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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO A SUPER-LINEAR THREE-POINT BOUNDARY-VALUE PROBLEM

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Abstract. In previous papers, degree theory for nonlinear operators has been used to study a class of three-point boundary-value problems for second order ordinary differential equations having a super-linear term, and existence of a sequence of solutions has been shown. In this paper, we forgo the previous approach for the shooting method, which gives a drastically simpler existence theory, with less assumptions, and easy calculation of solutions. We even obtain uniqueness in the simplest case.

1. Introduction

In the papers [2, 3, 7, 8, 9] the authors use degree theory to give existence of a sequence of solutions to a super-linear boundary value problem. More specifically, in [8, 9] they give existence of solutions to

$$x'' + g(x) = p(t, x, x') \tag{1.1}$$

$$x(0) = 0, \quad x(\eta) = \beta x(1) \tag{1.2}$$

Here $\eta \in (0, 1)$, making this a three point boundary value problem. The function g is assumed to be super-linear, that is, it satisfies $g(x)/x \to \infty$ as $|x| \to \infty$, and $\beta = 1$. In [1] the case $\beta \neq 1$ is argued along similar lines. In this paper, we obtain existence of solutions to (1.1), (1.2) for $\beta \neq 1$ via the intermediate value theorem, i.e. the shooting method, giving a drastically simpler existence theory, with less assumptions. Calculation of solutions numerically may be carried out by the shooting method. The shooting method is used theoretically in [5, 11, 13], and elsewhere.

Uniqueness is studied by Kwong in [13], which recovers results such as Moroney's theorem, giving uniqueness of a positive solution of a boundary-value problem involving a superlinear function. This builds on Kolodner's paper [11], which gave the exact number of solutions of a rotating string problem, given the angular velocity. Similarly, in [4], the boundary value problem

$$x'' + \lambda x^+ - \alpha x^- = \sin(t)$$
$$x(0) = x(\pi) = 0$$

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was shown to have exactly $2k$ solutions if $0 < \alpha < 1$ and $k^2 < \lambda < (k+1)^2$, and Dinca and Sanchez [5] pose the question of whether this uniqueness result can be obtained by elementary methods. Our uniqueness result, giving uniqueness of solutions to (1.1) and (1.2) in case $p = 0$, is elementary and presumably new. Our approach does not readily lend itself to the case of nonzero p, and this gives an open question.

There has been much recent work on 3-point boundary value problems, and much of it has concentrated on positive solutions, as in [6, 10, 12, 15]. He and Ge [6] give the existence of three positive solutions to the B.V.P. (1.1), (1.2), but the condition (4.2) of our uniqueness theorem and their conditions (D2), (D3) cannot hold at the same time. Thus their work cannot be used to show that Theorem 4.1 may not hold for all k.

Similarly Infante and Webb [10, Th 4.2] cannot be used because their conditions (S_1) and (S_2) are incompatible with (4.2).

Ma [15] shows that one can get existence of positive solutions to the B.V.P. (1.1), (1.2), assuming $g(x)/x \to 0$ as $x \to 0$, and $p = 0$, which does show that one can obtain existence theorems like our Theorem 3.1 for small k. Infante and Webb [10] show that one need not have positive coefficients in an m-point boundary value problem, and in this work we can indeed take β to be negative.

Capietto and Dambrosio [2] consider the case of asymmetric $g(x)$, superlinear for positive x, and give an extensive review of superlinear boundary value problems.

2. Assumptions and Preliminaries

A background on o.d.e.s involving functions satisfying Caratheodory's conditions is given in Chapter 18 of [14].

Assumption A: - Assume that $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous super-linear function, that is, it satisfies $\frac{g(x)}{x} \to \infty$ as $|x| \to \infty$. Let $p : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions, i.e. for every $(x,y) \in \mathbb{R}^2$, $p(t,x,y)$ is Lebesgue measurable in t, and for a.e. $t \in [0,1]$, $p(t,x,y)$ is continuous in (x,y). Suppose there exists an $M_1 : [0,1] \times [0,\infty) \mapsto [0,\infty)$ such that (a) for each $s \in [0,\infty)$, $M_1(\cdot,s)$ is integrable on $[0,1]$, (b) for each $t \in [0,1]$, $M_1(t,\cdot)$ is increasing on $[0,\infty)$ with $s^{-1}\int_0^1 M_1(t,s)dt \to 0$ as $s \to \infty$, and (c) for all $t \in [0,1]$, and $(x,y) \in \mathbb{R}^2$,

$$|p(t,x,y)| \le M_1(t, \max(|x|,|y|)).$$

We need the next result, proved in [1] as Lemma 2.

Lemma 2.1. *Let g, p, and M_1 satisfy Assumption A. Suppose that $\frac{g(x)}{x} \ge 1$ for $x \neq 0$. Suppose that $(x(t), y(t))$ is an absolutely continuous solution for the initial value problem*

$$x'(t) = y(t), \tag{2.1}$$

$$y'(t) + g(x(t)) = p(t, x(t), y(t)), \quad \text{for a.e. } t \in [0,1], \tag{2.2}$$

$$x(0) = 0\,, \tag{2.3}$$

$$y(0) = \alpha\,. \tag{2.4}$$

For $x \in \mathbb{R}$, let $G(x) = \int_0^x g(s)ds$. Let $\varepsilon > 0$ be given. Then for $\alpha > 0$, large enough, we have

$$|y(t)| \leq \alpha(1+\varepsilon) \tag{2.5}$$

$$2G(x(t)) \leq \alpha^2(1+\varepsilon), \tag{2.6}$$

for every $t \in [0,1]$. Moreover,

$$\Big|\frac{d}{dt}(y^2(t) + 2G(x(t)))\Big| \leq 2|y(t)|M_1(t, \max(|x(t)|, |y(t)|)), \tag{2.7}$$

for $t \in [0,1]$ a.e.

We note that if we assume g continuous and $g(x)/x \geq 1$ then the function $G(x) = \int_0^x g(s)ds$ is defined for $x \in \mathbb{R}$ and is such that G is strictly increasing on $[0,\infty)$ and is strictly decreasing on $(-\infty, 0]$. Also, $G(x) > 0$ for $x \in \mathbb{R}$, $x \neq 0$ and $G(0) = 0$. We denote the inverse of the function G restricted to $[0,\infty)$, $G\big|_{[0,\infty)}$, by G_+^{-1} and the inverse of the function $G\big|_{(-\infty,0]}$ by G_-^{-1}. We now need a new version of [1, Lemma 3], in which (2.8) and (2.9) replace (13) and (14) of [1].

Lemma 2.2. *Let $\varepsilon > 0$ be given and g, p, M_1 be as in Lemma 2.1. Then there exists an $A > 0$ such that if $(x(t), y(t))$ is a solution for the initial value problem (2.1), (2.2), (2.3), (2.4) and $t_0 \in (0,1]$ is such that $x(t_0) > 0$, $y(t_0) = 0$; then*

$$G_+^{-1}\Big(\frac{\alpha^2}{2(1+\varepsilon)}\Big) \leq x(t_0) \leq G_+^{-1}\Big(\frac{\alpha^2}{2}(1+\varepsilon)\Big) \tag{2.8}$$

if $|\alpha| > A$. Similarly, if $x(t_0) < 0$, $y(t_0) = 0$; then

$$G_-^{-1}\Big(\frac{\alpha^2}{2(1+\varepsilon)}\Big) \leq x(t_0) \leq G_-^{-1}\Big(\frac{\alpha^2}{2}(1+\varepsilon)\Big) \tag{2.9}$$

if $|\alpha| > A$. Also,

$$\min_{t\in[0,1]} \sqrt{x^2(t) + y^2(t)} \geq \frac{1}{2}\min\{G_+^{-1}(\frac{\alpha^2}{8}), \frac{\alpha}{2}\}.$$

Proof. We observe that the right inequality in (2.8) follows immediately from (2.6). Accordingly, it suffices to show that

$$\frac{\alpha^2}{2} \leq (1+\varepsilon)G(x(t_0)), \tag{2.10}$$

to prove that (2.8) holds. Let us choose $A > 0$, such that for $|\alpha| > A$, both (2.5) and (2.6) hold with α replaced by $|\alpha|$. With $h(t) := \sqrt{y^2(t) + 2G(x(t))}$, we get, by integrating (2.7) from 0 to t and using (2.5),

$$h^2(t) - \alpha^2 + 2|\alpha|(1+\varepsilon)\int_0^t M_1(s, \max(|x(s)|, |y(s)|))ds \geq 0.$$

