Electronic Journal of Differential Equations, Vol. 2005(2005), No. 27, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

KAMENEV-TYPE OSCILLATION CRITERIA FOR SECOND-ORDER QUASILINEAR DIFFERENTIAL EQUATIONS

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Abstract. We obtain Kamenev-type oscillation criteria for the second-order quasilinear differential equation

$$(r(t)|y'(t)|^{\alpha-1}y'(t))' + p(t)|y(t)|^{\beta-1}y(t) = 0\,.$$

The criteria obtained extend the integral averaging technique and include earlier results due to Kamenev, Philos and Wong.

1. Introduction

This paper concerns the oscillation of solutions to the second order quasilinear differential equation

$$(r(t)|y'(t)|^{\alpha-1}y'(t))' + p(t)|y(t)|^{\beta-1}y(t) = 0, \quad t \ge t_0 > 0, \tag{1.1}$$

where $r \in C^1([t_0,\infty),\mathbb{R}^+)$, $p \in C([t_0,\infty),\mathbb{R})$, and α, $\beta > 0$, $(\alpha \neq \beta)$, are constants.

In this paper we shall assume that the following conditions hold:

(A1) $R(t) := \int_{t_0}^t r^{-1/\alpha}(s)ds \to \infty$, as $t \to \infty$,

(A2) $\liminf_{t\to\infty} \int_{t_0}^t p(s)ds = -M_0 > -\infty$.

By a solution to (1.1), we mean a function $y \in C^1([T_y,\infty),\mathbb{R})$, $T_y \ge t_0$, which has the property $r(t)|y'(t)|^{\alpha-1}y'(t) \in C^1([T_y,\infty),\mathbb{R})$ and satisfies (1.1). We restrict our attention only to the nontrivial solutions of (1.1), i.e., to the solution $y(t)$ such that $\sup\{|y(t)| : t \ge T\} > 0$ for all $T \ge T_y$. A nontrivial solution of (1.1) is called oscillatory if it has arbitrary large zeros, otherwise, it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

When $\alpha = \beta$, Equation (1.1) reduces to second order half-linear differential equation

$$(r(t)|y'(t)|^{\alpha-1}y'(t))' + p(t)|y(t)|^{\alpha-1}y(t) = 0. \tag{1.2}$$

Oscillatory and nonoscilltory property of (1.2) have been widely discussed in the literatures (see, for example, [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 19, 20, 21] and the reference therein). However, relatively less attention [17] has been given to oscillation of (1.1). Some of the oscillation criteria [1, 12, 13, 15, 19] for (1.2)

2000 *Mathematics Subject Classification.* 34C10, 34C15.
Key words and phrases. Oscillation; second order quasilinear differential equation; integral operator.

Submitted August 11, 2004. Published March 6, 2005.

have been obtained by using the averaging technique from the papers of Kamenev [7] and Philos [16] for linear differential equation

$$(r(t)y'(t))' + p(t)y(t) = 0. \tag{1.3}$$

It is, therefore, natural to ask if it is possible to establish oscillation criteria for (1.1). Motivated by the idea of Wong [18], in this paper we extend the results of Kamenev [7], Philos [16], Wong [18] to (1.1) by general means given in [16, 18]. Our methodology is somewhat different from that of previous authors. We believe that our approach is simple and also provides a more unified account of the study of Kamenev-type oscillation theorems. We will also show that do not need any restriction on the sign of the function p.

2. Main results

First, we introduce the concept of general means [16, 18] and present some properties, which will be used in the proof of our results.

