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A MULTIPLICITY RESULT FOR QUASILINEAR PROBLEMS WITH CONVEX AND CONCAVE NONLINEARITIES AND NONLINEAR BOUNDARY CONDITIONS IN UNBOUNDED DOMAINS

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Abstract. We study the following quasilinear problem with nonlinear boundary conditions

$$\begin{gathered}-\Delta_p u=\lambda a(x)|u|^{p-2}u+k(x)|u|^{q-2}u-h(x)|u|^{s-2}u,\quad \text{in } \Omega,\\ |\nabla u|^{p-2}\nabla u\cdot\eta+b(x)|u|^{p-2}u=0\quad \text{on } \partial\Omega,\end{gathered}$$

where Ω is an unbounded domain in $\mathbb{R}^N$ with a noncompact and smooth boundary $\partial\Omega$, η denotes the unit outward normal vector on $\partial\Omega$, $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, a, k, h and b are nonnegative essentially bounded functions, $q<p<s$ and $p^*<s$. The properties of the first eigenvalue λ_1 and the associated eigenvectors of the related eigenvalue problem are examined. Then it is shown that if $\lambda<\lambda_1$, the original problem admits an infinite number of solutions one of which is nonnegative, while if $\lambda=\lambda_1$ it admits at least one nonnegative solution. Our approach is variational in character.

1. Introduction

Consider the problem

$$\begin{gathered}-\Delta_p u=\lambda a(x)|u|^{p-2}u+k(x)|u|^{q-2}u-h(x)|u|^{s-2}u,\quad x\in\Omega,\\ |\nabla u|^{p-2}\nabla u\cdot\eta+b(x)|u|^{p-2}u=0,\quad x\in\partial\Omega,\end{gathered}\tag{1.1}$$

on an unbounded domain $\Omega\subseteq\mathbb{R}^N$ with a noncompact smooth boundary $\partial\Omega$, where η is the unit outward normal vector on $\partial\Omega$ and $\Delta_p u=div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian.

Throughout this work the following hypotheses are assumed:

(D) $1<p<N$, $1<q<p$, $p^*:=\frac{Np}{N-p}<s<+\infty$.

(A) There exist positive constants α_1, A_1, A_2 with $\alpha_1\in(p,N)$, such that

$$\frac{A_1}{(1+|x|)^{\alpha_1}}\le a(x)\le\frac{A_2}{(1+|x|)^{\alpha_1}}\quad \text{a.e. in } \Omega.$$

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(K) $k(.) \geq 0$, $m\{x \in \Omega : k(x) > 0\} > 0$ and there exist positive constants K_1 and α_2, with $\frac{p}{q} < \frac{\alpha_1 - N}{\alpha_2 - N}$, such that
$$k(x) \leq \frac{K_1}{(1+|x|)^{\alpha_2}} \quad \text{a.e. in } \Omega.$$
(H) $h \in L^\infty(\Omega)$, $h \geq 0$ a.e. and $m\{x \in \Omega : h(x) > 0\} > 0$.
(B) $b \in C(\mathbb{R}^N)$ and
$$\frac{B_1}{(1+|x|)^{p-1}} \leq b(x) \leq \frac{B_2}{(1+|x|)^{p-1}},$$
where $B_1, B_2 > 0$.

The growing attention in the study of the p-Laplace operator Δ_p is motivated by the fact that it arises in various applications, e.g. non-Newtonian fluids, reaction-diffusion problems, flow through porus media, glacial sliding, theory of superconductors, biology etc. (see [14], [6], [10] and the references therein). The existence of nontrivial solutions to equations like (1) with a power like right hand side has received considerable attention since the work of Brezis and Nirenberg [5]. When Ω is bounded, $p = 2$ and $1 < q < s$, existence, nonexistence and multiplicity of solutions in $H_0^1(\Omega)$ was studied in [2] according to the integrability properties of the ratio k^{s-1}/h^{q-1}. If $p \neq 2$, $p < q < q^*$, $h = 0$, we refer to [8], where existence of two solutions in $W_0^{1,p}(\Omega)$ is provided for $\lambda \leq \lambda_1 + \varepsilon$ for some $\varepsilon > 0$. If $\Omega = \mathbb{R}^N$ and $h \geq 0$ we refer to [9] where it was shown that (1.1) admits an infinite number of solutions in $D^{1,p}(\mathbb{R}^N)$.

In this paper we study (1.1) in connection with the corresponding eigenvalue problem for the p-Laplacian:
$$-\Delta_p u = \lambda a(x)|u|^{p-2}u$$
subject to the nonlinear boundary condition in (1.1). We show that the first eigenvalue λ_1 is positive, simple and isolated, the associated eigenvectors do not change sign and form a vector space of dimension 1. Then we combine the method employed in [9] with the results in [11] in order to show that if $\lambda < \lambda_1$ then (1.1) admits an infinite number of solutions, while if $\lambda = \lambda_1$ we use the fibering method (which is also applicable in case $\lambda < \lambda_1$) to show that it admits at least one nonnegative solution. To be more specific, we establish the following

Theorem 1.1. *Suppose that (D), (A), (K), (H) and (B) are satisfied.*

(i) *If $\lambda < \lambda_1$ then* (1.1) *admits infinitely many solutions with negative energy. If in addition $k > 0$ a.e., then it also admits a nonnegative solution.*
(ii) *If $\lambda = \lambda_1$ and $k > 0$ a.e., then* (1.1) *admits at least one nonnegative solution with negative energy.*

The proof of Theorem 1.1 will be given in Sections 4 and 5.

