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UNIQUE CONTINUATION PROPERTY FOR THE KADOMTSEV-PETVIASHVILI (KP-II) EQUATION

MAHENDRA PANTHEE

Abstract. We generalize a method introduced by Bourgain in [2] based on complex analysis to address two spatial dimensional models and prove that if a sufficiently smooth solution to the initial value problem associated with the Kadomtsev-Petviashvili (KP-II) equation

$$(u_t + u_{xxx} + uu_x)_x + u_{yy} = 0, \quad (x, y) \in \mathbb{R}^2,\ t \in \mathbb{R},$$

is supported compactly in a nontrivial time interval then it vanishes identically.

1. Introduction

Let us consider the following initial value problem (IVP) associated with the Kadomtsev-Petviashvili (KP) equation,

$$\begin{gathered} (u_t + u_{xxx} + uu_x)_x = \alpha u_{yy}, \quad (x, y) \in \mathbb{R}^2,\ t \in \mathbb{R} \\ u(x, y, 0) = \phi(x, y), \end{gathered} \tag{1.1}$$

where $u = u(x, y, t)$ is a real valued function and $\alpha = \pm 1$. This model was derived by Kadomtsev and Petviashvili [16] to describe the propagation of weakly nonlinear long waves on the surface of fluid, when the wave motion is essentially one-directional with weak transverse effects along y-axis. Equation (1.1) is known as KP-I or KP-II equation according as $\alpha = 1$ or $\alpha = -1$ and is considered a two dimensional generalization of the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = 0, \quad x,\, t \in \mathbb{R}, \tag{1.2}$$

which arises in modeling the evolution of one dimensional surface gravity waves with small amplitude in a shallow channel of water. The KdV model is a widely studied model and arises in various physical contexts. It has very rich mathematical structure and can also be solved by using inverse scattering technique.

The next generalization of the KdV model in two space dimension is the Zakharov-Kuznetsov (ZK) equation

$$u_t + (u_{xx} + u_{yy})_x + uu_x = 0. \tag{1.3}$$

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The equation (1.3) derived by Zakharov and Kuznetsov in [28] models the propagation of nonlinear ion-acoustic waves in magnetized plasma. Much effort has been devoted to study several properties including well-posedness issues and existence and stability of solitary wave solutions for the ZK model, see for example [1], [7], [22] and references therein. In particular, the author in [22] considered the following question: If a sufficiently smooth real valued solution $u = u(x, y, t)$ to the IVP associated to (1.3) is supported compactly on a certain time interval, is it true that $u \equiv 0$? In some sense, it is a weak version of the unique continuation property (UCP) which is defined as follows (cf. [23]):

Definition 1.1. Let L be an evolution operator acting on functions defined on some connected open set Ω of $\mathbb{R}^n \times \mathbb{R}_t$. The operator L is said to have unique continuation property (UCP) if every solution u of $Lu = 0$ that vanishes on some nonempty open set $\Omega_1 \subset \Omega$ vanishes in the horizontal component of Ω_1 in Ω.

After Carleman [4] initiated studies of UCP based on the weighted estimates for the associated solutions, many authors improved and extended Carleman's method to address parabolic and hyperbolic operators (see [8] and [20]). As far as we know the first work dealing with the UCP for a general class of dispersive equations in one space dimension is due to Saut and Scheurer [23] that also includes the KdV equation and uses Carleman type estimates. Also, D. Tataru in [26] derived some Carleman type estimates to prove the UCP for Schrödinger equation. Further, Isakov in [15] considered UCP for a large class of evolution equations with nonhomogeneous principal part. Later, Zhang in [29] obtained a slightly stronger UCP for the KdV equation than that in [23] using inverse scattering theory. Using Miura's transformation Zhang in [29] obtained UCP for the modified KDV (mKdV) equation as well. Recently, Kenig, Ponce and Vega in [17] considered the generalized KdV equation and proved UCP for it by deriving new Carleman's type estimate. Further, Kenig, Ponce and Vega in [19] obtained much stronger type of UCP for generalized KdV equation by proving that, if $u_1, u_2 \in C(\mathbb{R}, H^s(\mathbb{R}))$ are two solutions of the generalized KdV equation with $s > 0$ large enough and if there exist $t_1 \neq t_2$ and $\alpha \in \mathbb{R}$ such that $u_1(x, t_j) = u_2(x, t_j)$ for any $x \in (\alpha, \infty)$ or $(-\infty, \alpha)$ for $j = 1, 2$ then $u_1 \equiv u_2$. Bourgain in [2] introduced a new approach to address a wider class of evolution equations using complex variables techniques. The method introduced in [2] is more general and can also be applied to models in higher spatial dimensions. Quite recently, Carvajal and Panthee [5, 6] extended the argument introduced in [2] and [17] to prove the UCP for the nonlinear Schrödinger-Airy equation. Also, it is worth to mention the works of Iório in [9, 10] and Kenig, Ponce, Vega in [18] dealing with the UCP for Benjamin-Ono type and nonlinear Schrödinger equation.