Let $D = \{(t,s) : t \ge s \ge t_0\}$ and $D_0 = \{(t,s) : t > s \ge t_0\}$. We will say that the function $H \in C(D,\mathbb{R})$ belongs to a class $\Im$ if

- (H1) $H(t,t) = 0$ for $t \ge t_0$, $H(t,s) > 0$ on D_0
- (H2) H has a continuous and non-positive partial derivative in D_0 with respect to the second variable
- (H3) There exist functions $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ and $h \in C(D,\mathbb{R})$ such that
$$\frac{\partial}{\partial s}[H(t,s)\rho(s)] = -H(t,s)h(t,s), \quad (t,s) \in D_0.$$

Let $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ and $H \in \Im$. We take the integral operator A, which is defined in [18], in terms of $H(t,s)$ and $\rho(s)$ as

$$A_T(\phi;t) := \int_T^t H(t,s)\phi(s)\rho(s)ds, \quad t \ge T \ge t_0, \tag{2.1}$$

where $\phi \in C([t_0,\infty),\mathbb{R})$. It is easily seen that the integral operator A satisfies the following properties:

$$A_T(\alpha_1 h_1 + \alpha_2 h_2; t) = \alpha_1 A_T(h_1,t) + \alpha_2 A_T(h_2,t), \tag{2.2}$$

$$A_T(h_3,t) \ge 0 \quad \forall\, h_3 \ge 0, \tag{2.3}$$

$$A_T(h_4'; t) = -H(t,T)h_4(T)\rho(T) + A_T(\rho^{-1}h_4 h; t). \tag{2.4}$$

Here $h_1, h_2, h_3 \in C([t_0,\infty),\mathbb{R})$, $h_4 \in C^1([t_0,\infty),\mathbb{R})$, and α_1, α_2 are real numbers.

For an arbitrary function $\xi \in C([t_0,\infty),\mathbb{R}^+)$, define the kernel

$$H(t,s) := \Big(\int_s^t \frac{du}{\xi(u)}\Big)^m, \quad m > 1, \tag{2.5}$$

with $\int_a^\infty 1/\xi(\tau)d\tau = \infty$. An important particular case is $\xi(\tau) = \tau^n$, where $n \le 1$ is real. When $\xi(\tau) = 1$ we have $H(t,s) = (t-s)^m$, and when $\xi(\tau) = \tau$ we have $H(t,s) = (\ln t/\ln s)^m$. It is easily verified that the kernel (2.5) satisfies (H1)–(H3).

We are now able to state and show the main results.

Theorem 2.1. *Suppose that there exist functions $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$, $H, h \in C(D,\mathbb{R})$ with $H \in \Im$ and for any $M > 0$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A_{t_0}(p - \theta g^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1}; t) = \infty, \tag{2.6}$$

where

$$\theta = (\alpha+1)^{-(\alpha+1)}, \quad g(t) = \frac{\beta M}{\alpha} r^{-1/\alpha}(t) R^{-1}(t).$$

Then (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we assume that $y(t) \neq 0$ for all $t \geq t_0$. Furthermore, we suppose that $y(t) > 0$ for $t \geq t_0$, since the substitution $u = -y$ when $y(t) < 0$ transforms (1.1) into an equation of the same form to the assumptions of the theorem. Now, we put

$$W(t) = \frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(t)|^{\beta-1}y(t)}. \tag{2.7}$$

Then it follows from (1.1) that

$$\begin{aligned} W'(t) &= -p(t) - \beta r(t)\frac{|y'(t)|^{\alpha+1}}{|y(t)|^{\beta+1}} \\ &= -p(t) - \beta r^{-1/\alpha}(t)|y(t)|^{(\beta-\alpha)/\alpha}|W(t)|^{(\alpha+1)/\alpha}, \quad \text{for } t \geq t_0, \end{aligned} \tag{2.8}$$

and consequently,

$$\frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(t)|^{\beta}} = C - \int_{t_0}^t p(s)ds - \beta\int_{t_0}^t r(s)\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds, \tag{2.9}$$

where $C = r(t_0)|y'(t_0)|^{\alpha-1}y'(t_0)/|y(t_0)|^{\beta}$.

