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LIE-GROUP ANALYSIS OF RADIATIVE AND MAGNETIC FIELD EFFECTS ON FREE CONVECTION AND MASS TRANSFER FLOW PAST A SEMI-INFINITE VERTICAL FLAT PLATE

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Abstract. In this paper, we discuss similarity reductions for problems of radiative and magnetic field effects on free-convection and mass-transfer flow past a semi-infinite vertical flat plate. Two cases are considered: Lie group analysis applied to uniform magnetic fields, and Scaling transformations applied to non-uniform magnetic fields. In particular, we determine new similarity reductions and find an analytical solution for the uniform magnetic field, by using Lie group method. Numerical results are presented and discussed for various values of the parameters governing the problem.

1. Introduction

The study of radiative and magnetic field effects have important applications in physics and engineering. The classical method for finding similarity reduction of PDEs is the Lie-group method of infinitesimal transformations [2, 3, 13, 14, 15, 16, 24, 25, 29]. Lakshmanan and Velan [20] studied Lie similarity reductions of certain (2+1)-dimensional nonlinear evolution equations. Group analysis of the Von Karman-Howarth equation was presented by Khabirov and Unal [19]. Clarkson [5] presented a new similarity reduction and Painleve analysis for the symmetric regularized long wave and modified Benjamin-Bona-Mahoney equations. Yurusoy and Pakdemirli [30] studied the group classification of the boundary layer equations of a non-Newtonian fluid model, in which the shear stress is arbitrary function of the velocity gradient. They used two different approaches for group classification (i) the classical approach and (ii) equivalence transformations. Clarkson and Kruskal [6] presented some new similarity reductions of the Boussinesq equation, which arises in several physical applications including shallow water waves. A new solution branch of similarity solutions were presented and discussed by Steinruck [27]. Ibragimov [16] discussed the properties of a perturbed nonlinear wave equation by using a group of transformations and derived the principal Lie algebra and its approximate equivalence transformation. The paper of Chupakhin [4] reviewed the main statements of the theory of differential invariants of continuous groups. Yurusoy [31] presented similarity solutions for the problem of the two-dimensional

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equations of motions for the slowly flowing second grade fluid with heat transfer. Soh [26] classified similarity solutions of a boundary-value problem for a nonlinear diffusion equation arising in the study of a charged power-law non-Newtonian fluid through a time-dependent transverse magnetic field. Ibragimov and *et al* [17] found for the equations the equivalence group generated by an infinitesimal Lie algebra involving two arbitrary functions of the variable x. In the paper of Gandarias *et al* [11], the complete Lie group classification of a non-linear wave equation was obtained. Fakhar *et al* [10] employed Lie theory on the axisymmetric flow. Radiative effects on magnetohydrodynamic natural convection flows saturated in porous media were studied by Mansour and El-Shaer [21]. Elbashbeshy and Dimian [9] studied effect of radiation on the flow and heat transfer over a wedge with variable viscosity. Abo-Eldahab and El Gendy [1] studied radiation effect on convective heat transfer in an electrically conducting fluid at a stretching surface with variable viscosity and uniform free stream. Radiation and mass transfer effects on flow of an incompressible viscous fluid past a moving vertical cylinder were presented by Ganesan and Loganathan [12]. The unsteady flow past a moving plate in the presence of free convection and radiation were studied by Mansour [22]. The effect of suction/injection on the flow and heat transfer for a continuous moving plate in a micropolar fluid in the presence of rendition was studied by El-Arabawy [8]. Ibrahim *et al* [18] studied radiative and thermal dispersion effects on non-Darcy natural convection with lateral mass flux for a non-Newtonian fluid from a vertical flat plate in a saturated porous medium. Dolapc and Pakdemirli [7] studied approximate symmetries of creeping flow equations of a second grade fluid. The purpose of this paper is to investigate the similarity reductions and to find similarity representations of radiative and magnetic field effects on free convection and mass transfer flow past a semi-infinite vertical flat plate, to study the effect of problem's parameters on the behavior of problem's variables and to represent exact solutions of the problem.

2. Analysis

We consider the flow along the x-axis, which is taken along the vertical flat plate in the upward direction, and the y-axis is taken normal to it as shown in Fig. 1. The plane is maintained at a constant temperature T_w higher than the constant temperature T_∞ of the surrounding fluid and the concentration C_w bigger than the constant concentration C_∞. The fluid properties are assumed to be constant. The governing equations for the problem under consideration can be written as

$$
\begin{gathered}
\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0\,, \\
\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \nu \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + g\beta(T - T_\infty) + g\beta^*(C - C_\infty) - \frac{\sigma B_{\bar{y}}^2}{\rho}\bar{u}\,, \\
\bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \frac{k_0}{\rho c_p}\frac{\partial^2 T}{\partial \bar{y}^2} - \frac{\alpha}{k_0}\frac{\partial q_r}{\partial \bar{y}}\,, \\
\bar{u}\frac{\partial C}{\partial \bar{x}} + \bar{v}\frac{\partial C}{\partial \bar{y}} = D\frac{\partial^2 C}{\partial \bar{y}^2}\,,
\end{gathered} \tag{2.1}
$$

with the boundary conditions

$$\begin{gathered}\bar{u}=0, \quad \bar{v}=0, \quad T=T_w, \quad C=C_w, \quad \text{as } \bar{y}=0, \\ \bar{u}=0, \quad T=T_\infty, \quad C=C_\infty, \quad \text{as } \bar{y}\to\infty.\end{gathered} \tag{2.2}$$

