem of difference equations and just is the problem considered in [11]. We will use Guo-Krasnoselskii's fixed-point theorem and Leggett-Williams fixed-point theorem to investigate the existence and multiplicity of positive solutions for the problem (1.3)-(1.4). Our main results extend the main results of Ma[10], Anderson[9], Ma and Raffoul[11].

The rest of the paper is arranged as follows: we state some basic time-scale definitions and prove several preliminary results in Section 2. Section 3 is devoted to the existence of a positive solution of (1.3)-(1.4), the main tool being the Guo-Krasnoselskii's fixed-point theorem. Next in Section 4, we give a multiplicity result by using the Leggett-Williams fixed-point theorem. Finally we give two examples to illustrate our results in Section 5.

2. Preliminaries

For convenience, we list here the following definitions which are needed later.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. The operators σ and ρ from $\mathbb{T}$ to $\mathbb{T}$, defined by [12],

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T},$$
$$\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \in \mathbb{T}$$

are called the *forward jump operator* and the *backward jump operator*, respectively. In this definition

$$\inf \emptyset := \sup \mathbb{T}, \quad \sup \emptyset := \inf \mathbb{T}.$$

The point $t \in \mathbb{T}$ is *left-dense, left-scattered, right-dense, right-scattered* if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$ (assume t is not left-scattered if $t = \sup \mathbb{T}$), then the *delta derivative of f at the point t* is defined to be the number $f^{\Delta}(t)$(provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq |\sigma(t) - s|, \quad \text{for all } s \in U.$$

Similarly, for $t \in \mathbb{T}$ (assume t is not right-scattered if $t = \inf \mathbb{T}$), the *nabla derivative of f at the point t* is defined in [1] to be the number $f^{\nabla}(t)$(provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq |\rho(t) - s|, \quad \text{for all } s \in U.$$

A function f is *left-dense continuous* (i.e. ld-continuous), if f is continuous at each left-dense point in $\mathbb{T}$ and its right-sided limit exists at each right-dense point in $\mathbb{T}$. It is well-known[13] that if f is ld-continuous, then there is a function $F(t)$ such that $F^{\nabla}(t) = f(t)$. In this case, it is defined that

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

For the rest of this article, $\mathbb{T}$ denotes a time scale with $0, T \in \mathbb{T}$. Also we denote the set of left-dense continuous functions from $[0,T] \subset \mathbb{T}$ to $E \subset \mathbb{R}$ by $C_{ld}([0,T],\ E)$, which is a Banach space with the maximum norm $\|u\| = \max_{t\in[0,T]} |u(t)|$. We now state and prove several lemmas before stating our main results.

Lemma 2.1. *Let $\beta \neq \frac{T-\alpha\eta}{T-\eta}$. Then for $y \in C_{ld}([0,T],\ \mathbb{R})$, the problem*

$$u^{\Delta\nabla}(t) + y(t) = 0, \quad t \in [0,T] \subset \mathbb{T}, \tag{2.1}$$

$$u(0) = \beta u(\eta), \quad u(T) = \alpha u(\eta) \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t) = -\int_0^t (t-s)y(s)\nabla s + \frac{(\beta-\alpha)t - \beta T}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^\eta (\eta-s)y(s)\nabla s \\ + \frac{(1-\beta)t + \beta\eta}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^T (T-s)y(s)\nabla s. \end{aligned} \tag{2.3}$$

Proof. From (2.1), we have

$$u(t) = u(0) + u^{\Delta}(0)t - \int_0^t (t-s)y(s)\nabla s \ := A + Bt - \int_0^t (t-s)y(s)\nabla s.$$

Since

$$\begin{aligned} u(0) &= A; \\ u(\eta) &= A + B\eta - \int_0^\eta (\eta-s)y(s)\nabla s; \\ u(T) &= A + BT - \int_0^T (T-s)y(s)\nabla s, \end{aligned}$$

by (2.2) from $u(0) = \beta u(\eta)$, we have

$$(1-\beta)A - B\beta\eta = -\beta \int_0^\eta (\eta-s)y(s)\nabla s;$$

from $u(T) = \alpha u(\eta)$, we have

$$(1-\alpha)A + B(T-\alpha\eta) = \int_0^T (T-s)y(s)\nabla s - \alpha \int_0^\eta (\eta-s)y(s)\nabla s.$$