We now take an $\varepsilon_1 > 0$ such that $2\varepsilon_1(1+\varepsilon)^2 \leq \min\{\frac{\varepsilon}{1+\varepsilon}, \frac{1}{2}\}$. Next, we use the assumption $s^{-1}\int_0^1 M_1(t,s)dt \to 0$ as $s \to \infty$, from Assumption A, to choose an $A > 0$ so that for $\alpha > A$ the inequalities (2.5), (2.6) hold for $s \in [0,1]$. When for $s \in [0,1]$

$$\max\{|x(s)|, |y(s)|\} \geq \frac{1}{2}\min\{G_+^{-1}(\frac{\alpha^2}{8}), \frac{\alpha}{2}\} \tag{2.11}$$

we have, with $M(x) := \int_0^1 M_1(t,x)dt$,

$$M(\max\{|x(s)|,|y(s)|\}) < \varepsilon_1 \max\{|x(s)|,|y(s)|\}.$$

For $\alpha > A$, we get on using the inequalities (2.5), (2.6), (2.12) and the assumption $\frac{g(x)}{x} \geq 1$ for $x \neq 0$, that

$$M(\max\{|x(s)|,|y(s)|\}) < \varepsilon_1 \alpha(1+\varepsilon), \tag{2.12}$$

and hence, using (2.11), we get

$$h^2(t) \geq \alpha^2(1 - 2\varepsilon_1(1+\varepsilon)^2), \tag{2.13}$$

provided (2.11) holds for all $s \in [0,t]$. Since we chose $\varepsilon_1 > 0$ such that $2\varepsilon_1(1+\varepsilon)^2 \leq \min\{\frac{\varepsilon}{1+\varepsilon}, \frac{1}{2}\}$, we see that

$$y^2(t) + 2G(x(t)) \geq \frac{\alpha^2}{2},$$

provided (2.11) holds for all $s \in [0,t]$. Accordingly, either $y^2(t) \geq \frac{\alpha^2}{4}$ or $|x(t)| \geq G_+^{-1}(\frac{\alpha^2}{8})$ and hence

$$\max\{|x(t)|,|y(t)|\} \geq \min\{G_+^{-1}(\frac{\alpha^2}{8}), \frac{\alpha}{2}\}, \tag{2.14}$$

provided (2.11) holds for all $s \in [0,t]$. We observe that (2.11) holds near $s = 0$ since $y(0) = \alpha$. Let us next assume that (2.11) holds for all $s \in [0,t]$, for some $t \in (0,1]$. If $0 < t < 1$, it follows from (2.14) that there exists a $t_1 > t$ such that (2.11) holds for all $s \in [0,t_1]$. Accordingly, it follows that (2.11) holds for all $s \in [0,1]$. Finally, if $y(t_0) = 0$, we see from (2.13) and the assumption that $2\varepsilon_1(1+\varepsilon)^2 \leq \min\{\frac{\varepsilon}{1+\varepsilon}, \frac{1}{2}\}$, that

$$\frac{\alpha^2}{2} \leq (1+\varepsilon)G(x(t_0)),$$

and (2.10) holds. This completes the proof that (2.8) holds. A similar proof works to prove that (2.9) holds. □

Definition 2.3. For $\mathbf{u}(t) = (x(t),y(t)) \in C^1([0,1],\mathbb{R}^2\backslash\{(0,0)\})$ we define

$$\varphi_1(\mathbf{u}) = \varphi_1(x,y) = -\int_0^1 \frac{x(t)y'(t) - y(t)x'(t)}{x^2(t)+y^2(t)}dt,$$

as the angle traversed clockwise from $\mathbf{u}(0)$ to $\mathbf{u}(1)$.

We need a variant of [7, Lemma 4.3] and of [9, Lemma 3] to show that the angle $\varphi_1(x,y) \to \infty$, for solutions (x,y) to (2.1)-(2.2), when $\min_{t\in[0,1]} \|(x(t),y(t))\| \to \infty$. We use the following assumption:

Assumption B: Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and super-linear. Let $p : [0,1] \times \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a function satisfying Caratheodory's conditions. Suppose that there exists a $\mu \in (0,1]$, $\beta \geq 0$, $0 \leq \gamma \in L^1[0,1]$, and $M_2 : \mathbb{R}^2 \to \mathbb{R}$, with $\frac{M_2(x,y)}{\|(x,y)\|} \to 0$ as $\|(x,y)\| \to \infty$, such that for a.e. $t \in [0,1]$, and $(x,y) \in \mathbb{R}^2$,

$$\operatorname{sign}(x)p(t,x,y) \leq (1-\mu)\operatorname{sign}(x)g(x) + \beta|y| + \gamma(t)M_2(x,y). \tag{2.15}$$

The inequality (2.15) corresponds to inequality (4.3) in [7], i.e.,

$$|p(t,x,y)| \leq (1-\mu)|g(x)| + \beta|y| + \gamma, \tag{2.16}$$

where $\gamma \in \mathbb{R}$.

We shall provide the slight modifications needed in the proof of Lemma 4.3 of [8], and Lemma 3 of [9] to cater for the difference between (2.15) and (2.16).

Lemma 2.4. *Suppose g and p satisfy Assumption B. Then for all $N \geq 0$ there exists an $R \geq 0$ such that for all absolutely continuous solutions $(x(t), y(t))$ for the system* (2.1)*,* (2.2) *with $\min_{t\in[0,1]} \|(x(t), y(t))\| \geq R$, we have $\varphi_1(x, y) \geq N$.*

Proof. Since $\varphi_1(\mathbf{u}) = -\int_0^1 \frac{x(t)y'(t)-y(t)x'(t)}{x^2(t)+y^2(t)}dt$, we see using (2.1),(2.2) that

$$-x(t)y'(t) + y(t)x'(t) = y^2(t) + x(t)g(x(t)) - x(t)p(t, x(t), y(t)).$$

Let us set

$$\theta(t) - \theta(0) = \int_0^t \frac{x(s)y'(s) - y(s)x'(s)}{x^2(s) + y^2(s)}ds,$$

so that

$$\begin{aligned}
-\theta'(t) &= -\frac{x(t)y'(t) - y(t)x'(t)}{x^2(t) + y^2(t)} \\
&= \frac{y^2(t) + x(t)g(x(t)) - x(t)p(t, x(t), y(t))}{x^2(t) + y^2(t)}.
\end{aligned}$$

Let $N > 0$ be given. Since g is super-linear, we have for $K > 0$ (to be chosen later) there is an $M = M(K)$ such that if $|x| \geq M$ then $\mu\frac{g(x)}{x} \geq K$. Hence

$$\mu\frac{g(x)}{x} + \frac{KM^2}{x^2} \geq K$$

for all $x \neq 0$, and $\mu xg(x) \geq Kx^2 - KM^2$ for all $x \in \mathbb{R}$. Hence,

$$\begin{aligned}
&y^2 + xg(x) - x(t)p(t, x, y) \\
&\geq y^2 + xg(x) - (1-\mu)xg(x) - \beta|x||y| - \gamma(t)|x|M_2(x, y) \\
&\geq y^2 + Kx^2 - KM^2 - \frac{\beta}{2}(\beta x^2 + \frac{y^2}{\beta}) - \gamma(t)|x|M_2(x, y) \\
&\geq \frac{y^2}{2} + (K - \frac{\beta^2}{2})x^2 - KM^2 - \gamma(t)|x|M_2(x, y) \\
&= \frac{y^2}{2} + \frac{k}{2}x^2 - KM^2 - \gamma(t)|x|M_2(x, y),
\end{aligned}$$

where $k = 2(K - \frac{\beta^2}{2})$. Then,

$$\begin{aligned}
-\theta'(t) &= \frac{y^2(t) + x(t)g(x(t)) - x(t)p(t, x(t), y(t))}{x^2(t) + y^2(t)} \\
&\geq \frac{\frac{y^2}{2} + \frac{k}{2}x^2 - KM^2 - \gamma(t)|x|M_2(x, y)}{x^2(t) + y^2(t)} \\
&\geq \frac{k}{2}(\frac{x^2 + k^{-1}y^2}{x^2 + y^2}) - \frac{1}{\sqrt{k}} - \frac{\gamma(t)M_2(x, y)}{\|(x, y)\|},
\end{aligned}$$

assuming $\min_t \|(x(t), y(t))\| \geq M\sqrt{K}\sqrt[4]{k}$. We can then write

$$-\theta'(t) \geq \frac{k}{2}(\cos^2\theta + k^{-1}\sin^2\theta) - \frac{1}{\sqrt{k}} - \frac{\gamma(t)M_2(x, y)}{\|(x, y)\|}.$$

Next we estimate

$$\int_{\theta(1)}^{\theta(0)} \frac{d\theta}{\cos^2\theta + k^{-1}\sin^2\theta},$$

rather than estimating the integral $\int_{\theta(1)}^{\theta(0)} d\theta$.

Since $\frac{1}{\cos^2\theta+k^{-1}\sin^2\theta}\le k$ we get

$$-\frac{\theta'(t)}{\cos^2\theta+k^{-1}\sin^2\theta}\ge\frac{k}{2}-\sqrt{k}-\frac{\gamma(t)M_2(x,y)k}{\|(x,y)\|}\ge\frac{k}{2}-\sqrt{k}-\gamma(t),$$

assuming $\frac{M_2(x,y)}{\|(x,y)\|}\le\frac{1}{k}$, which holds if $\min_t\|(x(t),y(t))\|\ge\xi(k)$, say. Note

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\cos^2\theta+k^{-1}\sin^2\theta}&=\int_0^{\frac{\pi}{2}}\frac{k\sec^2\theta d\theta}{k+\tan^2\theta}\\&=\int_0^\infty\frac{kdu}{k+u^2}\\&=\frac{k}{\sqrt{k}}\tan^{-1}(\frac{u}{\sqrt{k}})|_0^\infty\\&=\frac{\sqrt{k}\pi}{2}.\end{aligned}$$

Since $\cos^2\theta+k^{-1}\sin^2\theta$ has period π, given an interval (a,b) we write $b-a=(n-f)\pi$, where n is an integer and $f\in[0,1)$. Then

$$\begin{aligned}\int_a^b\frac{d\theta}{\cos^2\theta+k^{-1}\sin^2\theta}&\le\int_0^{n\pi}\frac{d\theta}{\cos^2\theta+k^{-1}\sin^2\theta}\\&\le 2n\frac{\sqrt{k}\pi}{2}\\&=\sqrt{k}\pi(\frac{b-a}{\pi}+f)\\&\le\sqrt{k}(b-a+\pi).\end{aligned}$$

In particular, we get

$$-\int_{\theta(0)}^{\theta(1)}\frac{d\theta}{\cos^2\theta+k^{-1}\sin^2\theta}\le\sqrt{k}(\theta(0)-\theta(1)+\pi).$$

Next we change the variable of integration from θ to t to get

$$\begin{aligned}\sqrt{k}(\varphi_1(x,y)+\pi)&\ge-\int_0^1\frac{\theta'(t)dt}{\cos^2\theta(t)+k^{-1}\sin^2\theta(t)}\\&\ge\int_0^1(\frac{k}{2}-\sqrt{k}-\gamma(t))dt\\&=\frac{k}{2}-\sqrt{k}-\int_0^1\gamma.\end{aligned}$$

This gives

$$\begin{aligned}\varphi_1(x,y)&\ge\frac{\sqrt{k}}{2}-1-\int_0^1\gamma-\pi,\quad\text{if } k\ge1,\\&=N,\quad\text{if } k=4(N+1+\int_0^1\gamma+\pi)^2\ge1.\end{aligned}$$

Since

$$k=2(K-\frac{\beta^2}{2}),$$

we choose $K = 2(N + 1 + \int_0^1 \gamma + \pi)^2 + \frac{\beta^2}{2}$. Finally choosing

$$R = \max(M\sqrt{K}\sqrt[4]{k}, \xi(k)),$$

we see that $\varphi_1(x, y) \geq N$ if $\min_{t\in[0,1]} \|(x(t), y(t))\| \geq R$. □

3. Existence of Solutions

To fix ideas we study the case $\beta > 1$, as in [1].