where, $\bar{u}$ and $\bar{v}$ are velocity components; $\bar{x}$ and $\bar{y}$ are space coordinates;T is the temperature; ν is the kinematic viscosity of the fluid; g is the acceleration due to gravity; β is the coefficient of thermal expansion; β^* is the coefficient of expansion with concentration; σ is the electric conductivity; $B_{\bar{y}}$ is the magnetic field strength in y direction; ρ is the density of the fluid; c_p is the specific heat of the fluid; α is thermal diffusivity; k_o is the thermal conductivity of fluid; D is the diffusion coefficient and q_r is the local radiative heat flux. The radiative heat flux term is simplified by using the Rosseland approximation (see Sparrow Cess [28]),

$$q_r=-\frac{4\sigma_0}{3k^*}\frac{\partial T^4}{\partial\bar{y}}, \tag{2.3}$$

where, σ_0 and k^* are the Stefan-Boltzman constant and mean absorption coefficient respectively.

We assume that the temperature differences within the flow are sufficiently small such that T^4 may be expressed as a linear function of temperature. This is accomplished by expanding T^4 in a Taylor series about T_∞ and neglecting higher-order terms, thus

$$T^4\cong 4T_\infty^3T-3T_\infty^4 \tag{2.4}$$

the following dimensionless parameters are defined:

$$\begin{gathered}x=\frac{\bar{x}}{L}, \quad y=\frac{\bar{y}}{L}, \quad u=\bar{u}\log\nu, \\ v=\frac{\bar{v}\log}{\nu}, \quad \theta=T-T_\infty T_w-T_\infty, \quad \phi=\frac{C-C_\infty}{C_w-C_\infty}.\end{gathered} \tag{2.5}$$

now we will study two cases:

Case 1: Non-uniform magnetic field. In this case the magnetic field strength is

$$B_y=\frac{B_0}{x}. \tag{2.6}$$

then, by using the non-dimensional variables (2.5) and (2.3), (2.4), (2.6), the system (2.1) becomes

$$\begin{gathered}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \\ u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{\partial^2u}{\partial y^2}-G_r\theta-G_c\phi+\frac{M}{x^2}u=0\,, \\ u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}-\frac{1}{P_r}(1+4R)\frac{\partial^2\theta}{\partial y^2}=0\,, \\ u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}-\frac{1}{S_c}\frac{\partial^2\phi}{\partial y^2}=0\,,\end{gathered} \tag{2.7}$$

where, $M=\frac{\sigma B_0^2}{\rho\nu}$ is the magnetic parameter; $G_c=\frac{g\beta^*(C_w-C_\infty)L^3}{\nu^2}$ is the mass Grashof number; $G_r=\frac{g\beta^*(T_w-T_\infty)L^3}{\nu^2}$ is the temperature Grashof number; $P_r=$

$\frac{\rho\nu c_p}{k_0}$ is the Prandtl number; $R = \frac{4\sigma_0 T_\infty^3}{3k_0 k^*}$ is the radiation parameter and $S_c = \frac{\nu}{D}$ is the Schmidt number. The boundary conditions (2.2) become

$$\begin{gathered} u = 0, \quad v = 0, \quad \theta = 1, \quad \phi = 1, \quad \text{at } y = 0, \\ u = 0, \quad \theta = 0, \quad \phi = 0, \quad \text{as } y \to \infty. \end{gathered} \tag{2.8}$$

In this case, the similarity solutions are obtained using the scaling transformations. The magnetic force in x-direction is $F_x = -\sigma B_y^2 u$. The system (2.1) and conditions (2.8) are invariant under the scaling transformations. We scale all independent and dependent variables as follows:

$$\begin{gathered} x^* = \lambda^{c_1} x, \quad y^* = \lambda^{c_2} y, \quad u^* = \lambda^{c_3} u, \\ v^* = \lambda^{c_4} v, \quad \theta^* = \lambda^{c_5} \theta, \quad \phi^* = \lambda^{c_6} \phi. \end{gathered}$$

Substituting these variables in (2.7), we obtain the invariance conditions:

$$c_2 = c_1, \quad c_3 = -c_1, \quad c_4 = -c_1, \quad c_5 = -3c_1, \quad c_6 = -3c_1 \,.$$

These relations lead to the following differential equations (characteristic equations) for similarity:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{-u} = \frac{dv}{-v} = \frac{d\theta}{-3\theta} = \frac{d\phi}{-3\phi} \,.$$

From these equalities, we find the similarity transformations

$$\eta = \frac{y}{x}, \quad u = \frac{F_1(\eta)}{x}, \quad v = \frac{F_2(\eta)}{x}, \quad \theta = \frac{F_3(\eta)}{x^3}, \quad \phi = \frac{F_4(\eta)}{x^3} \,.$$

Substituting these values in (2.1), we obtain

$$\begin{gathered} F_2' - \eta F_1' - F_1 = 0, \\ F_1'' - (F_2 - \eta F_1)F_1' + F_1^2 + G_r F_3 + G_c F_4 - M F_1 = 0, \\ \frac{1}{P_r}(1 + 4R)F_3'' - (F_2 - \eta F_1)F_3' + 3F_1 F_3 = 0, \\ \frac{1}{S_c} F_4'' - (F_2 - \eta F_1)F_4' + 3F_1 F_4 = 0, \end{gathered} \tag{2.9}$$

with boundary conditions

$$\begin{gathered} F_1 = 0, \quad F_2 = 0, \quad F_3 = 1, \quad F_4 = 1, \quad \text{at } \eta = 0 \\ F_1 = 0, \quad F_3 = 0, \quad F_4 = 0, \quad \text{as } \eta \to 0 \end{gathered} \tag{2.10}$$

Integrating the first equation in (2.9), we obtain $F_2 = \eta F_1$. Then, system (2.9) becomes

$$\begin{gathered} F_1'' + F_1^2 + G_r F_3 + G_c F_4 - M F_1 = 0, \\ \frac{1}{P_r}(1 + 4R)F_3'' + 3F_1 F_3 = 0, \\ \frac{1}{S_c} F_4'' + 3F_1 F_4 = 0, \end{gathered} \tag{2.11}$$

with boundary conditions (2.10).