Therefore,

$$\begin{aligned} A &= \frac{\beta\eta}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^T (T-s)y(s)\nabla s \\ &\quad - \frac{\beta T}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^\eta (\eta-s)y(s)\nabla s; \\ B &= \frac{1-\beta}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^T (T-s)y(s)\nabla s \\ &\quad - \frac{\alpha-\beta}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^\eta (\eta-s)y(s)\nabla s\,, \end{aligned}$$

from which it follows that

$$\begin{aligned} u(t) =& \frac{\beta\eta}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^T (T-s)y(s)\nabla s \\ &- \frac{\beta T}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^\eta (\eta-s)y(s)\nabla s \end{aligned}$$

$$
\begin{aligned}
&+ \frac{(1-\beta)t}{(T-\alpha\eta)-\beta(T-\eta)} \int_0^T (T-s)y(s)\nabla s \\
&- \frac{(\alpha-\beta)t}{(T-\alpha\eta)-\beta(T-\eta)} \int_0^\eta (\eta-s)y(s)\nabla s - \int_0^t (t-s)y(s)\nabla s \\
= &- \int_0^t (t-s)y(s)\nabla s + \frac{(\beta-\alpha)t-\beta T}{(T-\alpha\eta)-\beta(T-\eta)} \int_0^\eta (\eta-s)y(s)\nabla s \\
&+ \frac{(1-\beta)t+\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)} \int_0^T (T-s)y(s)\nabla s.
\end{aligned}
$$

The function u presented above is a solution to the problem (2.1)-(2.2), and the uniqueness of u is obvious. □

Lemma 2.2. *Let $0 < \alpha < \frac{T}{\eta}$, $0 \le \beta < \frac{T-\alpha\eta}{T-\eta}$. If $y \in C_{ld}([0,T],\ [0,\infty))$, then the unique solution u of the problem* (2.1)-(2.2) *satisfies*

$$u(t) \ge 0, \quad t \in [0,T] \subset \mathbb{T}.$$

Proof. It is known that the graph of u is concave down on $[0,T]$ from $u^{\Delta\nabla}(t) = -y(t) \le 0$, so

$$\frac{u(\eta)-u(0)}{\eta} \ge \frac{u(T)-u(0)}{T}.$$

Combining this with (2.2), we have

$$\frac{1-\beta}{\eta}u(\eta) \ge \frac{\alpha-\beta}{T}u(\eta).$$

If $u(0) < 0$, then $u(\eta) < 0$. It implies that $\beta \ge \frac{T-\alpha\eta}{T-\eta}$, a contradiction to $\beta < \frac{T-\alpha\eta}{T-\eta}$.

If $u(T) < 0$, then $u(\eta) < 0$, and the same contradiction emerges. Thus, it is true that $u(0) \ge 0$, $u(T) \ge 0$, together with the concavity of u, we have

$$u(t) \ge 0, \quad t \in [0,T] \subset \mathbb{T}.$$

as required. □

Lemma 2.3. *Let $\alpha\eta \ne T$, $\beta > \max\{\frac{T-\alpha\eta}{T-\eta}, 0\}$. If $y \in C_{ld}([0,T],[0,\infty))$, then problem* (2.1)-(2.2) *has no nonnegative solutions.*

Proof. Suppose that problem (2.1)-(2.2) has a nonnegative solution u satisfying $u(t) \ge 0, t \in [0,T]$ and there is a $t_0 \in (0,T)$ such that $u(t_0) > 0$.

If $u(T) > 0$, then $u(\eta) > 0$. It implies

$$u(0) = \beta u(\eta) > \frac{T-\alpha\eta}{T-\eta}u(\eta) = \frac{Tu(\eta)-\eta u(T)}{T-\eta},$$

that is

$$\frac{u(T)-u(0)}{T} > \frac{u(\eta)-u(0)}{\eta},$$

which is a contradiction to the concavity of u.

If $u(T) = 0$, then $u(\eta) = 0$. When $t_0 \in (0,\eta)$, we get $u(t_0) > u(\eta) = u(T)$, a violation of the concavity of u. When $t_0 \in (\eta,T)$, we get $u(0) = \beta u(\eta) = 0 = u(\eta) < u(t_0)$, another violation of the concavity of u. Therefore, no nonnegative solutions exist. □

Remark 2.4. When $\beta = 0$, the result similar to Lemma 2.3 has been obtained in Lemma 5 of [9] for $\alpha\eta > T$.

Lemma 2.5. *Let $0 < \alpha < \frac{T}{\eta}$, $0 < \beta < \frac{T-\alpha\eta}{T-\eta}$. If $y \in C_{ld}([0,T],[0,\infty))$, then the unique solution to the problem (2.1)-(2.2) satisfies*

$$\min_{t\in[0,T]} u(t) \geq \gamma\|u\|, \tag{2.4}$$

where

$$\gamma := \min\Big\{\frac{\alpha(T-\eta)}{T-\alpha\eta},\ \frac{\alpha\eta}{T},\ \frac{\beta(T-\eta)}{T},\ \frac{\beta\eta}{T}\Big\}. \tag{2.5}$$