2. Preliminaries

Let $C_\delta^\infty(\Omega)$ be the space of $C_0^\infty(\mathbb{R}^N)-$functions restricted on Ω. Then the weighted Sobolev space E_p is the completion of $C_\delta^\infty(\Omega)$ in the norm
$$|||u|||_p = \Big(\int_\Omega |\nabla u|^p \,dx + \int_\Omega \frac{1}{(1+|x|)^p} |u|^p \,dx \Big)^{1/p}.$$

By [11, Lemma 2] we see that if $b(\cdot)$ satisfies (B), then the norm

$$\|u\|_{1,p} = \Big(\int_\Omega |\nabla u|^p \, dx + \int_{\partial\Omega} b(x)\, |u|^p \, d\sigma(x) \Big)^{1/p} \tag{2.1}$$

is equivalent to $|||\cdot|||_p$ ($\sigma(\cdot)$ being the surface measure on $\partial\Omega$).

Let $w_\alpha(x) := \frac{1}{(1+|x|)^\alpha}$ where $\alpha \in \mathbb{R}$. If Σ is a measurable subset of $\mathbb{R}^N$, we assume that the weighted Lebesgue space

$$L^r(w_\alpha, \Sigma) := \{u : \int_\Sigma w_\alpha(x)|u(x)|^r dx < +\infty\},$$

$r \in (1, +\infty)$, is supplied with the norm

$$\|u\|_{w_\alpha, r} = \Big(\int_\Sigma w_\alpha(x)|u(x)|^r dx \Big)^{1/r}.$$

For a nonnegative measurable function $h : \Sigma \to \mathbb{R}$, the space $L^s(h, \Sigma)$ is similarly defined. We associate with it the seminorm $|u|_{h,s} = \big(\int_\Sigma h(x)|u(x)|^s dx \big)^{1/s}$.

Let $E = E_p \cap L^s(h, \Omega)$. Then E endowed with the norm $\|\cdot\|_E = \|\cdot\|_{1,p} + |\cdot|_{h,s}$ becomes a separable Banach space.

Lemma 2.1. (i) *If*

$$p \le r \le \frac{pN}{N-p} \quad \text{and} \quad N > \alpha \ge N - r\frac{N-p}{p},$$

then the embedding $E \subseteq L^r(w_\alpha, \Omega)$ is continuous. If the upper bound for r in the first inequality and the lower bound for α in the second are strict, then the embedding is compact.

(ii) *If*

$$p \le m \le \frac{p(N-1)}{N-p} \quad \text{and} \quad N > \beta \ge N - 1 - m\frac{N-p}{p},$$

then the embedding $E \subseteq L^m(w_\beta, \partial\Omega)$ is continuous. If the upper bound for m in the first inequality and the lower bound for β are strict, then the embedding is compact.

(iii) *If*

$$1 < q < p \quad \text{and} \quad \frac{\alpha_1 - N}{\alpha_2 - N} > \frac{p}{q},$$

then the embedding $L^p(w_{\alpha_1}, \Omega) \subseteq L^q(w_{\alpha_2}, \Omega)$ is continuous.

Proof. The first and second part of the lemma corresponds to [11, Theorem 1], while the third is a consequence of the following inequality

$$\int_\Omega \frac{1}{(1+|x|)^{\alpha_2}} |u|^q dx \le \Big(\int_\Omega \frac{1}{(1+|x|)^d} dx \Big)^{\frac{p-q}{p}} \Big(\int_\Omega \frac{1}{(1+|x|)^{\alpha_1}} |u|^p dx \Big)^{q/p},$$

where $d = (\alpha_2 p - \alpha_1 q)/(p-q)$. Note that the integral $\int_\Omega \frac{1}{(1+|x|)^d} dx$ converges since $d > N$. $\square$

The energy functional $\Phi_\lambda : E \to \mathbb{R}$ corresponding to our problem is

$$\begin{aligned} \Phi_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \frac{\lambda}{p} \int_\Omega a|u|^p dx - \frac{1}{q} \int_\Omega k|u|^q dx \\ + \frac{1}{s} \int_\Omega h|u|^s dx + \frac{1}{p} \int_{\partial\Omega} b|u|^p d\sigma(x). \end{aligned} \tag{2.2}$$

It is clear that if (D), (A), (K), (H) and (B) are satisfied, then $\Phi_\lambda(.)$ is continuously differentiable and its critical points correspond to solutions of (1.1).

3. The principal eigenvalue

In this section we examine the properties of the first eigenvalue λ_1 and the associated eigenvectors of the following problem

$$\begin{gathered} -\Delta_p u = \lambda a(x)|u|^{p-2}u \quad \text{in } \Omega \\ |\nabla u|^{p-2}\nabla u \cdot \eta + b(x)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.1}$$

Proposition 3.1. *Suppose that $1 < p < N$ and hypotheses (A) and (B) are satisfied. Then*

(i) *Problem* (3.1) *admits a positive principal eigenvalue λ_1.*
(ii) *The set E_1 of eigenfunctions corresponding to λ_1 is a vector space of dimension 1. The elements of E_1 are either positive or negative and of class $C^{1,\delta}_{\rm loc}(\Omega)$. A positive eigenfunction always corresponds to λ_1.*
(iii) *λ_1 is isolated in the sense that there exists $\xi > 0$ such that the interval $(0, \lambda_1 + \xi)$ does not contain any eigenvalue other than λ_1.*

Proof. (i) Let I, $J : E_p \to \mathbb{R}$ be defined by

$$I(u) = \int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} b(x)|u|^p d\sigma(x), \quad J(u) = \int_\Omega a(x)|u|^p dx.$$

Then the operators I, J are continuously Fréchet differentiable, $I(.)$ is coercive, J' is compact and $J'(u) = 0$ implies that $u = 0$. Theorem 6.3.2 in [4] implies the existence of a principal eigenvalue satisfying

$$\lambda_1 = \inf_{J(u)=1} I(u). \tag{3.2}$$

The positivity of $\lambda_1$follows by a standard argument.