Case 2: Uniform magnetic field. In this case the magnetic field strength is constant ($By = B_0$). Symmetry group and similarity solutions for the derived fundamental equations are obtained by employing Lie Group analysis. Lie algebra corresponding to the symmetries are constructed. After that, a special solution of the exact solution is obtained for a special symmetry. Then (2.7) becomes

$$\begin{gathered}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,,\\
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{\partial^2 u}{\partial y^2}-G_r\theta-G_c\phi+Mu=0\,,\\
u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}-\frac{1}{P_r}(1+4R)\frac{\partial^2 \theta}{\partial y^2}=0\,,\\
u\frac{\partial \phi}{\partial x}+v\frac{\partial \phi}{\partial y}-\frac{1}{S_c}\frac{\partial^2 \phi}{\partial y^2}=0\,,
\end{gathered}\tag{2.12}$$

These reduced equations, in two independent variables, can be further analyzed for its symmetry properties by looking at its own invariance property under the classical Lie group analysis. We introduce the vector field

$$\begin{aligned}
X=&\xi_1(x,y,u,v,\theta,\phi)\frac{\partial}{\partial x}+\xi_2(x,y,u,v,\theta,\phi)\frac{\partial}{\partial y}+\mu^1(x,y,u,v,\theta,\phi)\frac{\partial}{\partial u}\\
&+\mu^2(x,y,u,v,\theta,\phi)\frac{\partial}{\partial v}+\mu^3(x,y,u,v,\theta,\phi)\frac{\partial}{\partial \theta}+\mu^4(x,y,u,v,\theta,\phi)\frac{\partial}{\partial \phi}\,.
\end{aligned}\tag{2.13}$$

At this point, we assume that

$$\begin{gathered}
\Delta_1=u_1+v_2,\\
\Delta_2=uu_1+vu_2-u_{22}-G_r\theta-G_c\phi+Mu,\\
\Delta_3=u\theta_1+v\theta_2-\frac{1}{P_r}(1+4R)\theta_{22},\\
\Delta_4=u\phi_1+v\phi_2-\frac{1}{S_c}\phi_{22}.
\end{gathered}\tag{2.14}$$

To determine the infinitesimals ξ_1, ξ_2, μ^1, μ^2, μ^3, μ^4, the second prolongation of the operator (2.13) is applied to (2.14) and then substituted to the invariance criterion, i.e.,

$$X^{(2)}(\Delta_j)|_{\Delta_j=0}=0,\quad j=1,2,3,4\tag{2.15}$$

where $X^{(2)}$ stands for the second prolongation of the operator (2.13), which is defined by:

$$\begin{aligned}
X^{(2)}=&X+\mu_1^{(1)1}\frac{\partial}{\partial u_1}+\mu_2^{(1)1}\frac{\partial}{\partial u_2}+\mu_{11}^{(2)1}\frac{\partial}{\partial u_{11}}+\mu_{12}^{(2)1}\frac{\partial}{\partial u_{12}}+\mu_{22}^{(2)1}\frac{\partial}{\partial u_{22}}\\
&+\mu_1^{(1)2}\frac{\partial}{\partial v_1}+\mu_2^{(1)2}\frac{\partial}{\partial v_2}+\mu_{11}^{(2)2}\frac{\partial}{\partial v_{11}}+\mu_{12}^{(2)2}\frac{\partial}{\partial v_{12}}+\mu_{22}^{(2)2}\frac{\partial}{\partial v_{22}}\\
&+\mu_1^{(1)3}\frac{\partial}{\partial \theta_1}+\mu_2^{(1)3}\frac{\partial}{\partial \theta_2}+\mu_{11}^{(2)3}\frac{\partial}{\partial \theta_{11}}+\mu_{12}^{(2)3}\frac{\partial}{\partial \theta_{12}}+\mu_{22}^{(2)3}\frac{\partial}{\partial \theta_{22}}\\
&+\mu_1^{(1)4}\frac{\partial}{\partial \phi_1}+\mu_2^{(1)4}\frac{\partial}{\partial \phi_2}+\mu_{11}^{(2)4}\frac{\partial}{\partial \phi_{11}}+\mu_{12}^{(2)4}\frac{\partial}{\partial \phi_{12}}+\mu_{22}^{(2)4}\frac{\partial}{\partial \phi_{22}}.
\end{aligned}$$

The recursion relation for the higher order infinitesimals are

$$\mu_\alpha^{(m)k} = D_\alpha[\mu^m - \sum_{i=1}^{2} \xi_i \chi_i^m] + \sum_{i=1}^{2} \xi_i \chi_{\alpha i}^m$$

where, $(m = 1, 2, 3, 4)$, $(\alpha = 1, 2, 11, 12, 22)$, and χ^1, χ^2, χ^3, χ^4 stand for u, v, θ, ϕ respectively,

$$\chi_i^m = \frac{\partial \chi^m}{\partial x_i}, \quad \chi_{\alpha i}^m = \frac{\partial \chi_\alpha^m}{\partial x_i}, \quad i = 1, 2$$
$$x_1 = x, \quad x_2 = y,$$
$$D_\alpha = \frac{D}{D_\alpha} = \frac{\partial}{\partial x_\alpha} + \chi_i^m \frac{\partial}{\partial \chi^m} + \chi_{ij}^m \frac{\partial}{\partial \chi_j^m}. \quad i, j = 1, 2$$