Proof. It is known that the graph of u is concave down on $[0,T]$ from $u^{\Delta\nabla}(t) = -y(t) \leq 0$. We divide the proof into two cases.
Case 1. $0 < \alpha < 1$, then $\frac{T-\alpha\eta}{T-\eta} > \alpha$. For $u(0) = \beta u(\eta) = \frac{\beta}{\alpha}u(T)$, it may develop in the following two possible directions.
(i) $0 < \alpha \leq \beta$. It implies that $u(0) \geq u(T)$, so

$$\min_{t\in[0,T]} u(t) = u(T).$$

Assume $\|u\| = u(t_1)$, $t_1 \in [0,T)$, then either $0 \leq t_1 \leq \eta < \rho(T)$, or $0 < \eta < t_1 < T$. If $0 \leq t_1 \leq \eta < \rho(T)$, then

$$\begin{aligned}
u(t_1) &\leq u(T) + \frac{u(T)-u(\eta)}{T-\eta}(t_1 - T)\\
&\leq u(T) + \frac{u(T)-u(\eta)}{T-\eta}(0 - T)\\
&= \frac{Tu(\eta) - \eta u(T)}{T-\eta}\\
&= \frac{T-\alpha\eta}{\alpha(T-\eta)}u(T),
\end{aligned}$$

from which it follows that $\min_{t\in[0,T]} u(t) \geq \frac{\alpha(T-\eta)}{T-\alpha\eta}\|u\|$.
If $0 < \eta < t_1 < T$, from

$$\frac{u(\eta)}{\eta} \geq \frac{u(t_1)}{t_1} \geq \frac{u(t_1)}{T},$$

together with $u(T) = \alpha u(\eta)$, we have

$$u(T) > \frac{\alpha\eta}{T}u(t_1),$$

so that, $\min_{t\in[0,T]} u(t) \geq \frac{\alpha\eta}{T}\|u\|$.
(ii) $0 < \beta < \alpha$. It implies that $u(0) \leq u(T)$, so

$$\min_{t\in[0,T]} u(t) = u(0).$$

Assume $\|u\| = u(t_2)$, $t_2 \in (0,T]$, then either $0 < t_2 < \eta < \rho(T)$, or $0 < \eta \leq t_2 \leq T$. If $0 < t_2 < \eta < \rho(T)$, from

$$\frac{u(\eta)}{T-\eta} \geq \frac{u(t_2)}{T-t_2} \geq \frac{u(t_2)}{T},$$

together with $u(0) = \beta u(\eta)$, we have

$$u(0) \geq \frac{\beta(T-\eta)}{T}u(t_2),$$

hence, $\min_{t\in[0,T]} u(t) \geq \frac{\beta(T-\eta)}{T}\|u\|$.
If $0<\eta\leq t_2\leq T$, from

$$\frac{u(t_2)}{T}\leq\frac{u(t_2)}{t_2}\leq\frac{u(\eta)}{\eta},$$

together with $u(0)=\beta u(\eta)$, we have

$$u(0)\geq\frac{\beta\eta}{T}u(t_2),$$

so that, $\min_{t\in[0,T]} u(t)\geq\frac{\beta\eta}{T}\|u\|$.
Case 2. $\frac{T}{\eta}>\alpha\geq 1$, then $\frac{T-\alpha\eta}{T-\eta}\leq\alpha$. In this case, $\beta<\alpha$ is true. It implies that $u(0)\leq u(T)$. So,

$$\min_{t\in[0,T]} u(t)=u(0).$$

Assume $\|u\|=u(t_2),\ t_2\in(0,T]$ again. Since $\alpha\geq 1$, it is known that $u(\eta)\leq u(T)$, together with the concavity of u, we have $0<\eta\leq t_2\leq T$. Similar to the above discussion,

$$\min_{t\in[0,T]} u(t)\geq\frac{\beta\eta}{T}\|u\|.$$

Summing up, we have

$$\min_{t\in[0,T]} u(t)\geq\gamma\|u\|,$$

where

$$0<\gamma=\min\Big\{\frac{\alpha(T-\eta)}{T-\alpha\eta},\ \frac{\alpha\eta}{T},\ \frac{\beta(T-\eta)}{T},\ \frac{\beta\eta}{T}\Big\}<1.$$

This completes the proof. □

Remark 2.6. If $\beta=0$, Anderson obtained the inequality in [9, Lemma 7] that is

$$\min_{t\in[\eta,T]} u(t)\geq r\|u\|,$$

where

$$r:=\min\Big\{\frac{\alpha(T-\eta)}{T-\alpha\eta},\ \frac{\alpha\eta}{T},\ \frac{\eta}{T}\Big\}.$$

The following two theorems, Theorem 2.7 (Guo-Krasnoselskii's fixed-point theorem)and Theorem 2.8 (Leggett-Williams fixed-point theorem), will play an important role in the proof of our main results.