(ii) Let u_1 be an eigenfunction corresponding to λ_1. Since $|u_1|$ is also a minimizer in (3.2), we may assume that $u_1 \geq 0$. We will show first that $w_{\alpha_1}u_1$is essentially bounded in Ω. To that purpose for $M > 0$ define $u_M(x) := \min\{u_1(x), M\}$. Multiplying (3.1) by u_M^{kp+1}, $k > 0$, and integrating over Ω, we obtain

$$\int_\Omega |\nabla u_1|^{p-2}\,\nabla u_1 \cdot \nabla(u_M^{kp+1})\,dx + \int_{\partial\Omega} b(x)\,u_M^{(k+1)p}\,d\sigma(x) \leq \lambda_1 \int_\Omega a(x)\,u_1^{(k+1)p}\,dx\,. \tag{3.3}$$

Note that

$$\begin{aligned} \int_\Omega |\nabla u_1|^{p-2}\,\nabla u_1 \cdot \nabla(u_M^{kp+1})\,dx &= (kp+1)\int_\Omega |\nabla u_M|^p u_M^{kp} dx \\ &= \frac{kp+1}{(k+1)^p}\int_\Omega\ |\nabla u_M^{k+1}|^p\,dx,\ . \end{aligned}$$

So since $\frac{kp+1}{(k+1)^p} \leq 1$, it follows that

$$\begin{aligned} &\int_\Omega |\nabla u_1|^{p-2}\,\nabla u_1 \cdot \nabla(u_M^{kp+1})\,dx + \int_{\partial\Omega} b(x)\,u_M^{(k+1)p}\,d\sigma(x) \\ &\geq c_1 \frac{kp+1}{(k+1)^p}\Big(\int_\Omega \frac{1}{(1+|x|)^{\alpha_1}} u_M^{(k+1)p^*}\,dx\Big)^{p/p^*}, \end{aligned} \tag{3.4}$$

due to the embedding $E_p \subseteq L^{p^*}(w_{\alpha_1}, \Omega)$. By hypothesis (A), (3.3) and (3.4) we get that

$$
\begin{aligned}
&\Big(\int_\Omega \frac{1}{(1+|x|)^{\alpha_1}} u_M^{(k+1)p^*}\, dx\Big)^{1/p^*} \\
&\le \Big(\frac{\lambda_1 A_2 (k+1)^p}{c_3(kp+1)}\Big)^{1/p} \Big(\int_\Omega \frac{1}{(1+|x|)^{\alpha_1}}\, u_1^{(k+1)p} dx\Big)^{1/p},
\end{aligned}
$$

so

$$
\|u_M\|_{w_{\alpha_1},(k+1)p^*} \le \Big(\frac{\lambda_1 A_2 (k+1)^p}{c_3(kp+1)}\Big)^{1/((k+1)p)} \|u_1\|_{w_{\alpha_1},(k+1)p}.
$$

A bootstrap argument, as in the proof of [7, Lemma 3.2], shows that $w_{\alpha_1} u_1$ is essentially bounded. Theorems 1.9 and 1.11 in [7] imply that $u_1 \in C^{1,\delta}_{\rm loc}(\Omega)$ and $u_1 > 0$ in Ω.

We show next that E_1 is one dimensional by employing a technique similar to the one exposed in [1]. Namely, we shall prove that if for $\lambda > 0$, w_1 is a solution of

$$
-\Delta_p u \le \lambda a(x)|u|^{p-2}u \qquad \text{in } \Omega, \tag{3.5}
$$

and z_1 is a solution of

$$
-\Delta_p u \ge \lambda a(x)|u|^{p-2}u \qquad \text{in } \Omega, \tag{3.6}
$$

w_1, $z_1 > 0$ on Ω and satisfying the boundary condition in (1.1), then $z_1 = cw_1$ for some constant $c > 0$. For $\varepsilon > 0$ let $z_{1\varepsilon} = z_1 + \varepsilon$. If $\varphi \in C^\infty_\delta(\Omega)$, $\varphi \ge 0$, then $\frac{\varphi^p}{(z_{1\varepsilon})^{p-1}} \in E_p$. By Picone's identity [1], we get

$$
\begin{aligned}
0 &\le \int_\Omega |\nabla\varphi|^p dx - \int_\Omega \nabla\Big(\frac{\varphi^p}{z_{1\varepsilon}^{p-1}}\Big)\cdot|\nabla z_1|^{p-2}\nabla z_1 dx \\
&= \int_\Omega |\nabla\varphi|^p dx + \int_\Omega \frac{\varphi^p}{z_{1\varepsilon}^{p-1}}\Delta_p z_1 dx - \int_{\partial\Omega} \frac{\varphi^p}{z_{1\varepsilon}^{p-1}}|\nabla z_1|^{p-2}\nabla z_1\cdot\eta d\sigma(x) \\
&\le \int_\Omega |\nabla\varphi|^p dx - \lambda\int_\Omega \frac{\varphi^p}{z_{1\varepsilon}^{p-1}} a(x) z_1^{p-1} dx - \int_{\partial\Omega} \frac{\varphi^p}{z_{1\varepsilon}^{p-1}}|\nabla z_1|^{p-2}\nabla z_1\cdot\eta d\sigma(x)\,,
\end{aligned}
$$

while the boundary condition implies that

$$
0 \le \int_\Omega |\nabla\varphi|^p dx - \lambda\int_\Omega a(x)\frac{\varphi^p}{z_{1\varepsilon}^{p-1}} z_1^{p-1} dx + \int_{\partial\Omega} b(x)\frac{\varphi^p}{z_{1\varepsilon}^{p-1}} z_1^{p-1} d\sigma(x).
$$