Equations (2.15) furnish a set of constraints in the form of linear partial differential equations, which enable us to obtain the coefficients ξ_1, ξ_2, μ^1, μ^2, μ^3, μ^4. The system of linear partial differential equations is

$$\begin{gathered}
\mu_1^{(1)1} + \mu_2^{(1)2} = 0, \\
(M + u_1)\mu^1 + u_2\mu^2 - G_r\mu^3 - G_c\mu^4 + u\mu_1^{(1)1} + v\mu_2^{(1)1} - \mu_{22}^{(2)1} = 0, \\
\theta_1\mu^1 + \theta_2\mu^2 + u\mu_1^{(1)3} + v\mu_2^{(1)3} - \frac{1}{P_r}(1 + 4R)\mu_{22}^{(2)3} = 0, \\
\phi_1\mu^1 + \phi_2\mu^2 + u\mu_1^{(1)4} + v\mu_2^{(1)4} - \frac{1}{S_c}\mu_{22}^{(2)4} = 0.
\end{gathered} \tag{2.16}$$

where

$$\begin{gathered}
\mu_1^{(1)1} = \frac{\partial \mu^1}{\partial x_1} + u_1 \frac{\partial \mu^1}{\partial u} - (u_1 \frac{\partial \xi_1}{\partial x_1} + u_2 \frac{\partial \xi_2}{\partial x_1}), \\
\mu_2^{(1)1} = \frac{\partial \mu^1}{\partial x_2} + u_2 \frac{\partial \mu^1}{\partial u} - (u_1 \frac{\partial \xi_1}{\partial x_2} + u_2 \frac{\partial \xi_2}{\partial x_2}), \\
\mu_2^{(1)2} = \frac{\partial \mu^2}{\partial x_2} + v_2 \frac{\partial \mu^2}{\partial v} - (v_1 \frac{\partial \xi_1}{\partial x_2} + v_2 \frac{\partial \xi_2}{\partial x_2}), \\
\mu_1^{(1)3} = \frac{\partial \mu^3}{\partial x_1} + \theta_1 \frac{\partial \mu^3}{\partial \theta} - (\theta_1 \frac{\partial \xi_1}{\partial x_1} + \theta_2 \frac{\partial \xi_2}{\partial x_1}), \\
\mu_2^{(1)3} = \frac{\partial \mu^3}{\partial x_2} + \theta_2 \frac{\partial \mu^3}{\partial \theta} - (\theta_1 \frac{\partial \xi_1}{\partial x_2} + \theta_2 \frac{\partial \xi_2}{\partial x_2}), \\
\mu_1^{(1)4} = \frac{\partial \mu^4}{\partial x_1} + \phi_1 \frac{\partial \mu^4}{\partial \phi} - (\phi_1 \frac{\partial \xi_1}{\partial x_1} + \phi_2 \frac{\partial \xi_2}{\partial x_1}), \\
\mu_2^{(1)4} = \frac{\partial \mu^4}{\partial x_2} + \phi_2 \frac{\partial \mu^4}{\partial \phi} - (\phi_1 \frac{\partial \xi_1}{\partial x_2} + \phi_2 \frac{\partial \xi_2}{\partial x_2}), \\
\mu_{22}^{(2)1} = \frac{\partial \mu_2^{(1)1}}{\partial x_2} + u_2 \frac{\partial \mu_2^{(1)1}}{\partial u} + u_{21} \frac{\partial \mu_2^{(1)1}}{\partial u_1} + u_{22} \frac{\partial \mu_2^{(1)1}}{\partial u_2} - (u_{21} \frac{\partial \xi_1}{\partial x_2} + u_{22} \frac{\partial \xi_2}{\partial x_2}), \\
\mu_{22}^{(2)3} = \frac{\partial \mu_2^{(1)3}}{\partial x_2} + \theta_2 \frac{\partial \mu_2^{(1)3}}{\partial \theta} + \theta_{21} \frac{\partial \mu_2^{(1)3}}{\partial \theta_1} + \theta_{22} \frac{\partial \mu_2^{(1)3}}{\partial \theta_2} - (\theta_{21} \frac{\partial \xi_1}{\partial x_2} + \theta_{22} \frac{\partial \xi_2}{\partial x_2}), \\
\mu_{22}^{(2)4} = \frac{\partial \mu_2^{(1)4}}{\partial x_2} + \phi_2 \frac{\partial \mu_2^{(1)4}}{\partial \phi} + \phi_{21} \frac{\partial \mu_2^{(1)4}}{\partial \phi_1} + \phi_{22} \frac{\partial \mu_2^{(1)4}}{\partial \phi_2} - (\phi_{21} \frac{\partial \xi_1}{\partial x_2} + \phi_{22} \frac{\partial \xi_2}{\partial x_2}).
\end{gathered}$$

Substituting the above expressions in (2.16), we obtain

$$\begin{aligned}
\xi_1 &= 2C_1x + C_2y + C_3u + C_4v + h_1(\theta,\phi),\\
\xi_2 &= C_5x + C_6y + C_7v + h_2(u,\theta,\phi),\\
\mu^1 &= C_1u + C_2v + C_8,\\
\mu^2 &= C_5u + C_9v + C_{10},\\
\mu^3 &= C_1\theta + h_3(x,y,u,v,\phi),\\
\mu^4 &= C_1\phi + h_4(x,y,u,v,\phi).
\end{aligned}$$