Theorem 2.7 ([14])**.** *Let E be a Banach space, and let $K\subset E$ be a cone. Assume Ω_1,Ω_2 are open bounded subsets of E with $0\in\Omega_1,\ \overline{\Omega}_1\subset\Omega_2$, and let*

$$A:K\cap(\overline{\Omega}_2\setminus\Omega_1)\longrightarrow K$$

be a completely continuous operator such that either

(i) $\|Au\|\leq\|u\|,\quad u\in K\cap\partial\Omega_1$, *and* $\|Au\|\geq\|u\|,\quad u\in K\cap\partial\Omega_2$; *or*
(ii) $\|Au\|\geq\|u\|,\quad u\in K\cap\partial\Omega_1$, *and* $\|Au\|\leq\|u\|,\quad u\in K\cap\partial\Omega_2$

hold. Then A has a fixed point in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Theorem 2.8 ([15])**.** *Let P be a cone in the real Banach space E. Set*

$$P_c:=\{x\in P:\|x\|<c\}\,, \tag{2.6}$$

$$P(\psi,a,b):=\{x\in P:a\leq\psi(x),\|x\|\leq b\}. \tag{2.7}$$

Suppose $A : \overline{P}_c \to \overline{P}_c$ *be a completely continuous operator and* ψ *be a nonnegative continuous concave functional on* P *with* $\psi(x) \le \|x\|$ *for all* $x \in \overline{P}_c$. *If there exists* $0 < a < b < d \le c$ *such that the following conditions hold,*

(i) $\{x \in P(\psi, b, d) : \psi(x) > b\} \ne \emptyset$ *and* $\psi(Ax) > b$ *for all* $x \in P(\psi, b, d)$*;*
(ii) $\|Ax\| < a$ *for* $\|x\| \le a$*;*
(iii) $\psi(Ax) > b$ *for* $x \in P(\psi, b, c)$ *with* $\|Ax\| > d$.

Then A *has at least three fixed points* x_1, x_2 *and* x_3 *in* $\overline{P}_c$ *satisfying*

$$\|x_1\| < a, \quad \psi(x_2) > b, \quad a < \|x_3\| \quad \text{with } \psi(x_3) < b.$$

3. Existence of Positive Solutions

We assume the following hypotheses:

(A1) $f \in C([0,\infty), [0,\infty))$;
(A2) $a \in C_{ld}([0,T], [0,\infty))$ and there exists $t_0 \in (0,T)$, such that $a(t_0) > 0$.

Define

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

For the boundary-value problem (1.3)-(1.4), we establish the following existence theorem by using Theorem 2.7 (Guo-Krasnoselskii's fixed-point theorem).

Theorem 3.1. *Assume (A1), (A2) hold, and* $0 < \alpha < \frac{T}{\eta}$, $0 < \beta < \frac{T-\alpha\eta}{T-\eta}$. *If either*

(C1) $f_0 = 0$ *and* $f_\infty = \infty$ *(*f *is superlinear), or*
(C2) $f_0 = \infty$ *and* $f_\infty = 0$ *(*f *is sublinear),*

then problem (1.3)-(1.4) *has at least one positive solution.*

Proof. It is known that $0 < \alpha < \frac{T}{\eta}$, $0 < \beta < \frac{T-\alpha\eta}{T-\eta}$. From Lemma 2.1, u is a solution to the boundary-value problem (1.3)-(1.4) if and only if u is a fixed point of operator A, where A is defined by

$$\begin{aligned} &Au(t) \\ &= -\int_0^t (t-s)a(s)f(u(s))\nabla s + \frac{(\beta-\alpha)t - \beta T}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^\eta (\eta - s)a(s)f(u(s))\nabla s \\ &\quad + \frac{(1-\beta)t + \beta\eta}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^T (T-s)a(s)f(u(s))\nabla s. \end{aligned} \tag{3.1}$$

Denote

$$K = \{u \in C_{ld}([0,T], \mathbb{R}) : u \ge 0, \min_{t \in [0,T]} u(t) \ge \gamma\|u\|\},$$

where γ is defined in (2.5).

It is obvious that K is a cone in $C_{ld}([0,T], \mathbb{R})$. Moreover, from (A1), (A2), Lemma 2.2 and Lemma 2.5, $AK \subset K$. It is also easy to check that $A : K \to K$ is completely continuous.

Superlinear case. $f_0 = 0$ and $f_\infty = \infty$. Since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \le \epsilon u$, for $0 < u \le H_1$, where $\epsilon > 0$ satisfies

$$\epsilon \frac{T + \beta(T+\eta)}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^T (T-s)a(s)\nabla s \le 1.$$

Thus, if we let

$$\Omega_1 = \{u \in C_{ld}([0,T],\ \mathbb{R}) \ :\ \|u\| < H_1\},$$

then for $u \in K \cap \partial\Omega_1$, we get

$$\begin{aligned}
Au(t) &\le \frac{(\beta-\alpha)t-\beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta(\eta-s)a(s)f(u(s))\nabla s\\
&\quad+\frac{(1-\beta)t+\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T(T-s)a(s)f(u(s))\nabla s\\
&\le \frac{\beta t}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta(\eta-s)a(s)f(u(s))\nabla s\\
&\quad+\frac{t+\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T(T-s)a(s)f(u(s))\nabla s\\
&\le \frac{\beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta(\eta-s)a(s)f(u(s))\nabla s\\
&\quad+\frac{T+\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T(T-s)a(s)f(u(s))\nabla s\\
&\le \frac{T+\beta(T+\eta)}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T(T-s)a(s)f(u(s))\nabla s\\
&\le \epsilon\|u\|\frac{T+\beta(T+\eta)}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T(T-s)a(s)\nabla s \le \|u\|.
\end{aligned}$$