The author in [22] generalized the method introduced in [2] to address a bi-dimensional (spatial) model and provided with an affirmative answer to the question posed above for the ZK model. Although, employing this method, one can deduce UCP for the linear problem almost immediately, the same is not so simple for the nonlinear problem and is quite involved. The symbol associated with the linear operator and the appropriate choice of the parameter play important role in the approach we used. The positive result obtained for the ZK model motivated us to think for the similar result for the KP equation. Unlike ZK model, there is singularity in the associated symbol of the linear KP operator. So, one needs to handle the analysis with utmost care. The structure of the associated symbol has also influenced a lot in the well-posedness results for the Cauchy problem for the KP equation. In this sense, the KP-II equation is much better than the KP-I equation, see for example [3], [11]–[14], [21], [24], [25] and [27]. The structure of the associated symbol has also affected our result on UCP for the KP equation. Here, we are able to handle only the KP-II equation by choosing appropriate parameters, see Remark 3.1 below. Therefore, from here onwards, we concentrate our work on KP-II equation (i.e., the IVP (1.1) with $\alpha = -1$) and obtain UCP for it. More precisely, using the scheme employed for the Zakharov-Kuznetsov equation we prove the following theorem.

Theorem 1.2. *Let $u \in C(\mathbb{R}; H^s(\mathbb{R}^2))$ be a solution to the IVP associated with the KP-II equation with $s > 0$ large enough. If there exists a non trivial time interval $I = [-T, T]$ such that for some $B > 0$*

$$\operatorname{supp} u(t) \subseteq [-B, B] \times [-B, B], \quad \forall\, t \in I,$$

then $u \equiv 0$.

Remark 1.3. In the context of the KdV equation (1.2), several types unique continuation properties exist in the recent literature, see for example [17], [19], [23] and [29]. Since KP equation is known to be a two-dimensional version of the KdV equation, we believe that the other forms of the UCP as mentioned in the above references could be proved for the KP equation too, but this needs to be done. As far as we know, our result in this article is the first UCP for the KP type models.

To prove this theorem, using the principle of Duhamel, we write the IVP associated with the KP-II equation in the equivalent integral equation form,

$$u(t) = S(t)\phi - \int_0^t S(t-t')(uu_x)(t')\,dt', \tag{1.4}$$

where $S(t)$ given by,

$$S(t)f(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(t(\xi^3 - \frac{\eta^2}{\xi}) + x\xi + y\eta)}\hat{f}(\xi,\eta)\,d\xi d\eta, \tag{1.5}$$

is the unitary group which describes the solution to the linear problem

$$\begin{gathered}(u_t + u_{xxx})_x + u_{yy} = 0,\\ u(x,y,0) = f(x,y).\end{gathered} \tag{1.6}$$

Significant amount of work has been devoted to address the Cauchy problem associated with the KP equation, see for example [3], [11]–[14], [21], [24], [25], [27] and references therein. Here we are not going to deal with this problem. For our purpose H^1-well-posedness of the associated Cauchy problem is enough. Finally, let us record that the quantities

$$\int_{\mathbb{R}^2} u^2\,dx\,dy, \tag{1.7}$$

$$\frac{1}{2}\int_{\mathbb{R}^2}\big[u_x^2 - (\partial_x^{-1}u_y)^2 - \frac{1}{3}u^3\big]\,dx\,dy, \tag{1.8}$$

are conserved by the KP-II flow which are useful to get global solution to the associated Cauchy problem in certain Sobolev spaces.