If we let $\varepsilon \to 0$ and $\varphi \to w_1$ in E_p, we get

$$
0 \le \int_\Omega |\nabla w_1|^p dx - \lambda\int_\Omega a(x) w_1^p dx + \int_{\partial\Omega} b(x) w_1^p d\sigma(x). \tag{3.7}
$$

We can now work as in Theorem 2.1 in [1] to conclude that E_1 is a vector space of dimension 1. The same technique can be used to demonstrate that positive solutions in Ω correspond only to the first eigenvalue. Assume for instance, that there exists an eigenpair (λ^*, u_2) such that $\lambda^* > \lambda_1$ and $u_2 \ge 0$ a.e. in Ω. Then u_1 is a solution of (3.5) with $\lambda = \lambda_1$ and u_2 is a solution of (3.6) with $\lambda = \lambda^*$. But then $u_2 = cu_1$ for some $c > 0$, a contradiction.

(iii) Assume that there exists a sequence of eigenpairs (λ_n, u_n) with $\lambda_n \to \lambda_1$ and $\lambda_n \in (\lambda_1, \lambda_1 + \delta)$, $\delta > 0$, for every $n \in \mathbb{N}$. Without loss of generality, we may also assume that $\|u_n\|_{1,p} = 1$ for all $n \in \mathbb{N}$. Hence, there exists $\tilde u \in E_p$ such that $u_n \to \tilde u$ weakly in E_p. The simplicity of λ_1 implies that $\tilde u = u_1$ or $\tilde u = -u_1$. Let us suppose that $u_n \to u_1$ weakly in E_p. Multiplying (3.1) by $u_n - u_m$ and integrating by parts we get

$$\begin{aligned}
&\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)(\nabla u_n - \nabla u_m)\,dx \\
&+ \int_{\partial\Omega} b(x)(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m)\,d\sigma(x) \\
&= \lambda_n \int_\Omega a(x)\left(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m\right)(u_n - u_m)\,dx \\
&\quad + (\lambda_n - \lambda_m)\int_\Omega a(x)|u_m|^{p-2}u_m(u_n - u_m)\,dx\,.
\end{aligned}$$

Exploiting the compactness of the operator J and the monotonicity of the p-Laplacian operator, we obtain

$$\int_\Omega |\nabla u_n|^p\,dx \to \int_\Omega |\nabla u_1|^p\,dx.$$

The strict convexity of $L^p(\Omega)$ implies that $u_n \to u_1$ in E_p. For a fixed $n \in \mathbb{N}$ and for every $\phi \in E_p$ we have

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\phi\,dx + \int_{\partial\Omega} b(x)|u_n|^{p-2}u_n\phi\,d\sigma(x) = \lambda_n\int_\Omega a(x)|u_n|^{p-2}u_n\phi\,dx\,.$$

Let $\mathcal{U}_n^- =: \{x \in \overline{\Omega} : u_n(x) < 0\}$. By (iii) we must have $m(\mathcal{U}_n^-) > 0$. By choosing $\phi \equiv u_n^- = \min\{0, u_n\}$, it follows that

$$\int_{\mathcal{U}_n^-} |\nabla u_n^-|^p\,dx + \int_{\partial\Omega\cap\mathcal{U}_n^-} b(x)|u_n^-|^p\,dx = \lambda_n\int_{\mathcal{U}_n^-} a(x)|u_n^-|^p\,dx\,.$$

Thus

$$\|u_n^-\|_{1,p}^p \le A_2\,(\lambda_1 + \delta)\|u_n^-\|^p_{L^p(w_{\alpha_1},\mathcal{U}_n^-)}, \tag{3.8}$$

by (A). Denote by B_r the ball with radius $r > 0$ centered at $0 \in \mathbb{R}^n$. For $\varepsilon \in (0,1)$ there exists $r_{\varepsilon,n} > 0$ such that

$$\|u_n^-\|_{1,p}^p \le A_2\ (\lambda_1 + \delta)(\|u_n^-\|^p_{L^p(w_{\alpha_1},\mathcal{U}_n^-\cap B_{r_{\varepsilon,n}})} + \varepsilon\|u_n^-\|^p_{1,p})\,. \tag{3.9}$$

Apply once again the Hölder inequality to derive that

$$\begin{aligned}
&\|u_n^-\|^p_{L^p(w_{\alpha_1},\mathcal{U}_n^-\cap B_{r_{\varepsilon,n}})} \\
&\le \Big(\int_{\mathcal{U}_n^-\cap B_{r_{\varepsilon,n}}} \frac{1}{(1+|x|)^{\frac{\alpha_1 p^*}{p^*-p}}}\,dx\Big)^{\frac{p^*-p}{p^*}}\Big(\int_{\mathcal{U}_n^-\cap B_{r_{\varepsilon,n}}} |u_n^-|^{p^*}\,dx\Big)^{p/p^*}.
\end{aligned} \tag{3.10}$$