Thus $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\hat{H}_2 > 0$ such that $f(u) \ge \rho u$, for $u \ge \hat{H}_2$, where $\rho > 0$ is chosen so that

$$\rho\gamma\frac{T-\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T sa(s)\nabla s \ge 1.$$

Let $H_2 = \max\{2H_1, \frac{\hat{H}_2}{\gamma}\}$ and

$$\Omega_2 = \{u \in C_{ld}([0,T],\mathbb{R}) : \|u\| < H_2\}.$$

Then $u \in K \cap \partial\Omega_2$ implies

$$\min_{t\in[0,T]} u(t) \ge \gamma\|u\| = \gamma H_2 \ge \hat{H}_2,$$

and so

$$\begin{aligned}
&Au(\eta)\\
&= -\int_0^\eta(\eta-s)a(s)f(u(s))\nabla s + \frac{\beta\eta-\alpha\eta-\beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta(\eta-s)a(s)f(u(s))\nabla s\\
&\quad+\frac{\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T(T-s)a(s)f(u(s))\nabla s\\
&= \frac{-T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^\eta(\eta-s)a(s)f(u(s))\nabla s\\
&\quad+\frac{\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T(T-s)a(s)f(u(s))\nabla s\\
&\ge \frac{1}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T[-T(\eta-s)+\eta(T-s)]a(s)f(u(s))\nabla s
\end{aligned}$$

$$
\begin{aligned}
&= \frac{T-\eta}{(T-\alpha\eta)-\beta(T-\eta)} \int_0^T sa(s)f(u(s))\nabla s \\
&\geq \gamma\rho\|u\| \frac{T-\eta}{(T-\alpha\eta)-\beta(T-\eta)} \int_0^T sa(s)\nabla s \geq \|u\|.
\end{aligned}
$$

Hence, $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$. By the first part of Theorem 2.7, A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, such that $H_1 \leq \|u\| \leq H_2$. This completes the superlinear part of the theorem.

Sublinear case. $f_0 = \infty$ and $f_\infty = 0$. Since $f_0 = \infty$, choose $H_3 > 0$ such that $f(u) \geq Mu$ for $0 < u \leq H_3$, where $M > 0$ satisfies

$$
M\gamma \frac{T-\eta}{(T-\alpha\eta)-\beta(T-\eta)} \int_0^T sa(s)\nabla s \geq 1.
$$

Let

$$
\Omega_3 = \{u \in C_{ld}([0,T],\mathbb{R}) : \|u\| < H_3\},
$$

then for $u \in K \cap \partial\Omega_3$, we get

$$
\begin{aligned}
Ay(\eta) \geq& \frac{T-\eta}{(T-\alpha\eta)-\beta(T-\eta)} \int_0^T sa(s)f(u(s))\nabla s \\
\geq& M\gamma\|u\| \frac{T-\eta}{(T-\alpha\eta)-\beta(T-\eta)} \int_0^T sa(s)\nabla s \geq \|u\|.
\end{aligned}
$$

Thus, $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_3$. Now, since $f_\infty = 0$, there exists $\hat{H}_4 > 0$ so that $f(u) \leq \lambda u$ for $u \geq \hat{H}_4$, where $\lambda > 0$ satisfies

$$
\lambda \frac{T+\beta(T+\eta)}{(T-\alpha\eta)-\beta(T-\eta)} \int_0^T (T-s)a(s)\nabla s \leq 1.
$$

Choose $H_4 = \max\{2H_3, \frac{\hat{H}_4}{\gamma}\}$. Let

$$
\Omega_4 = \{u \in C_{ld}([0,T], \mathbb{R}) : \|u\| < H_4\},
$$

then $u \in K \cap \partial\Omega_4$ implies

$$
\min_{t\in[0,T]} u(t) \geq \gamma\|u\| = \gamma H_4 \geq \hat{H}_4 .
$$

Therefore,

$$
\begin{aligned}
Au(t) &\leq \frac{T+\beta(T+\eta)}{(T-\alpha\eta)-\beta(T-\eta)} \int_0^T (T-s)a(s)f(u(s))\nabla s \\
&\leq \lambda\|u\| \frac{T+\beta(T+\eta)}{(T-\alpha\eta)-\beta(T-\eta)} \int_0^T (T-s)a(s)\nabla s \leq \|u\|.
\end{aligned}
$$

Thus $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_4$.