3. Similarity generators and group-invariant solution

We will consider the following special case: Let $\xi_1 = \xi_1(x)$, $\xi_2 = \xi_2(y)$, $\mu^1 = \mu^1(u)$, $\mu^2 = \mu^2(v)$, $\mu^3 = \mu^3(\theta)$ and $\mu^4 = \mu^4(\phi)$, then we obtain

$$\begin{gathered}
\xi_1 = 2A_1x + A_2, \quad \xi_2 = A_3y + A_4, \quad \mu^1 = A_1u + A_5,\\
\mu^2 = A_6v + A_7, \quad \mu^3 = A_1\theta + A_8, \quad \mu^4 = A_1\phi + A_9.
\end{gathered}$$

where $A_1, A_2, \dots, A_9$ are constants. Then for this special case we can find the infinitesimal generators $X_1, X_2, \dots, X_9$ as follows:

$$\begin{gathered}
X_1 = 2x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + \theta\frac{\partial}{\partial \theta} + \phi\frac{\partial}{\partial \phi}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = y\frac{\partial}{\partial y},\\
X_4 = \frac{\partial}{\partial y}, \quad X_5 = \frac{\partial}{\partial u}, \quad X_6 = v\frac{\partial}{\partial v}, \quad X_7 = \frac{\partial}{\partial v}, \quad X_8 = \frac{\partial}{\partial \theta}, \quad X_9 = \frac{\partial}{\partial \phi}.
\end{gathered} \tag{3.1}$$

Commutator relations between these generators have been calculated according to the formula

$$[X_i, X_j] = X_i(X_j) - X_j(X_i)$$

and the results of such calculations are shown in Table 1.

Table 1. Commutator Table

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
X_1	0	$-2X_2$	0	0	$-X_5$	0	0	$-X_8$	$-X_9$
X_2	$2X_2$	0	0	0	0	0	0	0	0
X_3	0	0	0	$-X_4$	0	0	0	0	0
X_4	0	0	X_4	0	0	0	0	0	0
X_5	X_5	0	0	0	0	0	0	0	0
X_6	0	0	0	0	0	0	$-X_7$	0	0
X_7	0	0	0	0	0	X_7	0	0	0
X_8	X_8	0	0	0	0	0	0	0	0
X_9	X_9	0	0	0	0	0	0	0	0

From Table 1, it can be seen that the commutator relations of the operator (X_4 with X_3), (X_5, X_8, X_9 with X_1) and (X_7 with X_6) result again in itself. Therefore, it is the ideal of the equivalence algebra L. The commutator relations between X_2 with X_1 involves only X_2. Therefore, it is the sub algebra of L.

The one-parameter local groups G_i, $(i = 1, 2, , \dots, 9)$ associated with the generators (3.1) are obtained as follows:

$$\begin{gathered}
G_1 : (x, y, , u, v, \theta, \phi) \to (e^{2\varepsilon} x, y, e^{\varepsilon} u, v, e^{\varepsilon}\theta, e^{\varepsilon}\phi), \\
G_2 : (x, y, , u, v, \theta, \phi) \to (x + \varepsilon, y, u, v, \theta, \phi), \\
G_3 : (x, y, , u, v, \theta, \phi) \to (x, e^{\varepsilon} y, u, v, \theta, \phi), \\
G_4 : (x, y, , u, v, \theta, \phi) \to (x, y + \varepsilon, u, v, \theta, \phi), \\
G_5 : (x, y, , u, v, \theta, \phi) \to (x, y, u + \varepsilon, v, \theta, \phi), \\
G_6 : (x, y, , u, v, \theta, \phi) \to (x, y, u, e^{\varepsilon} v, \theta, \phi), \\
G_7 : (x, y, , u, v, \theta, \phi) \to (x, y, u, v + \varepsilon, \theta, \phi), \\
G_8 : (x, y, , u, v, \theta, \phi) \to (x, y, u, v, \theta + \varepsilon, \phi), \\
G_9 : (x, y, , u, v, \theta, \phi) \to (x, y, u, v, \theta, \phi + \varepsilon).
\end{gathered}$$

4. Reductions to ordinary differential equations and solutions

Now we look for the similarity solutions with respect to the generator

$$X_2 + X_4 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}. \tag{4.1}$$

The group representing translation symmetry for this generator is

$$G : (x, y, , u, v, \theta, \phi) \to (x + \varepsilon, y + \varepsilon, u, v, \theta, \phi), \tag{4.2}$$

where ε is the infinitesimal Lie group parameter. The similarity transformations of this group are

$$\eta = x - y, \quad u = F_1(\eta), \quad v = F_2(\eta), \quad \theta = F_3(\eta), \quad \phi = F_4(\eta). \tag{4.3}$$

Substituting these expression in (2.12), we obtain a system of non-linear ordinary differential equations:

$$\begin{gathered}
F_1' - F_2' = 0, \\
F_1'' + (F_2 - F_1)F_1' + G_r F_3 + G_c F_4 - M F_1 = 0, \\
\frac{1}{P_r}(1 + 4R)F_3'' + (F_2 - F_1)F_3' = 0, \\
\frac{1}{S_c}F_4'' + (F_2 - F_1)F_4' = 0.
\end{gathered}$$