Theorem 3.1. *Let $\eta \in (0, 1)$ and $\beta > 1$ be given. Let g and p satisfy Assumptions A and B. Then, for each k sufficiently large, there are (at least) two solutions $\mathbf{u}(t) = (x(t), y(t))$ of*

$$x'(t) = y(t) \tag{3.1}$$

$$y'(t) = -g(x(t)) + p(t, x(t), y(t)) \tag{3.2}$$

$$x(0) = 0 \tag{3.3}$$

$$x(\eta) = \beta x(1) \tag{3.4}$$

with $\varphi_1(\mathbf{u}) \in (\frac{\pi}{2}+k\pi, \frac{\pi}{2}+(k+1)\pi)$, one with $x'(0) > 0$ and the other with $x'(0) < 0$.

We may, when thinking of calculating these solutions, say that we have one sequence x_n of solutions to (1.1), (1.2) with $x_n'(0) \to \infty$ and another sequence x_n of solutions to (1.1), (1.2) with $x_n'(0) \to -\infty$, with the angles they traverse as above.

We break the proof of the theorem into two parts. We first prove existence of the solution x when p is smooth enough, and then we see that we can approximate p by a sequence of smooth p_n, giving solutions x_n, and then then take limits to obtain existence when p is not smooth. A sufficiently smooth p will satisfy the following Caratheodory-Lipschitz condition.

Definition 3.2. [14] Let U be open in $\mathbb{R}^n$, and let $[a, b]$ be an interval of real numbers. Let $F : [a, b] \times U \to \mathbb{R}^n$ be given. We say F satisfies a Caratheodory-Lipschitz condition if for all x, $t \mapsto F(t, x)$ is Lebesgue measurable, and for any $(t_0, x_0) \in D$, there are real valued integrable functions m and L, such that

$$\|F(t, x) - F(t, y)\| \leq L(t)\|x - y\| \tag{3.5}$$

$$\|F(t, x)\| \leq m(t) \tag{3.6}$$

for all x and y in some neighbourhood of x_0, and t a.e. in some neighbourhood of t_0.

We need the following definition.

Definition 3.3. For $\alpha \in \mathbb{R}$ let (x, y) be a solution of (2.1), (2.2), (2.3) and (2.4), and let $\eta \in (0, 1)$ and $\beta > 1$ be as in Theorem 3.1. We define a function $H : (0, \infty) \to \mathbb{R}$ by $H(\alpha) = \beta x(1) - x(\eta)$, for $\alpha \in (0, \infty)$.

Remark 3.4. Since the function $g : \mathbb{R} \to \mathbb{R}$ in Theorem 3.1 is assumed to be super-linear, we see that there exists an $M > 0$ such that $\frac{g(x)}{x} \geq 1$ for $|x| \geq M$. Let us, now, define a function $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{g}(x) = \begin{cases} g(x), & \text{for } x \geq M \\ \frac{g(M)}{M}x, & \text{for } 0 \leq x \leq M \\ \frac{g(-M)}{-M}x, & \text{for } -M \leq x \leq 0 \\ g(x), & \text{for } x \leq -M. \end{cases}$$

It then follows that $\frac{\widetilde{g}(x)}{x} \geq 1$ for all $x \neq 0$ and $g - \widetilde{g}$ is a bounded function on $\mathbb{R}$. Also, $p + \widetilde{g} - g$ satisfies the same conditions as p in Theorem 3.1. Accordingly, we shall assume in the following that the function g in Theorem 3.1 is such that $\frac{g(x)}{x} \geq 1$ for all $x \neq 0$, by replacing g by $\widetilde{g}$ and p by $p + \widetilde{g} - g$, if necessary.

Proof of Theorem: Smooth case. Here we assume, in addition to Assumption A, that p is Caratheodory-Lipschitz on $[0,1] \times \mathbb{R}^2$, g is locally Lipschitz, and for all nonzero x, $g(x)/x \geq 1$.

We first see from Lemma 2.4 and the last claim in Lemma 2.2, that $\varphi_1(x,y) \to \infty$, as $y(0) \to \infty$. Accordingly, for every positive integer k, sufficiently large, there exists an $h_k \in \mathbb{R}$ such that if (x,y) is a solution of

$$\begin{aligned} x' &= y \\ y' &= -g(x) + p(t,x,y) \\ x(0) &= 0 \\ y(0) &= h_k, \end{aligned} \tag{3.7}$$

then $\varphi_1(x,y) = \pi/2 + k\pi$. There may be more than one value for h_k, so we let h_k^{min} and h_k^{max} be the smallest and largest such numbers.

Then for the function H, defined in Definition 3.3, we claim $H(h_k) > 0$, if k is even. Since, now, $\varphi_1(x,y) = \pi/2 + k\pi$ and k is even, we see that $x(1) > 0$ and $y(1) = 0$, from the definition of $\varphi_1(x,y)$ (see Definition 2.3). Suppose, now, x is maximised at $\eta^* \in (0,1]$. We then get from (2.8) of Lemma 2.2 that $x(\eta) \leq x(\eta^*) \leq G_+^{-1}(\frac{h_k^2}{2}(1+\epsilon))$ and $\beta x(1) \geq \beta G_+^{-1}(\frac{h_k^2}{2(1+\epsilon)})$, since $x(1) > 0$ and $y(1) = 0$.

Now,

$$\begin{aligned} H(h_k) &= \beta x(1) - x(\eta) \\ &\geq \beta G_+^{-1}(\frac{h_k^2}{2(1+\epsilon)}) - G_+^{-1}(\frac{h_k^2}{2}(1+\epsilon)) \\ &= (\beta - 1) G_+^{-1}(\frac{h_k^2}{2(1+\epsilon)}) + G_+^{-1}(\frac{h_k^2}{2(1+\epsilon)}) - G_+^{-1}(\frac{h_k^2}{2}(1+\epsilon)). \end{aligned} \tag{3.8}$$

We may assume that $0 < \epsilon < 1$, and let us set

$$t = G_+^{-1}(\frac{h_k^2}{2(1+\epsilon)}) \tag{3.9}$$

and

$$t + \delta = G_+^{-1}(\frac{h_k^2}{2}(1+\epsilon)).$$

Then

$$G(t) = \frac{h_k^2}{2(1+\epsilon)} \quad \text{and} \quad G(t+\delta) = \frac{h_k^2}{2}(1+\epsilon).$$

Next, we see that

$$h_k^2 \epsilon \geq \frac{h_k^2}{2}(1+\epsilon) - \frac{h_k^2}{2(1+\epsilon)} = G(t+\delta) - G(t) = \int_t^{t+\delta} g(s)ds \geq t\delta, \tag{3.10}$$

in view of our assumption $\frac{g(x)}{x} \geq 1$ for all $x \neq 0$. It then follows from (3.8), (3.9), (3.10), the assumption $0 < \epsilon < 1$ and the fact that G_+^{-1} is an increasing function that

$$H(h_k) = \beta x(1) - x(\eta) \geq (\beta - 1)G_+^{-1}(\frac{h_k^2}{4}) - \frac{h_k^2 \epsilon}{G_+^{-1}(\frac{h_k^2}{4})} > 0$$

if $\epsilon > 0$ is chosen sufficiently small. Hence $H(h_k) > 0$. Similarly, $H(h_k) < 0$ when k is odd.

Now H is continuous, indeed the map $(0, \alpha) \mapsto (x(1), x(\eta))$ is locally Lipschitz by [14].

By the intermediate value theorem, there is an $\alpha \in (h_{2k}^{max}, h_{2k+1}^{min})$ and a solution (x, y) of (2.1), (2.2), (2.3), (2.4) such that $H(\alpha) = 0$. We claim $\varphi_1(x, y) \in (\pi/2 + 2k\pi, \pi/2 + (2k+1)\pi)$. Suppose $\varphi_1(x, y) \leq \pi/2 + 2k\pi$. Then by the intermediate value theorem there is an $h_{2k} > \alpha$, contradicting $\alpha > h_{2k}$. This concludes the proof of Theorem 3.1 in the smooth case. □

The next Lemma is needed in the proof for the non-smooth case. Writing $(x, y) = \mathbf{u}$, we mollify the function $p(t, \mathbf{u})$ with respect to the second variable $\mathbf{u}$.