By the second part of Theorem 2.7, A has a fixed point u in $K \cap (\overline{\Omega}_4 \setminus \Omega_3)$, such that $H_3 \leq \|u\| \leq H_4$. This completes the sublinear part of the theorem. Therefore, the problem (1.3)-(1.4) has at least one positive solution. It finishes the proof of Theorem 3.1. $\square$

4. Multiplicity of Positive Solutions

In this section, we discuss the multiplicity of positive solutions for the general boundary-value problem

$$u^{\Delta\nabla}(t) + f(t, u(t)) = 0, \quad t \in [0, T] \subset \mathbb{T}, \tag{4.1}$$

$$u(0) = \beta u(\eta), \quad u(T) = \alpha u(\eta), \tag{4.2}$$

where $\eta \in (0,\ \rho(T)) \subset \mathbb{T}$, $0 < \alpha < \frac{T}{\eta}$, $0 < \beta < \frac{T-\alpha\eta}{T-\eta}$ are given constants.

To state the next theorem we assume

(A3) $f \in C_{ld}([0, T] \times [0, \infty), [0, \infty))$.

Define constants

$$m := \Big(\frac{T + \beta(T + \eta)}{(T - \alpha\eta) - \beta(T - \eta)} \int_0^T (T - s)\nabla s\Big)^{-1}, \tag{4.3}$$

$$\begin{aligned} \delta := \min\Big\{ &\frac{\beta\eta}{(T - \alpha\eta) - \beta(T - \eta)} \int_\eta^T (T - s)\nabla s, \\ &\frac{\alpha\eta}{(T - \alpha\eta) - \beta(T - \eta)} \int_\eta^T (T - s)\nabla s\Big\} \end{aligned} \tag{4.4}$$

Note that $\delta > 0$ from $0 < \eta < \rho(T)$, $0 < \alpha < \frac{T}{\eta}$, $0 < \beta < \frac{T-\alpha\eta}{T-\eta}$. Using Theorem 2.8(the Leggett-Williams fixed-point theorem), we established the following existence theorem for the boundary-value problem (4.1)-(4.2).

Theorem 4.1. *Assume (A3) holds, and* $0 < \alpha < \frac{T}{\eta}$, $0 < \beta < \frac{T-\alpha\eta}{T-\eta}$. *Suppose there exists constants* $0 < a < b < b/\gamma \le c$ *such that*

(D1) $f(t, u) < ma$ *for* $t \in [0, T]$, $u \in [0, a]$*;*
(D2) $f(t, u) \ge \frac{b}{\delta}$ *for* $t \in [\eta, T]$, $u \in [b, \frac{b}{\gamma}]$*;*
(D3) $f(t, u) \le mc$ *for* $t \in [0, T]$, $u \in [0, c]$,

where γ, m, δ *are as defined in* (2.5), (4.3) *and* (4.4)*, respectively. Then the boundary-value problem* (4.1)-(4.2) *has at least three positive solutions* u_1, u_2 *and* u_3 *satisfying*

$$\|u_1\| < a, \quad \min_{t\in[0,T]} (u_2)(t) > b, \quad a < \|u_3\| \quad \text{with} \quad \min_{t\in[0,T]} (u_3)(t) < b.$$

Proof. It is known that $0 < \alpha < \frac{T}{\eta}$, $0 < \beta < \frac{T-\alpha\eta}{T-\eta}$. Define the cone $P \subset C_{ld}([0, T], \mathbb{R})$ by

$$P = \{u \in C_{ld}([0, T], \mathbb{R}) : u \text{ concave down and } u(t) \ge 0 \text{ on } [0, T]\}. \tag{4.5}$$

Let $\psi : P \to [0, \infty)$ be defined by

$$\psi(u) = \min_{t\in[0,T]} u(t), \quad u \in P. \tag{4.6}$$

then ψ is a nonnegative continuous concave functional and $\psi(u) \le \|u\|, u \in P$.

Define the operator $A : P \to C_{ld}([0, T], \mathbb{R})$ by

$$\begin{aligned} Au(t) = -\int_0^t (t - s)f(s, u(s))\nabla s + \frac{(\beta - \alpha)t - \beta T}{(T - \alpha\eta) - \beta(T - \eta)} \int_0^\eta (\eta - s)f(s, u(s))\nabla s \\ + \frac{(1 - \beta)t + \beta\eta}{(T - \alpha\eta) - \beta(T - \eta)} \int_0^T (T - s)f(s, u(s))\nabla s. \end{aligned} \tag{4.7}$$

Then the fixed points of A just are the solutions of the boundary-value problem (4.1)-(4.2) from Lemma 2.1. Since $(Au)^{\Delta\nabla}(t) = -f(t, u(t))$ for $t \in (0, T)$, together with (A3) and Lemma 2.2, we see that $Au(t) \geq 0,\ t \in [0, T]$ and $(Au)^{\Delta\nabla}(t) \leq 0,\ t \in (0, T)$. Thus $A : P \to P$. Moreover, A is completely continuous.