Integration of this system, we obtain the special solutions of the equations (2.12) as follows:

$$\begin{gathered}
u = B_1 e^{-S_4\eta} + B_2 e^{-S_5\eta} + E_1 e^{-k_1\eta} + E_2 e^{-k_2\eta} + E_3, \\
v = u - B_5, \\
\theta = B_3 - \frac{B_4}{k_1} e^{-k_1\eta}, \\
\theta = B_6 - \frac{B_7}{k_2} e^{-k_2\eta}.
\end{gathered} \tag{4.4}$$

where, $B_1, B_2, \dots, B_7$ are constants of integrations, and

$$k_1 = \frac{a_1 P_r}{1+4R}, \quad k_2 = a_1 S_{c,}$$
$$S_1 = \frac{B_4 G_r}{k_1}, \quad S_2 = \frac{B_7 M}{k_2}, S_3 = B_3 G_r + B_6 M,$$
$$E_1 = \frac{64}{S_{18}}(S_6 - S_{10} - M^2 S_1), \quad E_2 = \frac{64}{S_{18}}(S_7 - S_{11} - M^2 S_2),$$
$$E_3 = \frac{64}{S_{18}}(S_8 + S_9 + S_{15} - S_{12} - S_{13} - S_{14} - M^2 S_3).$$

Here,

$$S_4 = \frac{1}{2}(a_1 + \sqrt{a_1^2 - 4M}, \quad S_5 = -\frac{1}{2}(-a_1 + \sqrt{a_1^2 - 4M},$$
$$S_6 = a_1 k_2 M S_1 \log(e), \quad S_7 = a_1 k_1 M S_2 \log(e),$$
$$S_8 = \frac{S_3 S_7}{S_2}, \quad S_9 = \frac{S_3 S_6}{S_1}, \quad S_{10} = k_2^2 M S_1 [\log(e)]^2,$$
$$S_{11} = k_1^2 M S_2 [\log(e)]^2, \quad S_{12} = a_1^2 k_1 k_2 S_3 [\log(e)]^2,$$
$$S_{13} = \frac{S_3 S_{11}}{S_2}, \quad S_{14} = \frac{S_3 S_{10}}{S_1},$$
$$S_{15} = a_1 k_1 k_2 S_3 [\log(e)]^3, \quad S_{16} = \frac{k_2 S_{15}}{k_1},$$
$$S_{17} = k_1^2 k_2^2 S_3 [\log(e)]^4, \quad S_{18} = -64 S_4 S_5 S_{19} S_{20},$$
$$S_{19} = [S_4 - k_1[\log(e)]][-S_5 + k_1[\log(e)]],$$
$$S_{20} = [S_4 - k_2[\log(e)]][-S_5 + k_2[\log(e)]].$$

5. Discussion

The system of equations (2.11) with the boundary conditions (2.10) are solved numerically by the Runge-Kutta method with Shooting Techniques. Results are obtained for various values of the parameters governing the problem. Figures 2-6 display the results for velocity, temperature and concentration profiles for different values of the parameters associated with the governing problem. We observe that the velocity increases as M and G_r increase as shown in Figs. 2(a) and 4(a). In Fig. 6(a), we note that at $R = 0$ the velocity curve is higher than the other velocity curves, and at $R \neq 0$ the velocity increases as R increases. It is noticed that the velocity profiles not change as the Prandtl number P_r changes (see Fig. 3(a)). Fig. 5(a) shows that the velocity profiles slowly decrease as G_c increases until maximum velocity then increase far away the plate. Figures 2(b), 3(b), 4(b), 5(b) and 6(b) show the temperature profiles, there is no change in the temperature with different values of G_c in fig. 5(b), but in fig. 6(b) for $R \neq 0$ the temperature increases as R decreases. Figures 2(c), 3(c), 4(c), 5(c) and 6(c) display the results of the concentration profiles, we observe that in Fig. 2(c) the concentration increases as M increases but in Fig. 6(c) it increases as R decreases.

Also, in this work we have presented similarity reductions, which are Lie point transformation, since the infinitesimals $\xi_1, \xi_2, \mu^1, \mu^2, \mu^3,\ \mu^4$ depend only on the independent variables x, y and the dependent variables $u, v, \theta,\ \phi$, but not on the derivatives of the dependent variables (if the transformations also depend on the derivative of the dependent variables, then the associated symmetries are known as Lie-Backlund symmetries).

Explicit solutions of the boundary layer equations (2.12) are presented in (4.4).

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Figures

Figure 1 shows the coordinate system and the flow model.
Figures 2a-2c show the effect of the magnetic parameter M on: (a) F_1, (b) F_3 and $(c)F_4$, with $P_r = 0.733$, $G_r = 0.1$, $G_c = 0.1$, $S_c = 0.1$ and $R = 0$.
Figures 3a-3c show the effect of Prandtl number P_r on: (a) F_1, (b) F_3 and (c) F_4, with $M = 1$, $G_r = 0.01$, $G_c = 0.01$, $S_c = 1$ and $R = 1$.
Figures 4a-4c show the effect of Grashof number G_r on: (a) F_1, (b) F_3 and (c) F_4, with $M = 1$, $P_r = .733$, $G_c = 0.01$, $S_c = 1$ and $R = 1$.
Figures 5a-5c show the effect of mass Grashof number G_c on: ((a) F_1, (b) F_3 and (c) F_4, with $M = 1, G_r = 0.01, P_r = 0.733, S_c = 1$ and $R = 1$.
Figures 6a-6c show the effect of Radiation parameter R on: (a) F_1, (b) F_3 and (c) F_4, with $M = 0.1, G_r = 0.1$, $G_c = 0.1$, $S_c = 0.1$ and $P_r = 0.733$.