Next, we consider the following three cases for the behavior of $y'(t)$:

Case 1. $y'(t)$ is oscillatory. Then there exists a sequence $\{t_m\}$, $(m = 1, 2\dots)$, in $[t_0, \infty)$ with $\lim_{m\to\infty} t_m = \infty$ and such that $y'(t_m) = 0$, $(m = 1, 2, \dots)$. Thus, (2.9) gives

$$\beta\int_{t_0}^{t_m} r(s)\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds = C - \int_{t_0}^{t_m} p(s)ds, \quad m = 1, 2, \dots,$$

and hence, by (A2), we conclude that

$$\int_{t_0}^{\infty} r(s)\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds < \infty. \tag{2.10}$$

So, for some positive constant N, we have

$$\int_{t_0}^{t} r(s)\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds \leq N^{\alpha+1}, \quad \text{for } t \geq t_0.$$

By the Hölder inequality

$$\begin{aligned} \Big|\int_{t_0}^t \frac{y'(s)}{[y(s)]^{(\beta+1)/(\alpha+1)}}ds\Big| &\leq \Big[\int_{t_0}^t r(s)\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds\Big]^{1/(\alpha+1)}\Big[\int_{t_0}^t r^{-1/\alpha}(s)ds\Big]^{\alpha/(\alpha+1)} \\ &\leq NR^{\alpha/(\alpha+1)}(t). \end{aligned}$$

Hence

$$\big|[y(t)]^{(\alpha-\beta)/(\alpha+1)} - [y(t_0)]^{(\alpha-\beta)/(\alpha+1)}\big| \leq \frac{N(\alpha+1)}{|\alpha-\beta|}R^{\alpha/(\alpha+1)}(t).$$

So, there exist a $t_1 \geq t_0$ and a constant $M > 0$ so that for $t \geq t_1$

$$|y(t)|^{(\alpha-\beta)/(\alpha+1)} \leq M^{-\alpha/(\alpha+1)}R^{\alpha/(\alpha+1)}(t),$$

or

$$|y(t)|^{(\beta-\alpha)/\alpha} \geq MR^{-1}(t). \tag{2.11}$$

Substituting from (2.11) into (2.8), we have

$$\begin{aligned} W'(t) &\le -p(t) - \beta M r^{-1/\alpha}(t) R^{-1}(t)|W(t)|^{(\alpha+1)/\alpha} \\ &= -p(t) - \alpha g(t)|W(t)|^{(\alpha+1)/\alpha}. \end{aligned} \tag{2.12}$$

Applying the operator A_T, $(T \ge t_0)$, to (2.12), and using (2.4), we have

$$A_T(p;t) \le H(t,T)\rho(T)W(T) + A_T(\rho^{-1}|h||W|;t) - \alpha A_T(g|W|^{(\alpha+1)/\alpha};t). \tag{2.13}$$

The Young inequality gives

$$\rho^{-1}|h||W| \le \alpha g|W|^{(\alpha+1)/\alpha} + \theta g^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1}.$$

Substitute the above inequality into (2.13), we get

$$A_T(p;t) \le H(t,T)\rho(T)W(T) + \theta A_T(g^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t). \tag{2.14}$$

Set $T = t_0$ and divide (2.14) through by $H(t,t_0)$, so

$$\frac{1}{H(t,t_0)} A_{t_0}(p - \theta g^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t) \le \rho(t_0)W(t_0). \tag{2.15}$$

Taking limsup in (2.15) as $t \to \infty$, condition (2.6) gives a desired contradiction.

Case 2. $y'(t) > 0$ on $[T,\infty)$ for some $T \ge t_0$. In this case, from (2.9) it follows that (2.10) holds for $t \ge T$. Once again, we can complete the proof by the procedure of the proof of Case 1.