Lemma 3.5. *Suppose $p : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies (a) the Caratheodory-Lipschitz conditions, and $M_1 : [0, 1] \times [0, \infty) \to [0, \infty)$ satisfies:*

(b) *for all $t \in [0, 1]$, $M_1(t, \cdot)$ is increasing on $[0, \infty)$,*
(c) *for all $s \in [0, \infty)$, $M_1(\cdot, s)$ is integrable on $[0, 1]$, and*
(d) *$s^{-1} \int_0^1 M_1(t, s)dt \to 0$ as $s \to \infty$.*

Suppose that for all t and $\mathbf{u}$,

(e) *$|p(t, \mathbf{u})| \leq M_1(t, \|\mathbf{u}\|_\infty)$.*

Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ have support in $\{\mathbf{u} \in \mathbb{R}^2 : \|\mathbf{u}\|_\infty \leq 1\}$, $\varphi \geq 0$, $\int \varphi = 1$. Let $\epsilon > 0$ be given. Let $p^\epsilon(t, \mathbf{u}) = \int p(t, \mathbf{u} - \epsilon\mathbf{v})\varphi(\mathbf{v})d\mathbf{v}$, and let $M_1^\epsilon(t, s) = M_1(t, s + \epsilon)$. Then the pair of functions p^ϵ and ${M_1}^\epsilon$ satisfy conditions (a) through (e).

Proof. To show p^ϵ satisfies (a), we first let $\mathbf{u}$ be given, and claim $t \mapsto p^\epsilon(t, \mathbf{u})$ is Lebesgue measurable. That is,

$$t \mapsto \epsilon^{-2} \int p(t, \mathbf{x})\varphi(\frac{\mathbf{u} - \mathbf{x}}{\epsilon})dx_1 dx_2$$

is measurable. For a.e. $t \in [0, 1]$, $p(t, x)$ is continuous in x, and so the integral is a Riemann integral. Accordingly,

$$\epsilon^{-2} \int p(t, \mathbf{x})\varphi(\frac{\mathbf{u} - \mathbf{x}}{\epsilon})dx_1 dx_2 = \lim_{n\to\infty} \sum_{\mathbf{x}\in P(n)} p(t, \mathbf{x})\varphi(\frac{\mathbf{u} - \mathbf{x}}{\epsilon}),$$

where $\{P(n)\}$ is a sequence of partitions of $[0, 1] \times [0, 1]$. Each of these sums is a measurable function of t since p satisfies the Caratheodory conditions, Lebesgue measure is complete, and the Lebesgue measurable functions form a vector space. The limit of a sequence of measurable functions is measurable, and so we have proved the claim.

To show p^ϵ satisfies (a), let $(t_0, \mathbf{u}_0) \in D$ be given. We claim there are real valued integrable functions m and L, such that

$$\|p^\epsilon(t, \mathbf{u}) - p^\epsilon(t, \mathbf{w})\| \leq L(t)\|\mathbf{u} - \mathbf{w}\| \tag{3.11}$$

$$\|p^\epsilon(t, \mathbf{u})\| \leq m(t) \tag{3.12}$$

for all $\mathbf{u}$ and $\mathbf{w}$ in some neighbourhood of $\mathbf{u}_0$, and for a.e. t in some neighbourhood of t_0, i.e. (3.5) and (3.6) hold. Now

$$\|p^\epsilon(t,\mathbf{u}) - p^\epsilon(t,\mathbf{w})\| \le \epsilon^{-2}\int_{N(\epsilon)} p(t,\mathbf{x})|\varphi(\frac{\mathbf{u}-\mathbf{x}}{\epsilon}) - \varphi(\frac{\mathbf{w}-\mathbf{x}}{\epsilon})|dx_1dx_2$$

where $N(\epsilon)$ stands for $\{\mathbf{x} : \|\mathbf{x}-\mathbf{u}\| \le \epsilon\} \cup \{x : \|\mathbf{x}-\mathbf{w}\| \le \epsilon\}$. Note $N(\epsilon) \subset \{\mathbf{x} : \|\mathbf{x}\| \le \max(\|\mathbf{u}\|, \|\mathbf{v}\|) + \epsilon)$. By (b), with $K(\varphi)$ the Lipschitz constant of φ,

$$\begin{aligned} RHS &\le \epsilon^{-3}\int_{N(\epsilon)} M_1(t, \max(\|\mathbf{u}\|, \|\mathbf{v}\|) + \epsilon)K(\varphi)\|\mathbf{u}-\mathbf{w}\|dx_1dx_2 \\ &\le 2\epsilon^{-1}M_1(t, \max(\|\mathbf{u}\|, \|\mathbf{v}\|) + \epsilon)K(\varphi)\|\mathbf{u}-\mathbf{w}\|. \end{aligned}$$

Since M_1 is increasing, (b) shows that (3.11) holds for all $\mathbf{u}$ and $\mathbf{w}$ in any given bounded set, and all $t \in [0,1]$.

For (3.12), we check $\|p^\epsilon(t,0)\|$ is integrable, and this and (3.11) gives (3.12).

$$\begin{aligned} |p^\epsilon(t,0)| &= \|\int p(t,-\epsilon\mathbf{v})\varphi(\mathbf{v})d\mathbf{v}\| \\ &\le \int |p(t,-\epsilon\mathbf{v})\varphi(\mathbf{v})|d\mathbf{v} \\ &\le \int_{\|v\|\le 1} M_1(t,\epsilon)\varphi(\mathbf{v})dv, \quad \text{by (e)}, \\ &= M_1(t,\epsilon). \end{aligned}$$

To show (b) for ${M_1}^\epsilon$ we note that $s \mapsto M_1(t, s+\epsilon)$ is increasing on $[0,\infty)$. To show (c) for ${M_1}^\epsilon$ we note that for all s, $t \mapsto M_1(t, s+\epsilon)$ is integrable on $[0,1]$. To show (d) for ${M_1}^\epsilon$ we note that $s^{-1}\int_0^1 M_1(t, s+\epsilon)dt \to 0$ as $s \to \infty$. To show (e) for $p^\epsilon(t,0)$ and ${M_1}^\epsilon$ we note that

$$\begin{aligned} |p^\epsilon(t,\mathbf{u})| &\le \int |p(t,\mathbf{u}-\epsilon\mathbf{v})\varphi(\mathbf{v})d\mathbf{v} \\ &\le \int M_1(t,\|\mathbf{u}\|+\epsilon)\varphi(\mathbf{v})d\mathbf{v} \\ &\le \int M_1^\epsilon(t,\|\mathbf{u}\|)\varphi(\mathbf{v})d\mathbf{v} \\ &= M_1^\epsilon(t,\|\mathbf{u}\|). \end{aligned}$$

□

Proof of Theorem: Non-smooth case. Given g, we will, by adding a term to g and subtracting it from p, assume that $g(x)/x \ge 1$ for all $x \ne 0$. For $\epsilon > 0$, we take g^ϵ which is locally Lipschitz and such that $g^\epsilon(x) \to g(x)$ uniformly on bounded sets, and $g^\epsilon(x)/x \ge 1$ for all $x \ne 0$.

For each large integer k, and $\epsilon > 0$, we let (x_ϵ, y_ϵ) be a solution of (3.1) to (3.4) with g^ϵ and p^ϵ replacing g and p, satisfying $\varphi_1(x_\epsilon, y_\epsilon) \in (\frac{\pi}{2}+k\pi, \frac{\pi}{2}+(k+1)\pi)$, with $y_\epsilon(0) > 0$. Now we can check the (x_ϵ, y_ϵ) are uniformly bounded, and we can check they are equi-continuous, since their derivatives are uniformly bounded. By the Arzela-Ascoli Theorem, there is a sequence $\epsilon(n) \to 0$ with $(x_{\epsilon(n)}, y_{\epsilon(n)})$ converging to (x,y), say, in $C([0,1];\mathbb{R}^2)$. Now for all $t \in [0,1]$,

$$\begin{pmatrix} x_\epsilon \\ y_\epsilon \end{pmatrix}(t) = \begin{pmatrix} 0 \\ y_\epsilon(0) \end{pmatrix} + \int_0^t \begin{pmatrix} y_\epsilon(s) \\ -g^\epsilon(x_\epsilon(s)) + p^\epsilon(s, x_\epsilon(s), y_\epsilon(s)) \end{pmatrix} ds \tag{3.13}$$

We use the dominated convergence theorem to let $n \to \infty$ with $\epsilon = \epsilon(n)$.
(a) We claim $p^\epsilon(s, x_\epsilon(s), y_\epsilon(s))$ converges to $p(s, x(s), y(s))$ for s a.e. in $[0,1]$. Take s so that $p(s, \mathbf{u})$ is continuous in $\mathbf{u}$. Then we note

$$\int_{\mathbb{R}^2} p(s, x_{\epsilon(n)}(s) - \epsilon(n)v_1, y_{\epsilon(n)}(s) - \epsilon(n)v_2)\varphi(v_1, v_2)dv_1dv_2 \to p(s, x(s), y(s)),$$

proving the claim.
(b) We note $g^{\epsilon(n)}(s, x_{\epsilon(n)}(s))$ converges to $g(x(s))$ for any s.
(c) We claim $p^{\epsilon(n)}(s, x_{\epsilon(n)}(s), y_{\epsilon(n)}(s)) \le M_1(s, K)$ for some $K > 0$. Just take $K \ge \sup_{s\in[0,1]} \sup_n \max(|x_{\epsilon(n)}(s)|, |y_{\epsilon(n)}(s)|) + \max_n \epsilon(n)$.
(d) We note there is K such that for all n and x, $|g^{\epsilon(n)}(x_{\epsilon(n)}(s))| \le K$.
The dominated convergence theorem is applicable by (a) – (d). Hence

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} 0 \\ y(0) \end{pmatrix} + \int_0^t \begin{pmatrix} y(s) \\ -g(x(s)) + p(s, x(s), y(s)) \end{pmatrix} ds \tag{3.14}$$

Hence the o.d.e. (3.1) and (3.2) holds for (x, y). The boundary conditions (3.3) and (3.4) hold for (x, y), since they held for the approximations (x_ϵ, y_ϵ).