By Lemma 2.1 (i),

$$\Big(\int_{\mathcal{U}_n^-\cap B_{r_{\varepsilon,n}}} |u_n^-|^{p^*}\,dx\Big)^{p/p^*} \le c_2\|u_n^-\|^p_{1,p} \tag{3.11}$$

for some $c_2 > 0$. On combining (3.8)-(3.11) we get

$$1 - \varepsilon \le c_3\Big(\int_{\mathcal{U}_n^-\cap B_{r_{\varepsilon,n}}} \frac{1}{(1+|x|)^{\frac{\alpha_1 p^*}{p^*-p}}}\,dx\Big)^{\frac{p^*-p}{p^*}},$$

so $m(\mathcal{U}_n^- \cap B_{r_{\varepsilon,n}}) > c_4 > 0$, where the constant c_4 is independent of $n \in \mathbb{N}$. It is clear that there exists $R > 0$ such that

$$m(B_R \cap (\mathcal{U}_n^- \cap B_{r_{\varepsilon,n}})) > \frac{c_4}{2} \tag{3.12}$$

for every $n \in \mathbb{N}$. Since $u_n \to u_1$ in E_p we have that $u_n \to u_1$ in $L^{p^*}(w_{\alpha_1}, B_R \cap \Omega)$. By Egorov's Theorem, u_n converges uniformly to u_1 on $B_R \cap \Omega$ with the exception of a set with arbitrarily small measure. But this contradicts (3.12) and the conclusion follows. $\square$

Remark 3.2. If u_1 is continuous at $x_0 \in \partial\Omega$, then $u_1(x_0) > 0$. Indeed, if $u_1(x_0) = 0$, then by [16, Theorem 5] we would have $|\nabla u_1(x_0)|^{p-2}\nabla u_1(x_0)\cdot\eta(x_0) < 0$, contradicting (1.1).

4. The case $\lambda < \lambda_1$

We need the following lemma in order to show that Φ_λ is coercive.

Lemma 4.1. *If $\lambda < \lambda_1$ then the norm*

$$|||u|||_{1,p} := \Big(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} b|u|^p dx - \lambda \int_\Omega a|u|^p dx\Big)^{1/p}$$

is equivalent to $\|u\|_{1,p}$.

Proof. Suppose that there exists $u_n \in E_p$, $n \in \mathbb{N}$, such that $\|u_n\|_{1,p} = 1$ and

$$\int_\Omega |\nabla u_n|^p dx + \int_{\partial\Omega} b|u_n|^p d\sigma(x) - \lambda \int_\Omega a|u_n|^p dx \to 0.$$

In view of (3.2),

$$0 \le (\lambda_1 - \lambda) \int_\Omega a|u_n|^p dx \le \int_\Omega |\nabla u_n|^p dx + \int_{\partial\Omega} b|u_n|^p d\sigma(x) - \lambda \int_\Omega a|u_n|^p dx \to 0.$$

Hence, $\int_\Omega a|u_n|^p dx \to 0$, which shows that $\|u_n\|_{1,p} \to 0$. This is a contradiction with $\|u_n\|_{1,p} = 1$. $\square$

We can now prove our first result concerning (1.1).

Proof of Theorem 1.1(i). We will show that Φ_λ satisfies the Palais-Smale condition in E. So let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in E such that $\Phi_\lambda(u_n)$ is bounded and $\Phi'_\lambda(u_n) \to 0$. By Lemma 4.1 we get

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{p}\Big(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} b|u|^p d\sigma(x) - \lambda \int_\Omega a|u|^p dx\Big) \\
&\quad - \frac{1}{q}\int_\Omega k|u|^q dx + \frac{1}{s}\int_\Omega h|u|^s dx \\
&\ge \frac{1}{p}|||u|||_{1,p}^p - c_5|||u|||_{1,p}^q + \frac{1}{s}|u|_{h,s}^s,
\end{aligned}$$

implying that $\Phi_\lambda(.)$ is coercive. Thus $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E. Without loss of generality, we may assume that $u_n \to \overline{u}$ strongly in $L^p(w_{\alpha_1}, \Omega)$ and $L^q(w_{\alpha_2}, \Omega)$ and weakly in $L^p(w_{p-1},\partial\Omega)$, $E_p$and $L^s(h,\Omega)$. Thus

$$\int_\Omega a(x)|u_n-\overline{u}|^p dx\to 0,\quad \int_\Omega k(x)|u_n-\overline{u}|^q dx\to 0\,, \tag{4.1}$$

$$\int_{\partial\Omega} b(x)|\overline{u}|^{p-2}\overline{u}(u_n-\overline{u})d\sigma(x)\to 0,\quad \int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla(u_n-\overline{u})dx\to 0\,, \tag{4.2}$$

$$\int_\Omega h(x)|\overline{u}|^{s-2}\overline{u}(u_n-\overline{u})dx\to 0\,. \tag{4.3}$$

Therefore, by (4.1)-(4.3),

$$\langle \Phi'_\lambda(\overline{u}),u_n-\overline{u}\rangle\to 0.$$

Since $\Phi'_\lambda(u_n)\to 0$, we also have that

$$\langle \Phi'_\lambda(u_n)-\Phi'_\lambda(\overline{u}),u_n-\overline{u}\rangle\to 0.$$