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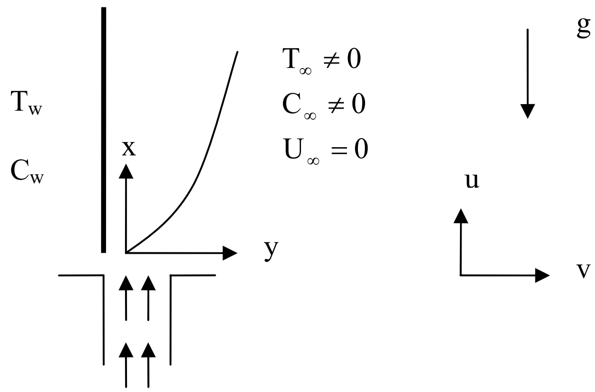
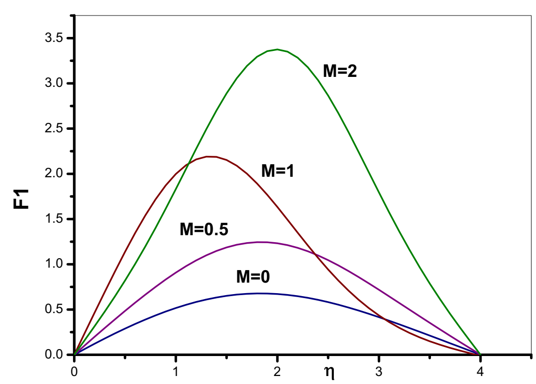


Fig1.: Coordinate system and flow model



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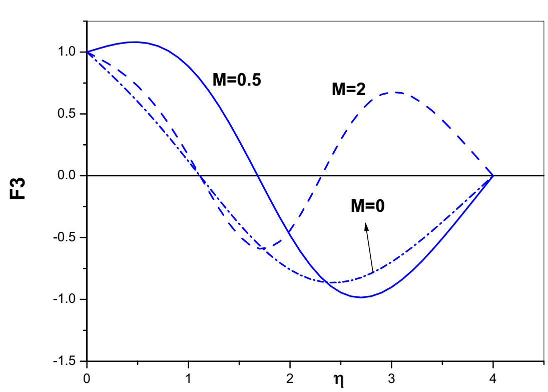

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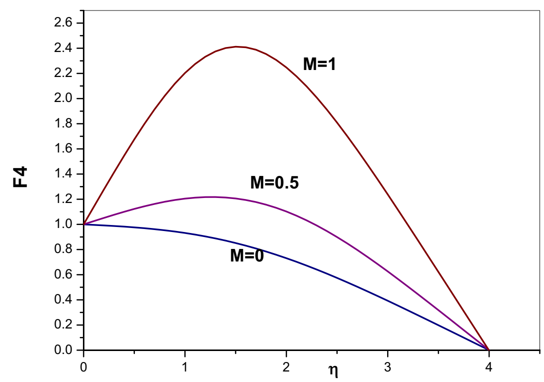

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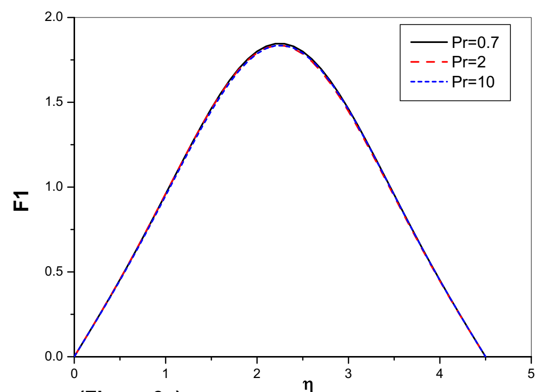

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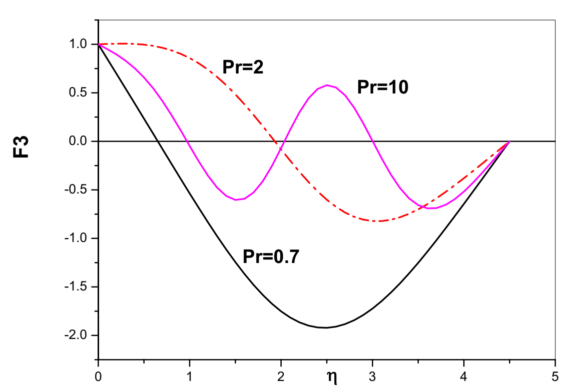

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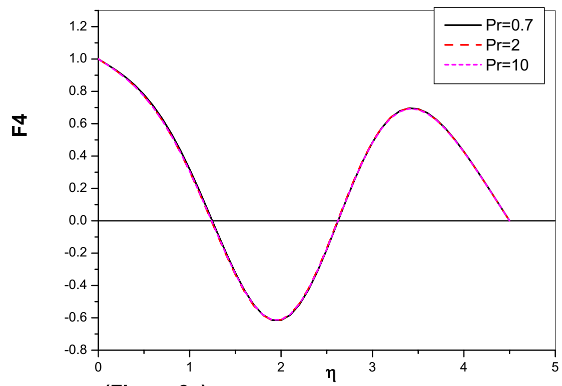

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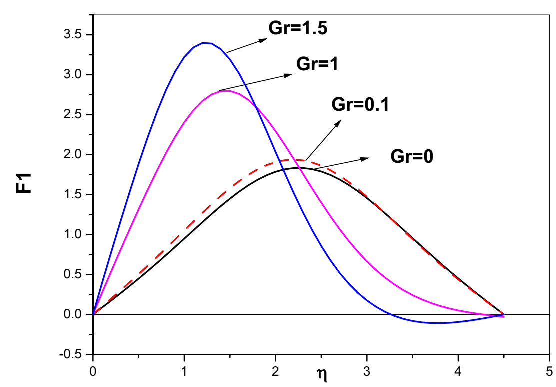