We note that $\varphi_1(x_{\epsilon(n)}, y_{\epsilon(n)}) \to \varphi_1(x, y)$, noting that for all n, $(x_{\epsilon(n)}, y_{\epsilon(n)})$ are outside some neighbourhood of $(0,0)$. Hence $\varphi_1(x, y) \in [\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi]$, since $\varphi_1(x_{\epsilon(n)}, y_{\epsilon(n)}) \in (\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi)$. Because of the boundary condition (3.4), $\varphi_1(x, y) \ne \frac{\pi}{2} + k\pi$ for all large k. This ends the proof of the theorem. $\square$

4. Uniqueness

We proved in Theorem 3.1 that the equations (3.1), (3.2), (3.3), (3.4) have at least one solution (x, x') with $\varphi_1(x, x') \in (\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi)$, and $x'(0) > 0$. In this section we shall show that the equations (3.1), (3.2), (3.3), (3.4) have exactly one solution (x, x') with $\varphi_1(x, x') \in (\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi)$ and $x'(0) > 0$, when $p \equiv 0$, g is like the function $x \mapsto |x|^s sgn(x)$, for some $s > 1$, and β, η satisfy a suitable inequality. The arguments can be easily modified to prove that the equations (3.1), (3.2), (3.3), (3.4) have exactly one solution (x, x') with $\varphi_1(x, x') \in (\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi)$, and $x'(0) < 0$. In Remark 4.5 we give a result for $\beta < 1$.

Theorem 4.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Suppose that there exist $p_0 > 0$, $p_1 > 0$, and an $s > 1$ such that for all $x \in \mathbb{R}$,*

$$p_0|x|^s \le g(x)\operatorname{sgn}(x) \le p_1|x|^s, \tag{4.1}$$

and there exists a $h > 0$ such that

$$\frac{g(x)}{x^{1+h}} \text{ is increasing on } (0, \infty) \text{ and } (-\infty, 0). \tag{4.2}$$

Let $\beta > 1$ and $\eta \in (0,1)$ be such that

$$\beta^2 > \sqrt{1 + \frac{\eta^4}{4}} + \frac{\eta^2}{2}. \tag{4.3}$$

Then, for k (an integer) sufficiently large, the solution of the system of equations

$$x'(t) = y(t), \tag{4.4}$$

$$y'(t) = -g(x(t)), \tag{4.5}$$

$$x(0) = 0, \tag{4.6}$$

$$x(\eta) = \beta x(1), \tag{4.7}$$

with $\varphi_1(x,x') \in (\frac{\pi}{2}+k\pi, \frac{\pi}{2}+(k+1)\pi)$*, and* $x'(0)>0$*, is unique.*

Remark 4.2. The existence of a solution for the system of equations (4.4), (4.5), (4.6), (4.7) is obtained using Theorem 3.1.

Proof of Theorem 4.1. Let $(x_\alpha(t), y_\alpha(t))$ be the solution of the equations (4.4), (4.5) with $x(0)=0$, $y(0)=\alpha$. Recalling $G(x)=\int_0^x g(t)dt$, we see, from (4.4), (4.5) with $(x,y)=(x_\alpha(t),y_\alpha(t))$, that

$$G(x)+\frac{y^2}{2}=\frac{\alpha^2}{2}=G(\gamma)=G(-\gamma^*),$$

where $\gamma=G_+^{-1}(\alpha^2/2)$ and $-\gamma^*=G_-^{-1}(\alpha^2/2)$. Note that both γ and γ^* are positive.

In the following, we shall use γ to parametrise the solution, giving $(x,y)=(x_\alpha(t),y_\alpha(t)):=(x(t,\gamma),y(t,\gamma))$. We define γ_k by setting $\gamma=\gamma_k$ when $\varphi_1(x,y)=\frac{\pi}{2}+k\pi$. This corresponds to $\alpha=h_k$ (see equation (3.7)). We next consider $\beta x(1,\gamma)$ and $x(\eta,\gamma)$ for $\gamma\in(\gamma_k,\gamma_{k+1})$. Now, from Theorem 3.1 we see for k sufficiently large that there exists a $\gamma_0\in(\gamma_k,\gamma_{k+1})$ such that $\beta x(1,\gamma_0)=x(\eta,\gamma_0)$. To show uniqueness of γ_0 it suffices to show that

$$|\beta\frac{\partial x}{\partial\gamma}(1,\gamma_0)|>|\frac{\partial x}{\partial\gamma}(\eta,\gamma_0)|. \tag{4.8}$$

Let us define $\widetilde{\varphi}(t,\gamma)$ by setting

$$\widetilde{\varphi}(t,\gamma)=\int_0^t\frac{x'(s)y(s)-y'(s)x(s)}{x^2(s)+y^2(s)}ds,$$

where $(x,y)=(x(t,\gamma),y(t,\gamma))$. Now, we define a function $\widetilde{t}(\varphi,\gamma)$ by

$$t=\widetilde{t}(\varphi,\gamma) \iff \varphi=\widetilde{\varphi}(t,\gamma). \tag{4.9}$$

We note that $\widetilde{t}(\varphi,\gamma)$ is the time taken for the solution $(x(t,\gamma),y(t,\gamma))$ to traverse the angle φ. For $t=1$, we then have

$$1=\widetilde{t}(\widetilde{\varphi}(1,\gamma),\gamma),$$

from (4.9). Next we use the implicit function theorem to get

$$\frac{\partial}{\partial\gamma}\widetilde{\varphi}(1,\gamma)=-\frac{\frac{\partial\widetilde{t}}{\partial\gamma}(1,\gamma)}{\frac{\partial\widetilde{t}}{\partial\varphi}(1,\gamma)}=-\frac{\partial\widetilde{t}}{\partial\gamma}(\widetilde{\varphi}(1,\gamma),\gamma)\frac{\partial\widetilde{\varphi}}{\partial t}(1,\gamma). \tag{4.10}$$

Let us define $\widetilde{x}(\varphi,\gamma)$ as follows: since traversing the angle φ clockwise along the curve $G(x)+\frac{y^2}{2}=G(\gamma)$ from $(0,\alpha)$ brings us to a point (x,y), we define

$$\widetilde{x}(\varphi,\gamma)=x.$$

Note

$$\widetilde{x}(\widetilde{\varphi}(1,\gamma),\gamma)=x(1,\gamma).$$

¿From the chain rule,

$$\frac{\partial x}{\partial\gamma}(1,\gamma)=\frac{\partial\widetilde{x}}{\partial\gamma}(\widetilde{\varphi}(1,\gamma),\gamma)+\frac{\partial\widetilde{x}}{\partial\varphi}(\widetilde{\varphi}(1,\gamma),\gamma)\frac{\partial\widetilde{\varphi}}{\partial\gamma}(1,\gamma). \tag{4.11}$$

Similarly, we get

$$\frac{\partial x}{\partial\gamma}(\eta,\gamma)=\frac{\partial\widetilde{x}}{\partial\gamma}(\widetilde{\varphi}(\eta,\gamma),\gamma)+\frac{\partial\widetilde{x}}{\partial\varphi}(\widetilde{\varphi}(\eta,\gamma),\gamma)\frac{\partial\widetilde{\varphi}}{\partial\gamma}(\eta,\gamma). \tag{4.12}$$

We express $\widetilde{t}(\varphi,\gamma)$ in terms of n, the number of times the solution (x,y) goes around the origin, and the time taken to traverse the angle 2π. Let

$$\widetilde{t}(\varphi,\gamma)=n(\varphi,\gamma)t_P(\gamma)+t_Q(\varphi,\gamma), \tag{4.13}$$

where t_P is the time for a full revolution or the time taken to traverse the angle 2π, and $t_Q\in[0,t_P)$. Note

$$\frac{n(\widetilde{\varphi}(\eta,\gamma),\gamma)}{n(\widetilde{\varphi}(1,\gamma),\gamma)}\to\eta \text{ as } \gamma\to\infty.$$

We differentiate $\widetilde{t}(\varphi,\gamma)$ with respect to γ in (4.13) when $\widetilde{\varphi}(1,\gamma)$ is not a multiple of 2π to get

$$\frac{\partial\widetilde{t}}{\partial\gamma}(\varphi,\gamma)=n\frac{\partial t_P}{\partial\gamma}+\frac{\partial t_Q}{\partial\gamma}. \tag{4.14}$$

Next, define

$$\varphi_\eta:=\widetilde{\varphi}(\eta,\gamma_0) \text{ and } \varphi_1:=\widetilde{\varphi}(1,\gamma_0).$$

We evaluate (4.14) at two points.

$$\frac{\partial\widetilde{t}}{\partial\gamma}(\varphi_1,\gamma_0)=n(\varphi_1,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\varphi_1,\gamma_0)+\frac{\partial t_Q}{\partial\gamma}(\varphi_1,\gamma_0). \tag{4.15}$$

$$\frac{\partial\widetilde{t}}{\partial\gamma}(\varphi_\eta,\gamma_0)=n(\varphi_\eta,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\varphi_\eta,\gamma_0)+\frac{\partial t_Q}{\partial\gamma}(\varphi_\eta,\gamma_0). \tag{4.16}$$

Now, by [1, Lemma 4] we see that

$$\frac{\partial}{\partial\gamma}\frac{\partial\widetilde{t}}{\partial\varphi}(\varphi,\gamma)\le 0, \text{ for all } (\varphi,\gamma).$$

Also, notice that $\frac{\partial\widetilde{t}}{\partial\gamma}(0,\gamma)=0$ for all γ. So we get

$$0\ge\frac{\partial t_Q}{\partial\gamma}(\varphi,\gamma_0)\ge\frac{\partial t_P}{\partial\gamma}(\gamma_0).$$