Thus

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla\overline{u}\right)(\nabla u_n-\nabla\overline{u})dx\\
&-\lambda\int_\Omega a(x)\left(|u_n|^{p-2}u_n-|\overline{u}|^{p-2}\overline{u}\right)(u_n-\overline{u})dx\\
&-\int_\Omega k(x)\left(|u_n|^{q-2}u_n-|\overline{u}|^{q-2}\overline{u}\right)(u_n-\overline{u})dx\\
&+\int_{\partial\Omega} b(x)\left(|u_n|^{p-2}u_n-|\overline{u}|^{p-2}\overline{u}\right)(u_n-\overline{u})d\sigma(x)\\
&+\int_\Omega h(x)\left(|u_n|^{s-2}u_n-|\overline{u}|^{s-2}\overline{u}\right)(u_n-\overline{u})dx\to 0\,.
\end{aligned}\tag{4.4}$$

On combining (4.1)-(4.4) we get

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla\overline{u}|^{p-2}\nabla\overline{u}\right)(\nabla u_n-\nabla\overline{u})dx\\
&+\int_{\partial\Omega} b(x)\left(|u_n|^{p-2}u_n-|\overline{u}|^{p-2}\overline{u}\right)(u_n-\overline{u})d\sigma(x)\\
&+\int_\Omega h(x)\left(|u_n|^{s-2}u_n-|\overline{u}|^{s-2}\overline{u}\right)(u_n-\overline{u})dx\to 0\,.
\end{aligned}$$

We can now use the inequality

$$\begin{aligned}
0&\le\Big\{\Big(\int_\Omega |f_1|^r dx\Big)^{1/r'}-\Big(\int_\Omega |f_2|^r dx\Big)^{1/r'}\Big\}\\
&\quad\times\Big\{\Big(\int_\Omega |f_1|^r dx\Big)^{1/r}-\Big(\int_\Omega |f_2|^r dx\Big)^{1/r}\Big\}\\
&\le\int_\Omega \left(|f_1|^{r-2}f_1-|f_2|^{r-2}f_2\right)(f_1-f_2)dx,
\end{aligned}$$

where $f_1,\ f_2\in L^r(\Omega)$, $r>1$, $r'=r/(r-1)$, to obtain

$$\|\nabla u_n\|_p\to\|\nabla\overline{u}\|_p,\quad \|h^{\frac1s}u_n\|_s\to\|h^{\frac1s}\overline{u}\|_s\,.$$

Exploiting the strict convexity of $L^p(\Omega)$ and $L^s(\Omega)$ we derive that $\nabla u_n\to\nabla\overline{u}$ in $(L^p(\Omega))^N$ and $u_n\to\overline{u}$ in $L^s(h,\Omega)$. Consequently, $u_n\to\overline{u}$ in E, proving the claim.

Now let $Z=\{x\in\Omega: k(x)=0\}$ and $E_0=\{u\in E: u(x)=0 \text{ a.e. in } Z\}$. Define a norm on E_0 by $\|u\|_{E_0}=\|k^{1/q}u\|_q$. Consider the family Σ of closed and symmetric subsets of $E\backslash\{0\}$. For $A \in \Sigma$ we define the genus $\gamma(A)$ of A as the minimum of the $n \in \mathbb{N}$ such that there exists a continuous function $\varphi : A \to \mathbb{R}^n\backslash\{0\}$ with $\varphi(-x) = -\varphi(x)$. If no such n exists, we define $\gamma(A) = +\infty$. We claim that for $n \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $\gamma(\{u \in E : \Phi_\lambda(u) \le -\varepsilon\}) \ge n$. It will be enough to show that the set $\{u \in E : \Phi_\lambda(u) \le -\varepsilon\}$ contains an n-dimensional sphere centered at $0 \in \mathbb{R}^N$. So let E_0^n be an n-dimensional subspace of E_0. Then

$$\begin{aligned}\Phi_\lambda(u) &= \frac{1}{p}\Big(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} b|u|^p d\sigma(x) - \lambda \int_\Omega a|u|^p dx\Big)\\ &\quad - \frac{1}{q}\int_\Omega k|u|^q dx + \frac{1}{s}\int_\Omega h|u|^s dx\\ &\le \frac{1}{p}|||u|||_{1,p}^p - \frac{1}{q}\|u\|_{E_0}^q + \frac{1}{s}|u|_{h,s}^s\,.\end{aligned}$$

Since all norms on E_0^n are equivalent, we have that $\Phi_\lambda(u) \le c_1'\|u\|_{E_0^n}^p + c_2'\|u\|_{E_0^n}^s - c_3'\|u\|_{E_0^n}^q$, so there exists $\varepsilon > 0$ and $\delta > 0$ such that $\Phi_\lambda(u) \le -\varepsilon$ for $\|u\|_{E_0^n} = \delta$. Thus $\{u \in E_0^n : \|u\|_X = \delta\} \subseteq \{u \in E : \Phi_\lambda(u) \le -\varepsilon\}$, implying that $\gamma(\{u \in E : \Phi_\lambda(u) \le -\varepsilon\}) \ge n$. Let $\Sigma_n = \{A \in \Sigma : \gamma(A) \ge n\}$. Then the numbers $c_n = \inf_{A\in\Sigma_n} \sup_{u\in A} \Phi_\lambda(u)$ are critical values of Φ_λ, providing an infinite sequence of critical points of Φ_λ. For more details we refer to [3]. For the existence of a nonnegative solution, see Remark 5.1 in the next section.