Case 3. $y'(t) < 0$ on $[T,\infty)$ for some $T \ge t_0$. If (2.10) holds, then we can arrive at a contradiction by the procedure of Case 1. So we suppose that

$$\int_{t_0}^{\infty} r(s)\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds = \infty.$$

Using (2.9), we have, for $t \ge T$

$$-\frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(t)|^{\beta}} \ge -(C+M_0) + \beta\int_T^t r(s)\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds. \tag{2.16}$$

By the assumption, we can choose $T_1 \ge T$ such that

$$\beta\int_T^{T_1} r(s)\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds = 1 + C + M_0,$$

and then for any $t \ge T_1$, we get

$$\frac{-\frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(t)|^{\beta}}\big(-\beta\frac{y'(t)}{y(t)}\big)}{-(C+M_0) + \beta\int_T^t r(s)\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds} \ge -\beta\frac{y'(t)}{y(t)}.$$

Integrate the above inequality from T_1 to t to obtain

$$\ln\big[-(C+M_0) + \beta\int_T^t r(s)\frac{|y'(s)|^{\alpha+1}}{|y(s)|^{\beta+1}}ds\big] \ge \ln\big[\frac{y(T_1)}{y(t)}\big]^{\beta},$$

which together with (2.16) yields

$$-\frac{r(t)|y'(t)|^{\alpha-1}y'(t)}{|y(t)|^{\beta-1}y(t)} \ge \big(\frac{y(T_1)}{y(t)}\big)^{\beta},$$

from which it follows that

$$y'(t) \le -y^{\beta/\alpha}(T_1)r^{-1/\alpha}(t), \quad \text{for } t \ge T_1,$$

then, by (A1),

$$y(t) \le y(T_1) - y^{\beta/\alpha}(T_1)\int_{T_1}^{t} r^{-1/\alpha}(s)ds \to -\infty, \quad \text{as } t \to \infty,$$

contradicting the assumption that $y(t) > 0$. This completes the proof. □

Corollary 2.2. *Replace condition* (2.6) *in Theorem 2.1 by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A_{t_0}(p;t) = \infty, \tag{2.17}$$

and assume that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A_{t_0}(g^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t) < \infty. \tag{2.18}$$

Then conclusion of Theorem 2.1 holds.

It is clear that (2.17) is a necessary condition for (2.6) to hold. In case (2.6) fails to satisfied, then the following theorem may be applicable.

Theorem 2.3. *Let ρ, H and h be as in Theorem 2.1. Suppose that there exists $\phi_1, \phi_2 \in C([t_0,\infty),\mathbb{R})$ and for all $T \ge t_0$, $M > 0$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} A_T(p;t) \ge \phi_1(T) \tag{2.19}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} A_T(g^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t) \le \phi_2(T), \tag{2.20}$$

where ϕ_1 and ϕ_2 satisfy

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} A_T(g\rho^{-(\alpha+1)/\alpha}(\phi_1 - \theta\phi_2)_+^{(\alpha+1)/\alpha};t) = \infty, \tag{2.21}$$

where θ and g are the same as in Theorem 2.1. Then (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a non-oscillatory solution of (1.1), say $y(t) > 0$ for $t \ge t_0$, and let $W(t)$ be as defined in the proof of Theorem 2.1 for all $t \ge t_0$, we get (2.8). As in the proof of Theorem 2.1, we consider three cases of the behavior of $y'(t)$.

Case 1. $y'(t)$ is oscillatory. Proceeding as the proof of Theorem 2.1 (Case 1), (2.13) and (2.14) hold. Then by (2.14), we have, for all $T \ge t_0$,

$$\frac{1}{H(t,T)} A_T(p;t) - \frac{\theta}{H(t,T)} A_T(g^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t) \le \rho(T)W(T).$$

Taking limsup in above inequality as $t \to \infty$ and applying (2.19) and (2.20), we obtain

$$\phi_1(T) - \theta\phi_2(T) \le \rho(T)W(T),$$

from which it follows that

$$\frac{1}{H(t,T)} A_T\big(g\rho^{-(\alpha+1)/\alpha}(\phi_1 - \theta\phi_2)_+^{(\alpha+1)/\alpha};t\big) \le \frac{1}{H(t,T)} A_T\big(g|W|^{(\alpha+1)/\alpha};t\big). \tag{2.22}$$