We organize this article as follows. In Section 2 we establish some basic estimates needed in the proof of the main result. The proof of the Theorem 1.2 will be presented in Section 3.

Before leaving this section let us introduce some notations that will be used throughout this article. We use $\hat{f}$ to denote the Fourier transform of f and is defined as,

$$\hat{f}(\lambda) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\lambda} f(\mathbf{x})\, d\mathbf{x}.$$

We denote the L^2-based Sobolev space of order s by H^s. The various constants whose exact values are immaterial will be denoted by c. We use the notation $A \lesssim B$ if there exists a constant $c > 0$ such that $A \leq cB$.

2. Basic Estimates

This section is devoted to establish some basic estimates that will play fundamental role in our analysis. These estimates are not new and can be found in [2] and the author's previous work in [22]. We will not give the details of the proofs rather we just sketch the idea of the proof. Let us begin with the following result.

Lemma 2.1. *Let $u \in C([-T,T]; H^s(\mathbb{R}^2))$ be a sufficiently smooth solution to the IVP (1.1). If for some $B > 0$, $\operatorname{supp} u(t) \subseteq \mathcal{B} := [-B,B] \times [-B,B]$, then for all $\lambda = (\xi,\eta), \sigma = (\theta,\delta) \in \mathbb{R}^2$, we have*

$$|\widehat{u(t)}(\lambda + i\sigma)| \lesssim e^{c|\sigma|B}. \tag{2.1}$$

Where we have used $|(x,y)| = \max\{|x|,|y|\}$.

Proof. The proof follows using the Cauchy-Schwarz inequality and the conservation law (1.7) with the argument similar to the one given in the proof of Lemma 2.1 in [22]. □

For $\lambda = (\xi,\eta)$ and $\lambda' = (\xi',\eta')$ define

$$u^*(\lambda) = \sup_{t\in I} |\widehat{u(t)}(\lambda)|, \tag{2.2}$$

$$a(\lambda) = \sup_{\substack{|\xi'|\geq|\xi| \\ |\eta'|\geq|\eta|}} |u^*(\lambda')|. \tag{2.3}$$

Considering ϕ sufficiently smooth and taking in to account the well-posedness theory for the IVP (1.1) (see for example, [3]), we have the following result.

Lemma 2.2. *Let $u \in C([-T,T]; H^s(\mathbb{R}^2))$ be a sufficiently smooth solution to the IVP (1.1) with $\operatorname{supp} u(t) \subseteq \mathcal{B}$, $t \in I$, then for some constant B_1, we have,*

$$a(\lambda) \lesssim \frac{B_1}{1+|\lambda|^4}. \tag{2.4}$$

Proof. The Cauchy-Schwarz inequality and the conservation law (1.7) yield

$$\int_{\mathbb{R}^2} |u(t)(\lambda)|\, d\lambda \leq |\mathcal{B}|^{1/2} \|u(t)\|_{L^2} \lesssim 1. \tag{2.5}$$

Now, using properties of the Fourier transform and (2.5) we get

$$|\widehat{u(t)}(\lambda)| \leq \|\widehat{u(t)}\|_{L^\infty} \leq c\|u(t)\|_{L^1} \lesssim 1. \tag{2.6}$$

Therefore,

$$\sup_t |\widehat{u(t)}(\lambda)| \leq c. \tag{2.7}$$

From the local well-posedness result (see [3]), we have

$$\|D^s u(t)\|_{L^\infty_T L^2_{xy}} \le c. \tag{2.8}$$

Since, $\operatorname{supp} u(t) \subseteq \mathcal{B}$ and

$$|\lambda|^s \widehat{u(t)}(\lambda) = \widehat{D^s u(t)}(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}^2} D^s u(t)(x,y) e^{-i(x\xi + y\eta)}\, dxdy,$$

using the Cauchy-Schwarz inequality and the estimate (2.8) we obtain

$$|\lambda|^s |\widehat{u(t)}(\lambda)| \le c\int_{\mathbb{R}^2} |D^s u(t)(x,y)|\, dxdy \le c\Big(\int_{\mathbb{R}^2} |D^s u(t)(x,y)|^2\, dxdy\Big)^{1/2} \le c_1. \tag{2.9}$$