5. The case $\lambda = \lambda_1$

In this section we apply the fibering method introduced by Pohozaev [12], [13] in order to show that (1.1) admits at least one nonnegative solution.

Proof of Theorem 1.1 (ii). We decompose the function $u \in E$ as $u(x) = rv(x)$ with $r \in \mathbb{R}$ and $v \in E$. By (2.2) we have that

$$\begin{aligned}\Phi_{\lambda_1}(rv) &= \frac{|r|^p}{p}\Big(\int_\Omega |\nabla v|^p - \lambda_1\int_\Omega a|v|^p + \int_{\partial\Omega} b|v|^p d\sigma(x)\Big)\\ &\quad - \frac{|r|^q}{q}\int_\Omega k|v|^q + \frac{|r|^s}{s}\int_\Omega h|v|^s.\end{aligned}$$

If u is a critical point of Φ_{λ_1}, then $\frac{\partial \Phi_{\lambda_1}}{\partial r} = 0$, so we will search for the critical points of Φ_{λ_1} among the ones which satisfy this equation, that is

$$\begin{aligned}&|r|^{p-q}\Big(\int_\Omega |\nabla v|^p dx - \lambda_1\int_\Omega a|v|^p dx + \int_{\partial\Omega} b|v|^p d\sigma(x)\Big) + |r|^{s-q}\int_\Omega h|v|^s dx\\ &= \int_\Omega k|v|^q dx\,.\end{aligned} \tag{5.1}$$

Since $k > 0$ a.e., for every $v \in E\backslash\{0\}$ there exists a unique $r = r(v) > 0$ satisfying (5.1). By using the implicit function theorem [17, Thm. 4.B, p.150], we see that the function $v \to r(v)$ is continuously differentiable for $v \neq 0$. Clearly,

$$r(\mu v)\mu v = r(v)v \quad \text{for every } \mu > 0\,. \tag{5.2}$$

Also, in view of (5.1)

$$\Phi_{\lambda_1}(r(v)v) = \big(\frac{r^q}{p} - \frac{r^q}{q}\big)\int_\Omega k|v|^q dx + \big(\frac{r^s}{s} - \frac{r^s}{p}\big)\int_\Omega h|v|^s dx \le 0\,. \tag{5.3}$$

Let
$$H(v)=\int_\Omega |\nabla v|^p dx-\lambda_1\int_\Omega a|v|^p dx+\int_{\partial\Omega} b|v|^p d\sigma(x)+\int_\Omega h|v|^s dx.$$
The variational characterization of λ_1 and hypothesis (H) imply that $H(v)\geq 0$ for every $v\in E$. Let $W=\{v\in E: H(v)=1\}$. By (3.2), W is bounded in $L^s(h,\Omega)$. Since
$$(H'(v),v)=p\Big(\int_\Omega |\nabla v|^p dx-\lambda_1\int_\Omega a|v|^p dx+\int_{\partial\Omega} b|v|^p d\sigma(x)\Big)+s\int_\Omega h|v|^s dx$$
we see that $(H'(v),v)\neq 0$ for $v\in W$. In view of [8, Lemma 3.4], any conditional critical point of the function $\widehat{\Phi}_{\lambda_1}(v):=\Phi_{\lambda_1}(r(v)v)$ subject to $H(v)=1$ provides a critical point $r(v)v$ of Φ_{λ_1}. Consider the problem
$$M_1=\inf\{\Phi_{\lambda_1}(r(v)v): v\in W\}.$$
Suppose that $\{v_n\}_{n\in\mathbb{N}}$ is a minimizing sequence in W, that is
$$\Phi_{\lambda_1}(r(v_n)v_n)\to M_1$$
and
$$H(v_n)=\Big(\int_\Omega |\nabla v_n|^p dx-\lambda_1\int_\Omega a|v_n|^p dx+\int_{\partial\Omega} b|v_n|^p d\sigma(x)\Big)+\int_\Omega h|v_n|^s dx=1.$$
Assume that $\|v_n\|_{1,p}\to+\infty$and let $u_n=\dfrac{v_n}{a_n}$ where $a_n=\|v_n\|_{1,p}$. Then
$$a_n^p\Big(\int_\Omega |\nabla u_n|^p dx-\lambda_1\int_\Omega a|u_n|^p dx+\int_{\partial\Omega} b|u_n|^p d\sigma(x)\Big)+a_n^s\int_\Omega h|u_n|^s dx=1,$$
so, by (3.2),
$$0\leq\int_\Omega |\nabla u_n|^p dx-\lambda_1\int_\Omega a|u_n|^p dx+\int_{\partial\Omega} b|u_n|^p d\sigma(x)\leq\frac{1}{a_n^p}\to 0 \tag{5.4}$$
and
$$0\leq\int_\Omega h|u_n|^s dx\leq\frac{1}{a_n^s}\to 0. \tag{5.5}$$
Thus
$$\lim_{n\to\infty}\lambda_1\int_\Omega a|u_n|^p dx=1. \tag{5.6}$$
Since $\|u_n\|_{1,p}=1$, by passing to a subsequence if necessary, we may assume that $u_n\to u$ weakly in E_p. In view of (5.6) we get
$$\lambda_1\int_\Omega a|u|^p dx=1,$$
so $u\neq 0$. The lower semicontinuity of the norm of E_p implies that
$$\int_\Omega |\nabla u|^p dx+\int_{\partial\Omega} b|u|^p d\sigma(x)\leq 1,$$
and (5.4) gives
$$\int_\Omega |\nabla u|^p dx+\int_{\partial\Omega} b|u|^p d\sigma(x)=\lambda_1\int_\Omega a|u|^p dx.$$
Thus $u$is an eigenfunction corresponding to λ_1. But then
$$\int_\Omega h|u|^s dx\leq\liminf_{n\to\infty}\int_\Omega h|u_n|^s dx=0\,,$$