We now verify that all of the conditions of Theorem 2.8 are satisfied. Since

$$\psi(u) = \min_{t\in[0,T]} u(t), \quad u \in P.$$

we have $\psi(u) \leq \|u\|$. Now we show $A : \overline{P_c} \to \overline{P_c}$, where P_c is given in (2.6). If $u \in \overline{P_c}$, then $0 \leq u \leq c$, together with (D3), we find $\forall\ t \in [0, T]$,

$$\begin{aligned}
Au(t) &\leq \frac{(\beta-\alpha)t - \beta T}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^\eta (\eta - s) f(s, u(s))\nabla s \\
&\quad + \frac{(1-\beta)t + \beta\eta}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^T (T - s) f(s, u(s))\nabla s \\
&\leq \frac{T + \beta(T+\eta)}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^T (T - s) f(s, u(s))\nabla s \\
&\leq mc\frac{T + \beta(T+\eta)}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^T (T - s)\nabla s = c.
\end{aligned}$$

Thus, $A : \overline{P_c} \to \overline{P_c}$.

By (D1) and the argument above, we can get that $A : \overline{P_a} \to P_a$. So, $\|Au\| < a$ for $\|u\| \leq a$, the condition (ii) of Theorem 2.8 holds.

Consider the condition (i) of Theorem 2.8 now. Since $\psi(b/\gamma) = b/\gamma > b$, let $d = b/\gamma$, then $\{u \in P(\psi, b, d) : \psi(u) > b\} \neq \emptyset$. For $u \in P(\psi, b, d)$, we have $b \leq u(t) \leq b/\gamma,\ t \in [0, T]$. Combining with (D2), we get

$$f(t, u) \geq \frac{b}{\delta}, \quad t \in [\eta, T].$$

Since $u \in P(\psi, b, d)$, then there are two cases that either $\psi(Au)(t) = Au(0)$, or $\psi(Au)(t) = Au(T)$. As the former holds, we have

$$\begin{aligned}
\psi(Au)(t) &= \frac{-\beta T}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^\eta (\eta - s) f(s, u(s))\nabla s \\
&\quad + \frac{\beta\eta}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^T (T - s) f(s, u(s))\nabla s \\
&= \frac{\beta\eta}{(T-\alpha\eta) - \beta(T-\eta)} \int_\eta^T T f(s, u(s))\nabla s \\
&\quad + \frac{\beta T}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^\eta s f(s, u(s))\nabla s \\
&\quad - \frac{\beta\eta}{(T-\alpha\eta) - \beta(T-\eta)} \int_0^T s f(s, u(s))\nabla s \\
&> \frac{\beta\eta}{(T-\alpha\eta) - \beta(T-\eta)} \int_\eta^T T f(s, u(s))\nabla s \\
&\quad - \frac{\beta\eta}{(T-\alpha\eta) - \beta(T-\eta)} \int_\eta^T s f(s, u(s))\nabla s
\end{aligned}$$

$$\geq \frac{b\beta\eta}{\delta[(T-\alpha\eta)-\beta(T-\eta)]}\int_{\eta}^{T}(T-s)\nabla s \geq b.$$

As the later holds, we have

$$\begin{aligned}
&\psi(Au)(t)\\
&= -\int_0^T (T-s)f(s,u(s))\nabla s + \frac{(\beta-\alpha)T-\beta T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^{\eta}(\eta-s)f(s,u(s))\nabla s\\
&\quad + \frac{(1-\beta)T+\beta\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)f(s,u(s))\nabla s\\
&= \frac{\alpha\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T (T-s)f(s,u(s))\nabla s\\
&\quad - \frac{\alpha T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^{\eta}(\eta-s)f(s,u(s))\nabla s\\
&= \frac{\alpha\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_{\eta}^T Tf(s,u(s))\nabla s\\
&\quad - \frac{\alpha\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^T sf(s,u(s))\nabla s\\
&\quad + \frac{\alpha T}{(T-\alpha\eta)-\beta(T-\eta)}\int_0^{\eta} sf(s,u(s))\nabla s\\
&> \frac{\alpha\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_{\eta}^T Tf(s,u(s))\nabla s\\
&\quad - \frac{\alpha\eta}{(T-\alpha\eta)-\beta(T-\eta)}\int_{\eta}^T sf(s,u(s))\nabla s\\
&\geq \frac{b\alpha\eta}{\delta[(T-\alpha\eta)-\beta(T-\eta)]}\int_{\eta}^T (T-s)\nabla s \geq b.
\end{aligned}$$

So, $\psi(Au) > b,\ u \in P(\psi, b, b/\gamma)$, as required.

For the condition (iii) of the Theorem 2.8, we can verify it easily under our assumptions using Lemma 2.5. Here

$$\psi(Au) = \min_{t\in[0,T]} Au(t) \geq \gamma\|Au\| > \gamma\frac{b}{\gamma} = b$$

as long as $u \in P(\psi, b, c)$ with $\|Au\| > b/\gamma$.