On the other hand, by (2.13), we have

$$\begin{aligned}
&\frac{\alpha}{H(t,T)} A_T(g|W|^{(\alpha+1)/\alpha};t) - \frac{1}{H(t,T)} A_T(\rho^{-1}|h||W|;t)\\
&\le \rho(T)W(T) - \frac{1}{H(t,T)} A_T(p;t).
\end{aligned}$$

Thus, by (2.19),

$$
\begin{aligned}
&\liminf_{t\to\infty}\Big\{\frac{\alpha}{H(t,T)}A_T(g|W|^{(\alpha+1)/\alpha};t)-\frac{1}{H(t,T)}A_T(\rho^{-1}|h||W|;t)\Big\}\\
&\le \rho(T)W(T)-\phi_1(T)\le C_0.
\end{aligned}\tag{2.23}
$$

where C_0 is a constant. Now, we claim that

$$
\liminf_{t\to\infty}\frac{1}{H(t,T)}A_T\big(g|W|^{(\alpha+1)/\alpha};t\big)<\infty. \tag{2.24}
$$

If this inequality does not hold, then there exists a sequence $\{t_j\}_{j=1}^{\infty}\in[t_0,\infty)$ with $\lim_{j\to\infty}t_j=\infty$ such that

$$
\liminf_{j\to\infty}\frac{1}{H(t_j,T)}A_T\big(g|W|^{(\alpha+1)/\alpha};t_j\big)=\infty. \tag{2.25}
$$

Hence, by (2.23), for j large enough, we have

$$
\frac{\alpha}{H(t_j,T)}A_T\big(g|W|^{(\alpha+1)/\alpha};t_j\big)-\frac{1}{H(t_j,T)}A_T(\rho^{-1}|h||W|;t_j)\le C_0+1.
$$

This and (2.25) give, for j large enough,

$$
\frac{A_T(\rho^{-1}|h||W|;t_j)}{A_T(g|W|^{(\alpha+1)/\alpha};t_j)}-\alpha\ge-\frac{\alpha}{2},
$$

that is

$$
A_T(\rho^{-1}|h||W|;t_j)\ge\frac{\alpha}{2}A_T(g|W|^{(\alpha+1)/\alpha};t_j),\quad\text{for all large } j. \tag{2.26}
$$

By the Hölder inequality

$$
\begin{aligned}
&A_T(\rho^{-1}|h||W|;t_j)\\
&\le\big[A_T(g|W|^{(\alpha+1)/\alpha};t_j)\big]^{\alpha/(\alpha+1)}\big[A_T(g^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t_j)\big]^{1/(\alpha+1)}.
\end{aligned}\tag{2.27}
$$

From (2.26) and (2.27), we obtain

$$
\frac{1}{H(t_j,T)}A_T\big(g^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t_j\big)\ge\big(\frac{\alpha}{2}\big)^{\alpha+1}\frac{1}{H(t_j,T)}A_T(g|W|^{(\alpha+1)/\alpha};t_j). \tag{2.28}
$$

By (2.20), the left-hand side of (2.28) is bounded, which contradicts (2.25). Therefore, (2.24) holds. Hence by (2.22),

$$
\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{H(t,T)}A_T(g\rho^{-(\alpha+1)/\alpha}(\phi_1-\theta\phi_2)_+^{(\alpha+1)/\alpha};t)\\
&\le\liminf_{t\to\infty}\frac{1}{H(t,T)}A_T(g|W|^{(\alpha+1)/\alpha};t)<\infty,
\end{aligned}
$$

which contradicts (2.21).

Case 2. $y'(t)>0$ on $[T,\infty)$ for some $T\ge t_0$. In this case, from (2.9), it follows (2.10) holds for $t\ge T$. Once again, we can compete the proof by the procedure of the proof of Case 1.