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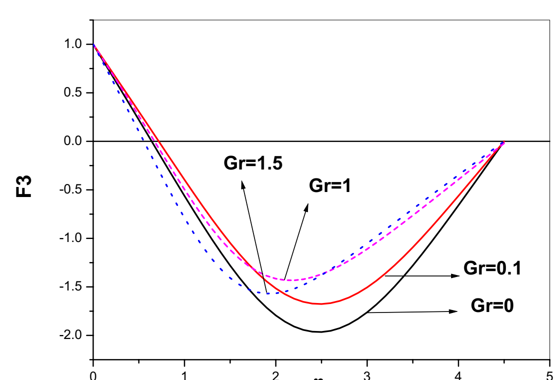

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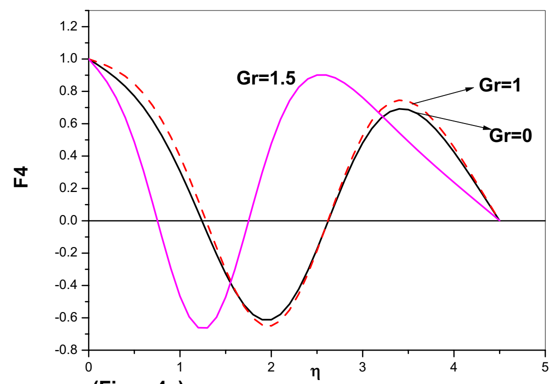

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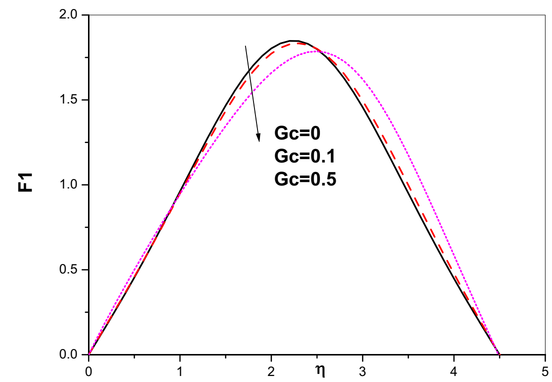

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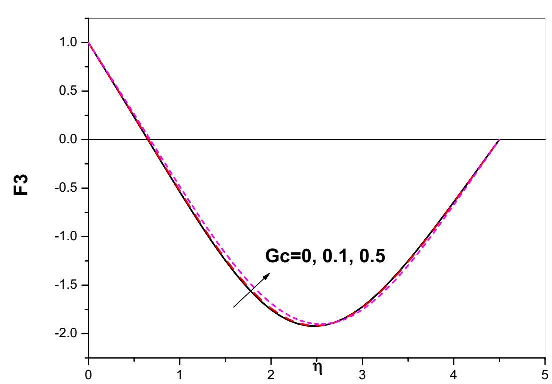

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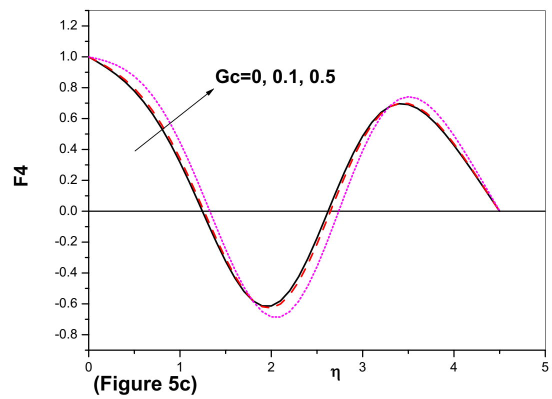

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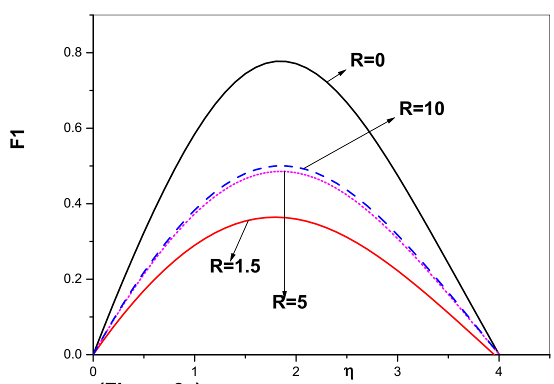

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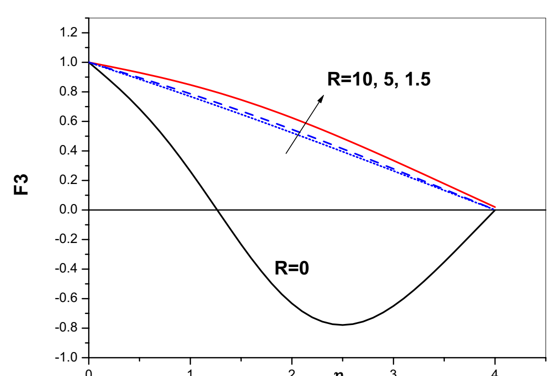

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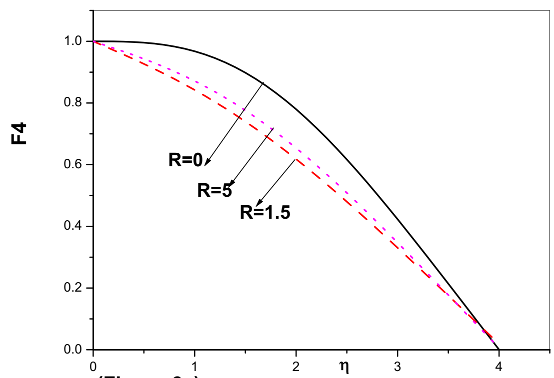

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