Since all conditions of Theorem 2.8 are satisfied. We say the problem (4.1)-(4.2) has at least three positive solutions u_1, u_2, u_3 with

$$\|u_1\| < a, \quad \psi(u_2) > b, \quad a < \|u_3\| \quad \text{with } \psi(u_3) < b.$$

□

5. Examples

Example 5.1. *Let $\mathbb{T} = [0,1] \cup [2,3]$. Considering the boundary-value problem on $\mathbb{T}$*

$$u^{\Delta\nabla}(t) + tu^p = 0, \quad t \in [0,3] \subset \mathbb{T}, \tag{5.1}$$

$$u(0) = \frac{1}{2}u(2), \quad u(3) = u(2), \tag{5.2}$$

where $p \neq 1$. When taking $T = 3$, $\eta = 2$, $\alpha = 1$, $\beta = \frac{1}{2}$, and

$$a(t) = t, \quad t \in [0,3] \subset \mathbb{T}; \quad f(u) = u^p, \quad u \in [0,\infty),$$

we prove the solvability of problem (5.1)-(5.2) by means of Theorem 3.1. It is clear that $a(\cdot)$ and $f(\cdot)$ satisfy (A1) and (A2). We can also show that

$$0 < \alpha\eta = 2 < 3 = T, \quad 0 < \beta(T-\eta) = \frac{1}{2} < T - \alpha\eta = 1.$$

Now we consider the existence of positive solutions of the problem (5.1)-(5.2) in two cases.
Case 1: $p > 1$. In this case,

$$\lim_{u\to 0^+} \frac{f(u)}{u} = \lim_{u\to 0^+} u^{p-1} = 0, \quad \lim_{u\to\infty} \frac{f(u)}{u} = \lim_{u\to\infty} u^{p-1} = \infty,$$

and (C1) of Theorem 3.1 holds. So the problem (5.1)-(5.2) has at least one positive solution by Theorem 3.1.
Case 2. $p < 1$. In this case,

$$\lim_{u\to 0^+} \frac{f(u)}{u} = \lim_{u\to 0^+} \frac{1}{u^{1-p}} = \infty, \quad \lim_{u\to\infty} \frac{f(u)}{u} = \lim_{u\to\infty} \frac{1}{u^{1-p}} = 0,$$

and (C2) of Theorem 3.1 holds. So the problem (5.1)-(5.2) has at least one positive solution by Theorem 3.1. Therefore, the boundary-value problem (5.1)-(5.2) has at least one positive solution when $p \neq 1$.

Example 5.2. *Let $\mathbb{T} = \{0\} \cup \{1/2^n : n \in \mathbb{N}_0\}$. Considering the boundary-value problem on $\mathbb{T}$*

$$u^{\Delta\nabla}(t) + \frac{2005u^3}{u^3 + 5000} = 0, \quad t \in [0,1] \subset \mathbb{T}, \tag{5.3}$$

$$u(0) = \frac{1}{3}u(\frac{1}{16}), \quad u(1) = 8u(\frac{1}{16}), \tag{5.4}$$

When taking $T = 1$, $\eta = 1/16$, $\alpha = 8$, $\beta = 1/3$, and

$$f(t,u) = f(u) = \frac{2005u^3}{u^3 + 5000}, \quad u \geq 0,$$

we prove the solvability of the problem (5.1)-(5.2) by means of Theorem 4.1. It is clear that $f(\cdot)$ is continuous and increasing on $[0,\infty)$. We can also seen that

$$0 < \alpha\eta = \frac{1}{2} < 1 = T, \quad 0 < \beta(T-\eta) = \frac{5}{16} < T - \alpha\eta = \frac{1}{2}.$$

Now we check that (D1), (D2) and (D3) of Theorem 4.1 are satisfied. By (2.5), (4.3) and (4.4), we get $\gamma = 1/48$, $m = 27/65$, $\delta = 35/1152$. Let $c = 5000$, we have

$$f(u) \leq 2005 < mc \approx 2076.92, \quad u \in [0,c]$$

from $\lim_{u\to\infty} f(u) = 2005$, so that (D3) is met. Note that $f(10) \approx 334.17$, when we set $b = 10$,

$$f(u) \geq \frac{b}{\delta} \approx 329.14, \quad u \in [b, 48b]$$

holds. It means that (D2) are satisfied. To verify (D1), as $f(\frac{1}{5}) \approx 0.0032$, we take $a = 1/5$, then

$$f(u) < ma \approx 0.083, \quad u \in [0,a],$$

and (D1) holds. Summing up, there exists constants $a = 1/5$, $b = 10$, $c = 5000$ *satisfying*

$$0 < a < b < \frac{b}{\gamma} \le c$$

such that (D1), (D2) and (D3) of Theorem 4.1 hold. So the boundary-value problem (5.3)-(5.4) *has at least three positive solutions* u_1, u_2 *and* u_3 *satisfying*

$$\|u_1\| < \frac{1}{5}, \quad \min_{t\in[0,T]}(u_2)(t) > 10, \quad \frac{1}{5} < \|u_3\| \quad \textit{with} \quad \min_{t\in[0,T]}(u_3)(t) < 10.$$

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