Case 3. $y'(t)<0$ on $[T,\infty)$ for some $T\ge t_0$. The proof is exactly the same as for the same case in Theorem 2.1, and hence is omitted. $\square$

Remark 2.4. It is easy to check that condition (A2) can be replaced by

$$\liminf_{t\to\infty}\int_{t_0}^{t} p(s)ds > -\infty,$$

and still the conclusion of Theorems 2.1 and 2.2 hold.

Remark 2.5. The results in this paper are presented in a form with a high degree of generality, and thus they give many possibilities for oscillation criteria with an appropriate choice of functions H and ρ, we omit the details.

3. Examples

In this section, we provide two examples to illustrate the results obtained in this paper. Note that criteria reported in the references do not apply to these equations. For simplicity in these two examples, we take

$$H(t,s) = (t-s)^2, \quad \rho(t) = 1,$$

then

$$h(t,s) = \frac{2}{t-s}.$$

Example 3.1. Consider the quasilinear differential equation

$$(t^{-\nu}|y'(t)|^{\alpha-1}y'(t))' + t^{\lambda-1}(\lambda(2-\sin t) - t\cos t)|y(t)|^{\beta-1}y(t) = 0, \tag{3.1}$$

for $t \ge t_0 > 0$, where $\nu, \lambda, \alpha, \beta$ are arbitrary positive constants with $\alpha \ne \beta$, $\alpha \ne 2$, and for any $M > 0$

$$g(t) = \frac{\beta M(\nu+\alpha)}{\alpha^2} t^{\nu/\alpha}\big[t^{(\nu+\alpha)/\alpha} - t_0^{(\nu+\alpha)/\alpha}\big]^{-1}.$$

Then, for any $t \ge t_0$, we have

$$\int_{t_0}^{t} p(s)ds = \int_{t_0}^{t} d[s^\lambda(2-\sin s)] = t^\lambda(2-\sin t) - k_1 \ge t^\lambda - k_1,$$

where $k_1 = t_0^\lambda(2-\sin t_0)$. Moreover

$$\begin{aligned}
&\frac{1}{H(t,t_0)} A_{t_0}(p - \theta g^{-\alpha}\rho^{-(\alpha+1)}|h|^{\alpha+1};t)\\
&= \frac{1}{(t-t_0)^2}\int_{t_0}^{t}\big\{(t-s)^2 p(s) - k_2\theta(t-s)^{1-\alpha}s^{-\nu}\big[s^{(\nu+\alpha)/\alpha} - t_0^{(\nu+\alpha)/\alpha}\big]^\alpha\big\}ds\\
&= \frac{2}{(t-t_0)^2}\int_{t_0}^{t}(t-s)\int_{t_0}^{s} p(\tau)d\tau ds - \frac{k_2\theta}{(t-t_0)^2}\int_{t_0}^{t}(t-s)^{1-\alpha}s^{-\nu}\\
&\quad \big[s^{(\nu+\alpha)/\alpha} - t_0^{(\nu+\alpha)/\alpha}\big]^\alpha ds\\
&\ge \frac{2}{(t-t_0)^2}\int_{t_0}^{t}(t-s)(s^\lambda - k_1)ds - \frac{k_2\theta}{(t-t_0)^2}\int_{t_0}^{t}(t-s)^{1-\alpha}s^\alpha ds\\
&\ge \frac{2}{(t-t_0)^2}\Big[\frac{t^{\lambda+2}}{(\lambda+1)(\lambda+2)} - \frac{t\,t_0^{\lambda+1}}{\lambda+1} + \frac{t_0^{\lambda+2}}{\lambda+2} - \frac{k_1t^2}{2} + k_1 t\,t_0 + \frac{k_1t_0^2}{2}\Big]\\
&\quad - \frac{k_2\theta t^2}{(2-\alpha)(t-t_0)^2}(1-\frac{t_0}{t})^{2-\alpha}.
\end{aligned}$$

where $k_2 = 2^{\alpha+1}\big(\frac{\alpha^2}{\beta M(\nu+\alpha)}\big)^\alpha$. Consequently, (2.6) is satisfied. Hence, (3.1) is oscillatory by Theorem 2.1.