Now, to prove (4.8) we compute the ratio of (4.12) to (4.11). This gives

$$\frac{\frac{\partial x}{\partial\gamma}(\eta,\gamma)}{\frac{\partial x}{\partial\gamma}(1,\gamma)}=\frac{\frac{\partial\widetilde{x}}{\partial\gamma}(\widetilde{\varphi}(\eta,\gamma),\gamma)+\frac{\partial\widetilde{x}}{\partial\varphi}(\widetilde{\varphi}(\eta,\gamma),\gamma)\frac{\partial\widetilde{\varphi}}{\partial\gamma}(\eta,\gamma)}{\frac{\partial\widetilde{x}}{\partial\gamma}(\widetilde{\varphi}(1,\gamma),\gamma)+\frac{\partial\widetilde{x}}{\partial\varphi}(\widetilde{\varphi}(1,\gamma),\gamma)\frac{\partial\widetilde{\varphi}}{\partial\gamma}(1,\gamma)}.$$

Using (4.10) we then get

$$\frac{\frac{\partial x}{\partial\gamma}(\eta,\gamma)}{\frac{\partial x}{\partial\gamma}(1,\gamma)}=\frac{\frac{\partial\widetilde{x}}{\partial\gamma}(\widetilde{\varphi}(\eta,\gamma),\gamma)-\frac{\partial\widetilde{x}}{\partial\varphi}(\widetilde{\varphi}(\eta,\gamma),\gamma)\frac{\partial\widetilde{t}}{\partial\gamma}(\widetilde{\varphi}(\eta,\gamma),\gamma)\frac{\partial\widetilde{\varphi}}{\partial t}(\eta,\gamma)}{\frac{\partial\widetilde{x}}{\partial\gamma}(\widetilde{\varphi}(1,\gamma),\gamma)-\frac{\partial\widetilde{x}}{\partial\varphi}(\widetilde{\varphi}(1,\gamma),\gamma)\frac{\partial\widetilde{t}}{\partial\gamma}(\widetilde{\varphi}(1,\gamma),\gamma)\frac{\partial\widetilde{\varphi}}{\partial t}(1,\gamma)}. \tag{4.17}$$

We note that

$$\frac{\partial x}{\partial t}(t,\gamma)=\frac{\partial\widetilde{x}}{\partial\varphi}(\widetilde{\varphi}(t,\gamma),\gamma)\frac{\partial\widetilde{\varphi}}{\partial t}(t,\gamma) \text{ for all } (t,\gamma).$$

Hence, (4.17) becomes

$$\frac{\frac{\partial x}{\partial\gamma}(\eta,\gamma)}{\frac{\partial x}{\partial\gamma}(1,\gamma)}=\frac{\frac{\partial\widetilde{x}}{\partial\gamma}(\widetilde{\varphi}(\eta,\gamma),\gamma)-\frac{\partial\widetilde{t}}{\partial\gamma}(\widetilde{\varphi}(\eta,\gamma),\gamma)\frac{\partial x}{\partial t}(\eta,\gamma)}{\frac{\partial\widetilde{x}}{\partial\gamma}(\widetilde{\varphi}(1,\gamma),\gamma)-\frac{\partial\widetilde{t}}{\partial\gamma}(\widetilde{\varphi}(1,\gamma),\gamma)\frac{\partial x}{\partial t}(1,\gamma)}.$$

Now, by (4.15) and (4.16) we get

$$\frac{\frac{\partial x}{\partial \gamma}(\eta,\gamma_0)}{\frac{\partial x}{\partial \gamma}(1,\gamma_0)} = \frac{NUM}{DENOM}, \tag{4.18}$$

where

$$NUM = \frac{\partial \widetilde{x}}{\partial \gamma}(\widetilde{\varphi}(\eta,\gamma_0),\gamma_0) - \frac{\partial x}{\partial t}(\eta,\gamma_0)[n(\varphi_\eta,\gamma_0)\frac{\partial t_P}{\partial \gamma}(\gamma_0) + \frac{\partial t_Q}{\partial \gamma}(\varphi_\eta,\gamma_0)], \tag{4.19}$$

and

$$DENOM = \frac{\partial \widetilde{x}}{\partial \gamma}(\widetilde{\varphi}(1,\gamma_0),\gamma_0) - \frac{\partial x}{\partial t}(1,\gamma_0)[n(\varphi_1,\gamma_0)\frac{\partial t_P}{\partial \gamma}(\gamma_0) + \frac{\partial t_Q}{\partial \gamma}(\varphi_1,\gamma_0)]. \tag{4.20}$$

Next, we need to show that the first term in (4.20) is small compared to the second term. To do this we need the following lemma. □

Lemma 4.3. *Suppose that the assumptions (4.1) and (4.2) of Theorem 4.1 hold. Then*

$$\frac{|\frac{\partial \widetilde{x}}{\partial \gamma}(\varphi_1,\gamma_0)|}{n(\varphi_1,\gamma_0)|\frac{\partial t_P}{\partial \gamma}(\gamma_0)|\cdot|\frac{\partial x}{\partial t}(1,\gamma_0)|} \to 0, \tag{4.21}$$

as $\gamma_0 \to \infty$, (note that γ_0 depends on k and as $k \to \infty$, $\gamma_0 \to \infty$).

Proof. For better readability we break the proof into a sequence of items.
Item 1:- We show that

$$\frac{\partial \widetilde{x}}{\partial \gamma}(\varphi,\gamma) = \frac{xg(\gamma)}{xg(x)+2(G(\gamma)-G(x))} = \frac{xg(\gamma)}{xg(x)+y^2}, \tag{4.22}$$

$$\frac{\partial \widetilde{y}}{\partial \gamma}(\varphi,\gamma) = \frac{yg(\gamma)}{xg(x)+2(G(\gamma)-G(x))} = \frac{yg(\gamma)}{xg(x)+y^2}. \tag{4.23}$$

We first note that as we hold φ constant, we hold y/x constant i.e.

$$\frac{\sqrt{2(G(\gamma)-G(x))}}{x}$$

is constant, which in turn implies that

$$\frac{\partial}{\partial \gamma}\Big(\frac{G(\gamma)-G(x)}{x^2}\Big) = 0.$$

Hence,

$$(g(\gamma) - g(x)\frac{\partial \widetilde{x}}{\partial \gamma})x^2 - 2x\frac{\partial \widetilde{x}}{\partial \gamma}(G(\gamma)-G(x)) = 0,$$

i.e.

$$\frac{\partial \widetilde{x}}{\partial \gamma}[g(x)x^2 + 2x(G(\gamma)-G(x))] = x^2 g(\gamma),$$

and so (4.22) holds. Also, since we hold φ constant, y/x is constant and we get

$$\frac{\partial \widetilde{y}}{\partial \gamma} = \frac{y}{x}\frac{\partial \widetilde{x}}{\partial \gamma},$$

giving (4.23).
Item 2:- There exists a constant $K_0 = K_0(s,p_0,p_1)$ such that

$$|\frac{\partial \widetilde{x}}{\partial \gamma}| \le K_0(s,p_0,p_1),$$

for all (φ, γ). We first claim that for all $x \in \mathbb{R}$,

$$xg(x) - 2G(x) \geq 0.$$

Indeed, both for $x > 0$ and for $x < 0$,

$$G(x) = \int_0^x g(t)dt \leq \int_0^x \frac{tg(x)}{x}dt = \frac{xg(x)}{2},$$

since $\frac{g(x)}{x}$ is increasing. Hence the claim. Next, we claim that for all x,

$$\frac{p_1|x|^{s+1}}{s+1} \geq G(x) \geq \frac{p_0|x|^{s+1}}{s+1}. \tag{4.24}$$

We see from (4.1) that for $x > 0$,

$$G(x) \geq \int_0^x p_0 t^s dt = \frac{p_0 x^{s+1}}{s+1}.$$

Similarly, we get for $x < 0$,

$$G(x) \geq \frac{p_0|x|^{s+1}}{s+1}.$$

The proof of the left half of the inequality in (4.24) is similar. ¿From (4.22) we have

$$\begin{aligned} \frac{\partial \widetilde{x}}{\partial \gamma}(\varphi, \gamma) &= \frac{xg(\gamma)}{xg(x) + 2(G(\gamma) - G(x))} \\ &\leq \frac{|x|g(\gamma)}{2G(\gamma)}, \text{ by (4.24)}, \\ &\leq \max(\gamma, \gamma^*)\frac{p_1\gamma^s(s+1)}{2p_0\gamma^{s+1}}. \end{aligned} \tag{4.25}$$

Next, we see from (4.24) that there exists a constant K, depending on p_0, p_1, and s such that

$$\gamma^* \leq K\gamma \quad \text{and} \quad \gamma \leq K\gamma^*. \tag{4.26}$$

Using (4.26) in (4.25) the proof of Item 2 is immediate.
Item 3:- There exists a constant $K_1 = K_1(s, p_0, p_1) > 0$ such that

$$|\frac{\partial x}{\partial t}(1, \gamma_0)| \geq K_1 \gamma_0^{\frac{s+1}{2}}, \tag{4.27}$$

for all γ_0. Since $g(x)/x$ is increasing, we have $G(x)/x$ is increasing by [1, Lemma 3]. So

$$G(\frac{\gamma}{\beta}) \leq \frac{G(\gamma)}{\beta}.$$

Hence, assuming $x(1, \gamma_0) \geq 0$, so that $x(1, \gamma_0) \leq \frac{\gamma}{\beta}$, we have

$$\begin{aligned} \frac{1}{2}(\frac{\partial x}{\partial t}(1, \gamma_0))^2 &= G(\gamma_0) - G(x(1, \gamma_0)) \\ &\geq G(\gamma_0) - G(\frac{\gamma_0}{\beta}) \\ &\geq (1 - \frac{1}{\beta})G(\gamma_0) \\ &\geq (1 - \frac{1}{\beta})\frac{p_0\gamma^{s+1}}{s+1}, \end{aligned}$$

by (4.24). Taking square roots, we get (4.27), proving Item 3.
Item 4:- Let t_R be the time taken to go from $(0,\alpha)$ to $(\gamma,0)$. Also let t_L be the time taken to go from $(0,-\alpha)$ to $(-\gamma^*,0)$. We shall show that

$$\frac{dt_R}{d\gamma}=\int_0^{\frac{\pi}{2}}\frac{xg(\gamma)(g(x)-xg'(x))}{(xg(x)+y^2)^3}(x^2+y^2)d\varphi. \tag{4.28}$$