by (5.5), a contradiction. Thus $\{v_n\}_{n\in\mathbb{N}}$ is bounded in E_p. Since $\{v_n\}_{n\in\mathbb{N}}$ is also bounded in $L^s(h,\Omega)$ we conclude that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in E. Going back to (5.1) we get that $r(W)$ is also bounded. Consequently, $I=\{\Phi_{\lambda_1}(r(v)v) : v\in W\}$ is a bounded interval in $\mathbb{R}$ with endpoints A,B, $A<B\le 0$. We will show that $A\in I$. To that purpose let $\{v_n\}_{n\in\mathbb{N}}\in W$ such that $\Phi_{\lambda_1}(r(v_n)v_n)\to A$. Without loss of generality we may assume that $v_n\to v_0$ weakly in E_p and in $L^s(h,\Omega)$. Furthermore, we may also assume that $r_n=r(v_n)\to d$, $d\in\mathbb{R}$. Clearly $r_nv_n\to dv_0$ weakly in E_p. Since $\Phi_{\lambda_1}(.)$ is weakly lower semicontinuous we have

$$\Phi_{\lambda_1}(dv_0)\le \liminf_{n\to+\infty}\Phi_{\lambda_1}(r_nv_n)=A\,,$$

so $dv_0\neq 0$. By lemma 2.1, $r(v_n)v_n\to dv_0$ strongly in $L^p(w_{\alpha_1},\Omega)$ and in $L^q(w_{\alpha_2},\Omega)$. Exploiting the lower semicontinuity of the norms in the relation $H(v_n)=1$ and in (5.1) we get

$$\Big(\int_\Omega|\nabla v_0|^pdx+\int_{\partial\Omega}b|v_0|^pd\sigma(x)-\lambda_1\int_\Omega a|v_0|^pdx\Big)+\int_\Omega h|v_0|^sdx\le 1$$

and

$$\begin{aligned}&d^{p-q}\Big(\int_\Omega|\nabla v_0|^pdx+\int_{\partial\Omega}b|v_0|^pd\sigma(x)-\lambda_1\int_\Omega a|v_0|^pdx\Big)+d^{s-q}\int_\Omega h|v_0|^sdx\\&\le\int_\Omega k|v_0|^qdx.\end{aligned}\tag{5.7}$$

Thus $d\le r(v_0)$. We will show that $d=r(v_0)$. So assume that $d<r(v_0)$and define $G(r)=\Phi_{\lambda_1}(rv_0)$. For $r\in[0,\,r(v_0))$ we have

$$\begin{aligned}\frac{G'(r)}{r^{q-1}}&=r^{p-q}\Big(\int_\Omega|\nabla v_0|^pdx-\lambda_1\int_\Omega a|v_0|^pdx+\int_{\partial\Omega}b|v_0|^pd\sigma(x)\Big)\\&\quad+r^{s-q}\int_\Omega h|v_0|^sdx-\int_\Omega k|v_0|^qdx<0,\end{aligned}$$

by (5.1). Thus $G(\cdot)$ is strictly decreasing on $[0,\,r(v_0))$. Consequently,

$$\Phi_{\lambda_1}(dv_0)=G(d)>G(r(v_0))=\Phi_{\lambda_1}(r(v_0)v_0).\tag{5.8}$$

Let $\gamma\ge 1$ be such that

$$\Big(\int_\Omega|\nabla\gamma v_0|^pdx+\int_{\partial\Omega}b|\gamma v_0|^pd\sigma(x)-\lambda_1\int_\Omega a|\gamma v_0|^pdx\Big)+\int_\Omega h|\gamma v_0|^sdx=1,\tag{5.9}$$

implying that $\gamma v_0\in W$. On combining (5.2), (5.8) and (5.9) we obtain

$$\Phi_{\lambda_1}(r(\gamma v_0)\gamma v_0)=\Phi_{\lambda_1}(r(v_0)v_0)<\Phi_{\lambda_1}(dv_0)\le\liminf_{n\to+\infty}\Phi_{\lambda_1}(r(v_n)v_n)=A,$$

that is $\Phi_{\lambda_1}(r(\gamma v_0)\gamma v_0)<A$, a contradiction. So $d=r(v_0)$. By taking $\gamma\ge1$ as in (5.9) we get

$$\Phi_{\lambda_1}(r(\gamma v_0)\gamma v_0)=\Phi_{\lambda_1}(r(v_0)v_0)\le\liminf_{n\to+\infty}\Phi_\lambda(r_nv_n)=A,$$

so $\widehat{\Phi}_{\lambda_1}(v_0)=\Phi_{\lambda_1}(r(v_0)v_0)=A$. Since $|v_0|$ is also a minimizer, we may assume that $v_0\ge 0$. [8, Lemma 3.4] guarantees that $w_0=r(v_0)v_0$is a nontrivial nonnegative solution of (1.1).

Remark 5.1. It is easy to see that the proof of Theorem 1.1(ii) can be applied for the case $\lambda < \lambda_1$. Therefore (1.1) admits also a nonnegative solution for $\lambda < \lambda_1$. If, in addition, $h \equiv 0$, then working as in Proposition 3.1 we see that this solution is positive in Ω.

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