Example 3.2. Consider the quasilinear differential equation

$$(t^{\nu}|y'(t)|^{\alpha-1}y'(t))' + (t^{\lambda}\cos t)|y(t)|^{\beta-1}y(t) = 0, \tag{3.2}$$

for $t \ge t_0 > 0$, where $\nu, \lambda, \alpha, \beta$ are arbitrary constants with $\lambda \le 0$, $0 < \alpha < 2$, $\beta > 0$, $\alpha \ne \beta$, $\nu < \alpha$, and for any $M > 0$

$$g(t) = \frac{\beta M(\alpha-\nu)}{\alpha^2} t^{-\nu/\alpha}\big[t^{(\alpha-\nu)/\alpha} - t_0^{(\alpha-\nu)/\alpha}\big]^{-1}.$$

Moveover, for $t > s \ge T \ge t_0$, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{(t-T)^2}\int_T^t (t-s)^2 g^{-\alpha}(s)|h(t,s)|^{\alpha+1}ds\\
&= k_3 \limsup_{t\to\infty} \frac{1}{(t-T)^2}\int_T^t (t-s)^{1-\alpha}s^{\nu}\big[s^{(\alpha-\nu)/\alpha} - t_0^{(\alpha-\nu)/\alpha}\big]^{\alpha}ds\\
&\le k_3 \limsup_{t\to\infty} \frac{1}{(t-T)^2}\int_T^t (t-s)^{1-\alpha}s^{\alpha}ds\\
&\le \frac{k_3}{2-\alpha}\limsup_{t\to\infty}\frac{t^{\alpha}}{(t-T)^{\alpha}} = \frac{k_3}{2-\alpha},
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{(t-T)^2}\int_T^t (t-s)^2 p(s)ds &= \limsup_{t\to\infty} \frac{1}{(t-T)^2}\int_T^t (t-s)^2 s^{\lambda}\cos s\,ds\\
&\ge -T^{\lambda}\sin T,
\end{aligned}$$

where $k_3 = 2^{\alpha+1}\big(\frac{\alpha^2}{\beta M(\alpha-\nu)}\big)^{\alpha}$. Let

$$\phi(s) = \phi_1(s) - \theta\phi_2(s) = -s^{\lambda}\sin s - \varepsilon,$$

where $\varepsilon = \theta k_3/(2-\alpha)$. Consider an integer N such that $2N\pi + \frac{5}{4}\pi \ge (1+\sqrt{2}\varepsilon)^{1/\lambda}$. Then, for all integers $n \ge N$, we have

$$\phi(s) \ge \frac{1}{\sqrt{2}}, \quad \forall s \in [2n\pi + \frac{5}{4}\pi, 2n\pi + \frac{11}{8}\pi],$$

which implies

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{(t-T)^2}\int_T^t (t-s)^2 g(s)(\phi_1(s) - \theta\phi_2(s))_+^{(\alpha+1)/\alpha}ds\\
&\ge \frac{k_4}{(t-T)^2}\sum_{n=N}^{\infty}\int_{2n\pi+\frac{5}{4}\pi}^{2n\pi+\frac{11}{8}\pi}(t-s)^2 s^{-\nu/\alpha}[s^{(\alpha-\nu)/\alpha} - t_0^{(\alpha-\nu)/\alpha}]^{-1}ds\\
&\ge k_4\sum_{n=N}^{\infty}\int_{2n\pi+\frac{5}{4}\pi}^{2n\pi+\frac{11}{8}\pi} s^{-1}ds = \infty,
\end{aligned}$$

where $k_4 = \frac{\beta M(\alpha-\nu)}{\alpha^2}(\frac{1}{\sqrt{2}})^{(\alpha+1)/\alpha}$. Hence, by Theorem 3.2, (3.2) is oscillatory.

Acknowledgement. The authors are grateful to the referee for her/his suggestions and comments on the original manuscript.

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