We note that since

$$\begin{gathered} x'=y,\\ y'=-g(x), \end{gathered}$$

we have

$$\frac{\partial\widetilde{\varphi}}{\partial t}(t,\gamma)=-\frac{xy'-yx'}{x^2+y^2}=\frac{xg(x)+y^2}{x^2+y^2}.$$

Accordingly,

$$t_R(\gamma)=\int_0^{\frac{\pi}{2}}\frac{x^2+y^2}{xg(x)+y^2}d\varphi.$$

Using Leibnitz's rule, (4.22) and (4.23), we get

$$\begin{aligned} \frac{dt_R}{d\gamma}&=\int_0^{\frac{\pi}{2}}\frac{1}{(xg(x)+y^2)^2}\{(2x\frac{xg(\gamma)}{xg(x)+y^2}+2y\frac{yg(\gamma)}{xg(x)+y^2})(xg(x)+y^2)\\ &\quad-(x^2+y^2)[(xg'(x)+g(x))\frac{xg(\gamma)}{xg(x)+y^2}+2y\frac{yg(\gamma)}{xg(x)+y^2}\}d\varphi\\ &=\int_0^{\frac{\pi}{2}}\frac{2x^2g(\gamma)+2y^2g(\gamma)-\frac{x^2+y^2}{xg(x)+y^2}[(xg'(x)+g(x))xg(\gamma)+2y^2g(\gamma)]}{(xg(x)+y^2)^2}d\varphi\\ &=\int_0^{\frac{\pi}{2}}\frac{2g(\gamma)(xg(x)+y^2)-(xg'(x)+g(x))xg(\gamma)-2y^2g(\gamma)}{(xg(x)+y^2)^3}(x^2+y^2)d\varphi, \end{aligned}$$

which gives (4.28).
Item 5:- We change variables from φ to u to calculate (4.28). We parametrise the curve

$$\frac{y^2}{2}+G(x)=G(\gamma),$$

in the first quadrant, by setting

$$\begin{gathered} y=\sqrt{2G(\gamma)}\sin u\\ x=G_+^{-1}(G(\gamma)\cos^2 u). \end{gathered}$$

So, with γ kept fixed,

$$\begin{aligned} \frac{dy}{du}&=\sqrt{2G(\gamma)}\cos u\\ g(x)\frac{dx}{du}&=-2G(\gamma)\cos u\sin u, \end{aligned}$$

i.e.

$$\frac{dx}{du}=-\frac{2G(\gamma)\cos u\sin u}{g(x)}.$$

Now,

$$(x^2+y^2)\frac{d\varphi}{du}=y\frac{dx}{du}-x\frac{dy}{du}.$$

Hence,

$$(x^2+y^2)\frac{d\varphi}{du} = -\frac{(2G(\gamma))^{\frac{3}{2}}\cos^2 u\sin u}{g(x)} - x\sqrt{2G(\gamma)}\cos u.$$

Therefore,

$$\frac{dt_R}{d\gamma} = \int_0^{\frac{\pi}{2}} \frac{xg(\gamma)(xg'(x)-g(x))}{(xg(x)+y^2)^3}\Big[\frac{(2G(\gamma))^{\frac{3}{2}}\cos^2 u\sin u}{g(x)} + x\sqrt{2G(\gamma)}\cos u\Big]du, \tag{4.29}$$

noting that as φ varies from 0 to $\pi/2$, u also varies from 0 to $\pi/2$.

In the next items we estimate the various terms in (4.29).

Item 6:- There exist constants $K_3 > 0$ and $K_4 > 0$ depending on p_0, p_1, and s (but not u, γ) such that

$$x \le K_3\gamma(\cos u)^{\frac{2}{s+1}}, \tag{4.30}$$

$$x \ge K_4\gamma(\cos u)^{\frac{2}{s+1}}. \tag{4.31}$$

Indeed,

$$\frac{p_1\gamma^{s+1}}{s+1}\cos^2 u \ge G(\gamma)\cos^2 u = G(x) \ge \frac{p_1 x^{s+1}}{s+1}.$$

The inequality given by the two outside terms above gives (4.30). We obtain (4.31) similarly.

Item 7:- There exists a constant $K_5 > 0$ depending on p_0, p_1, and s (but not γ) such that

$$xg(x) + y^2 \le K_5\gamma^{s+1}. \tag{4.32}$$

Indeed,

$$xg(x) \le p_1 x^{s+1} \le \frac{p_1}{p_0}(s+1)G(x) \le \frac{p_1}{p_0}(s+1)G(\gamma) \le \frac{p_1^2}{p_0}\gamma^{s+1},$$

$$y^2 = 2G(\gamma)\sin^2 u \le 2p_1\frac{\gamma^{s+1}}{s+1}\sin^2 u,$$

and (4.32) follows.

Item 8:- There exists a constant $K_6 > 0$ depending on p_0, p_1, and s (but not γ) such that

$$xg'(x) - g(x) \ge K_6 x^s. \tag{4.33}$$

We see from (4.2) that $\frac{g(x)}{x^{1+h}}$ is increasing, which implies $\log\frac{g(x)}{x^{1+h}}$ is increasing, which implies $\frac{d}{dx}(\log\frac{g(x)}{x^{1+h}}) \ge 0$, which implies $\frac{g'(x)}{g(x)} - \frac{1+h}{x} \ge 0$, which implies $xg'(x) - g(x) \ge hg(x) \ge kp_0x^s$, thereby proving (4.33).

Item 9:- We find a lower bound for $\frac{(2G(\gamma))^{\frac{3}{2}}\cos^2 u\sin u}{g(x)} + x\sqrt{2G(\gamma)}\cos u$, using (4.24), (4.30) and (4.31). Indeed,

$$
\begin{aligned}
&\frac{(2G(\gamma))^{\frac{3}{2}}\cos^2 u\sin u}{g(x)}+x\sqrt{2G(\gamma)}\cos u\\
&\geq 2\sqrt{2}\frac{p_0^{\frac{3}{2}}\gamma^{\frac{3(s+1)}{2}}}{(s+1)^{\frac{3}{2}}p_1x^s}\cos^2 u\sin u+K_4\gamma(\cos u)^{\frac{2}{s+1}}\sqrt{2\frac{p_0}{s+1}}\gamma^{\frac{s+1}{2}}\cos u\\
&\geq 2\sqrt{2}\frac{p_0^{\frac{3}{2}}\gamma^{\frac{3(s+1)}{2}}}{(s+1)^{\frac{3}{2}}p_1K_3^s\gamma^s(\cos u)^{\frac{2s}{s+1}}}\cos^2 u\sin u+K_4\gamma(\cos u)^{\frac{2}{s+1}}\sqrt{2\frac{p_0}{s+1}}\gamma^{\frac{s+1}{2}}\cos u\\
&\geq K_7\gamma^{\frac{s+3}{2}}[(\cos u)^{\frac{2}{s+1}}\sin u+(\cos u)^{\frac{s+3}{s+1}}].
\end{aligned}
$$

Item 10:- Now we use our estimates to find a lower bound for (4.29).

$$
\begin{aligned}
&\frac{dt_R}{d\gamma}\\
&=\int_0^{\frac{\pi}{2}}\frac{xg(\gamma)(xg'(x)-g(x))}{(xg(x)+y^2)^3}\Big[\frac{(2G(\gamma))^{\frac{3}{2}}\cos^2 u\sin u}{g(x)}+x\sqrt{2G(\gamma)}\cos u\Big]du\\
&\geq\int_0^{\frac{\pi}{2}}\frac{K_4\gamma(\cos u)^{\frac{2}{s+1}}g(\gamma)(K_6x^s)}{(K_5\gamma^{s+1})^3}K_7\gamma^{\frac{s+3}{2}}[(\cos u)^{\frac{2}{s+1}}\sin u+(\cos u)^{\frac{s+3}{s+1}}]du\\
&\geq\int_0^{\frac{\pi}{2}}\frac{K_4\gamma(\cos u)^{\frac{2}{s+1}}p_0\gamma^sK_6(K_4\gamma(\cos u)^{\frac{2}{s+1}})^s}{(K_5\gamma^{s+1})^3}\\
&\quad\times K_7\gamma^{\frac{s+3}{2}}\big[(\cos u)^{\frac{2}{s+1}}\sin u+(\cos u)^{\frac{s+3}{s+1}}\big]du\\
&\geq K_8\gamma^{-\frac{s+1}{2}}.
\end{aligned}
$$

We note in a similar way that

$$\frac{dt_L}{d\gamma}\geq K_9\gamma^{-\frac{s+1}{2}},$$

and hence

$$\frac{dt_P}{d\gamma}\geq K_{10}\gamma^{-\frac{s+1}{2}}.$$

Next, to complete the proof of the lemma we use the various estimates obtained above, in (4.21).

$$\frac{|\frac{\partial\tilde{x}}{\partial\gamma}(\varphi_1,\gamma_0)|}{n(\varphi_1,\gamma_0)|\frac{\partial t_P}{\partial\gamma}(\gamma_0)|\cdot|\frac{\partial x}{\partial t}(1,\gamma_0)|}\leq\frac{K_{11}}{n(\varphi_1,\gamma_0)\gamma_0^{\frac{s+1}{2}}\gamma_0^{-\frac{s+1}{2}}}=\frac{K_{11}}{n(\varphi_1,\gamma_0)}\to 0,$$

as $\gamma_0\to\infty$. $\square$

Next, we find an upper bound for $\frac{|y(\eta,\gamma_0)|}{|y(1,\gamma_0)|}$, using (4.3).

Lemma 4.4. *Suppose g is continuous, super-linear and $\frac{g(x)}{x}$ is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$. Suppose (4.3) holds. Then there exists a $\beta_0<\beta$ such that*

$$\frac{|y(\eta,\gamma_0)|}{|y(1,\gamma_0)|}\leq\frac{\beta_0}{\eta}. \tag{4.34}$$

Proof. We have $x(1,\gamma_0) = \frac{x(\eta,\gamma_0)}{\beta}$ giving

$$-\frac{\gamma_0^*}{\beta} \le x(1,\gamma_0) \le \frac{\gamma_0}{\beta},$$

where $\gamma_0^* > 0$ is defined by $G(-\gamma_0^*) = G(\gamma_0)$. Suppose $x(\eta,\gamma_0) \ge \frac{\gamma_0}{\beta}$. Then $x(\eta,\gamma_0) \ge x(1,\gamma_0)$, giving $|y(\eta,\gamma_0)| \le |y(1,\gamma_0)|$, and (4.34) holds for any $\beta_0 \in [\eta,\beta)$. Similarly, if $x(\eta,\gamma_0) \le -\frac{\gamma_0^*}{\beta}$, then (4.34) holds. Hence we assume that

$$-\frac{\gamma_0^*}{\beta} \le x(\eta,\gamma_0) \le \frac{\gamma_0}{\beta},$$

giving

$$-\frac{\gamma_0^*}{\beta^2} \le x(1,\gamma_0) \le \frac{\gamma_0}{\beta^2}.$$

This implies if $x(1,\gamma_0) \ge 0$, that

$$|y(1,\gamma_0)| \ge \sqrt{2(G(\gamma_0) - G(\frac{\gamma_0}{\beta^2})}).$$

Similarly, if $x(1,\gamma_0) \le 0$, then

$$|y(1,\gamma_0)| \ge \sqrt{2(G(\gamma_0) - G(-\frac{\gamma_0^*}{\beta^2})}).$$

Since $|y(\eta,\gamma_0)| \le \sqrt{2G(\gamma_0)}$, (4.34) holds if

$$\frac{\beta_0^2}{\eta^2} \ge \frac{G(\gamma_0)}{G(\gamma_0) - G(\frac{\gamma_0}{\beta^2})},$$

i.e.

$$(\beta_0^2 - \eta^2)G(\gamma_0) \ge \beta_0^2 G(\frac{\gamma_0}{\beta^2}), \tag{4.35}$$

and also

$$(\beta_0^2 - \eta^2)G(\gamma_0) \ge \beta_0^2 G(-\frac{\gamma_0^*}{\beta^2}). \tag{4.36}$$

We claim that for all $z \in \mathbb{R}$ and $\delta \ge 1$,

$$G(\delta z) \ge \delta^2 G(z). \tag{4.37}$$

Suppose $s > 0$. Then $\frac{g(\delta s)}{\delta s} \ge \frac{g(s)}{s}$. For $z > 0$,

$$G(\delta z) = \int_0^{\delta z} g(t)dt = \int_0^z g(\delta s)\delta ds \ge \delta^2 \int_0^z g(s)ds,$$

proving the claim. Similarly the claim can be proved for $z < 0$.

In (4.37) put $z = \frac{\gamma_0}{\beta^2}$, and $\delta = \beta^2$, giving $G(\gamma_0) \ge \beta^4 G(\frac{\gamma_0}{\beta^2})$ and $G(\gamma_0) \ge \beta^4 G(-\frac{\gamma_0^*}{\beta^2})$. Hence, (4.35) and (4.36) hold if

$$(1 - \frac{\eta^2}{\beta_0^2})\beta^4 \ge 1. \tag{4.38}$$

Taking $\beta_0 = (1-\varepsilon)\beta$, (4.38) holds if

$$\beta^2 \ge \frac{(\frac{\eta}{1-\varepsilon})^2}{2} + \sqrt{\frac{(\frac{\eta}{1-\varepsilon})^4}{4} + 1}, \tag{4.39}$$

and (4.39) holds for some $\varepsilon > 0$ by (4.3). $\square$

Proof of Theorem 4.1, continued. We divide NUM from (4.19) and $DENOM$ from (4.20) by $\frac{\partial x}{\partial t}(1,\gamma_0)n(\varphi_1,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\gamma_0)$ to define

$$NEWNUM = \frac{NUM}{\frac{\partial x}{\partial t}(1,\gamma_0)n(\varphi_1,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\gamma_0)} \quad \text{and}$$
$$NEWDENOM = \frac{DENOM}{\frac{\partial x}{\partial t}(1,\gamma_0)n(\varphi_1,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\gamma_0)}.$$

Now, $NEWDENOM \to -1$ as $\gamma_0 \to \infty$ since the first summand converges to zero by Lemma 4.3 and the second summand is

$$-(1 + \frac{\frac{\partial t_Q}{\partial\gamma}(\varphi_1,\gamma_0)}{n(\varphi_1,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\gamma_0)}),$$

which converges to -1 since

$$\frac{\frac{\partial t_Q}{\partial\gamma}(\varphi_1,\gamma_0)}{\frac{\partial t_P}{\partial\gamma}(\gamma_0)} \le 1$$

and $n(\varphi_1,\gamma_0)\to\infty$. Now,

$$\begin{aligned} NEWNUM = & \frac{\frac{\partial\widetilde{x}}{\partial\gamma}(\widetilde{\varphi}(\eta,\gamma_0),\gamma_0)}{\frac{\partial x}{\partial t}(1,\gamma_0)n(\varphi_1,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\gamma_0)} \\ & - \frac{\frac{\partial x}{\partial t}(\eta,\gamma_0)[n(\varphi_\eta,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\gamma_0) + \frac{\partial t_Q}{\partial\gamma}(\varphi_\eta,\gamma_0)]}{\frac{\partial x}{\partial t}(1,\gamma_0)n(\varphi_1,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\gamma_0)}, \end{aligned}$$

and we set

$$NEWNUM1 = \frac{\frac{\partial\widetilde{x}}{\partial\gamma}(\widetilde{\varphi}(\eta,\gamma_0),\gamma_0)}{\frac{\partial x}{\partial t}(1,\gamma_0)n(\varphi_1,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\gamma_0)},$$
$$NEWNUM2 = \frac{\frac{\partial x}{\partial t}(\eta,\gamma_0)[n(\varphi_\eta,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\gamma_0) + \frac{\partial t_Q}{\partial\gamma}(\varphi_\eta,\gamma_0)]}{\frac{\partial x}{\partial t}(1,\gamma_0)n(\varphi_1,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\gamma_0)}.$$

Item 2 of Lemma 4.3 shows that $|\frac{\partial\widetilde{x}}{\partial\gamma}(\widetilde{\varphi}(\eta,\gamma_0),\gamma_0)| \le K_0$, and hence Lemma 4.3 applies to show that $NEWNUM1$ converges to zero as $\gamma_0\to\infty$. We rewrite $NEWNUM2$ as

$$\frac{\frac{\partial x}{\partial t}(\eta,\gamma_0)}{\frac{\partial x}{\partial t}(1,\gamma_0)}(\frac{n(\varphi_\eta,\gamma_0)}{n(\varphi_1,\gamma_0)} + \frac{\frac{\partial t_Q}{\partial\gamma}(\varphi_\eta,\gamma_0)}{n(\varphi_1,\gamma_0)\ \frac{\partial t_P}{\partial\gamma}(\gamma_0)}).$$

Hence, we get from Lemma 4.3 that

$$|NEWNUM2| \le \frac{\beta_0}{\eta}\Big(\frac{n(\varphi_\eta,\gamma_0)}{n(\varphi_1,\gamma_0)} + \Big|\frac{\frac{\partial t_Q}{\partial\gamma}(\varphi_\eta,\gamma_0)}{n(\varphi_1,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\gamma_0)}\Big|\Big).$$

Since $\frac{n(\varphi_\eta,\gamma_0)}{n(\varphi_1,\gamma_0)}\to\eta$ and

$$\Big|\frac{\frac{\partial t_Q}{\partial\gamma}(\varphi_\eta,\gamma_0)}{n(\varphi_1,\gamma_0)\frac{\partial t_P}{\partial\gamma}(\gamma_0)}\Big| \to 0$$

as $\gamma_0\to\infty$, we have $|\frac{NEWNUM2}{\beta}| < 1$ for γ_0 sufficiently large. Hence, (4.8) holds and the proof of the theorem is complete. $\square$

Remark 4.5. Given $\eta \in (0,1)$, and $\beta \in (0,1)$, satisfying

$$\beta < \frac{\eta}{\sqrt{0.5+\sqrt{0.25+\eta^4}}}, \tag{4.40}$$

the solution of (4.4) - (4.7) satisfying $\varphi_\eta(x,x') \in (\frac{\pi}{2}+k\pi, \frac{\pi}{2}+(k+1)\pi)$ exists and is unique, if k is large. Note we have replaced 1 by η in Definition 2.3, to give

$$\varphi_\eta(x,y) = -\int_0^\eta \frac{x(t)y'(t)-y(t)x'(t)}{x^2(t)+y^2(t)}dt\,.$$

The inequality (4.40) follows from (4.3) by replacing η and β by their inverses. The change of variable $\tau = \eta^{-1}t$ leads to this, using the fact that Theorem 4.1 holds for $\eta > 1$ too.

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