Therefore,

$$|\widehat{u(t)}(\lambda)| \le \frac{c_1}{|\lambda|^s}. \tag{2.10}$$

If we consider $s = 4$ (which is possible, since we have local well-posedness for the IVP (1.1), for eg., in H^1) and combine (2.7) and (2.10) we get

$$\sup_t |\widehat{u(t)}(\lambda)| \le \frac{B_1}{1+|\lambda|^4}. \tag{2.11}$$

If λ' is such that $|\xi'| \ge |\xi|$ and $|\eta'| \ge |\eta|$, then $\frac{1}{1+|\lambda|^4} \ge \frac{1}{1+|\lambda'|^4}$. Hence

$$a(\lambda) = \sup_{\substack{|\xi'|\ge|\xi| \\ |\eta'|\ge|\eta|}} \sup_{t\in I} |\widehat{u(t)}(\lambda')| \le \sup_{\substack{|\xi'|\ge|\xi| \\ |\eta'|\ge|\eta|}} \frac{B_1}{1+|\lambda'|^4} \le \frac{B_1}{1+|\lambda|^4},$$

as required. □

Proposition 2.3. *Let $u(t)$ be compactly supported and suppose that there exists $t \in I$ with $u(t) \neq 0$. Then there exists a number $c > 0$ such that for any large number $Q > 0$ there are arbitrary large $|\lambda|$-values such that*

$$a(\lambda) > c(a * a)(\lambda), \tag{2.12}$$

$$a(\lambda) > e^{-\frac{|\lambda|}{Q}}. \tag{2.13}$$

Proof. The main ingredient in the proof is the estimate (2.4) in Lemma 2.2. The argument is similar to the one given in the proof of lemma in page 440 in [2], so we omit it. □

Using the definition of $a(\lambda)$ and Proposition 2.3 we choose $|\lambda|$ large (with $|\xi|$, $|\eta|$ large) and $t_1 \in I$ such that

$$|\widehat{u(t_1)}(\lambda)| = u^*(\lambda) = a(\lambda) > c(a*a)(\lambda) + e^{-\frac{|\lambda|}{Q}}. \tag{2.14}$$

Now we prove some estimates for derivative of an entire function. Let us begin with the following lemma whose proof is given in [2].

Lemma 2.4. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function which is bounded and integrable on the real axis and satisfies*

$$|\phi(\xi + i\theta)| \lesssim e^{|\theta| B}, \quad \xi,\, \theta \in \mathbb{R}.$$

Then, for $\lambda_1 \in \mathbb{R}^+$ we have

$$|\phi'(\lambda_1)| \lesssim B\Big(\sup_{\xi' \ge \lambda_1} |\phi(\xi')|\Big)\Big[1 + \Big|\log\Big(\sup_{\xi'\ge\lambda_1} |\phi(\xi')|\Big)\Big|\Big]. \tag{2.15}$$

In the sequel we use this lemma to obtain some more estimates which are crucial in the proof of our main result. The details of the proof of these estimates can be found in [22]. For the sake of clarity, we mention main ingredients and sketch of the proof.

Lemma 2.5. *Let $\Phi : \mathbb{C}^2 \to \mathbb{C}$ be an entire function satisfying*
$$|\Phi(\lambda + i\sigma)| \lesssim e^{c|\sigma|B} \qquad \lambda,\, \sigma \in \mathbb{R}^2,$$
such that for z_2 fixed, $\Phi_1(z_1) := \Phi(z_1, z_2)$ and for z_1 fixed, $\Phi_2(z_2) := \Phi(z_1, z_2)$ are bounded and integrable on the real axis. Then for $\lambda_1,\, \lambda_2 \in \mathbb{R}^+$ we have
$$|\nabla\Phi(\lambda_1, \lambda_2)| \lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\xi', \eta')|\Big)\Big[1 + \Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\xi', \eta')|\Big)\Big|\Big]. \tag{2.16}$$

Proof. Let us take $\lambda' = (\xi', \eta')$. First we apply Lemma 2.4 to Φ_1 for fixed z_2 to get
$$|\Phi_1'(\lambda_1)| \lesssim B\Big(\sup_{\xi'\geq\lambda_1} |\Phi_1(\xi')|\Big)\Big[1 + \Big|\log\Big(\sup_{\xi'\geq\lambda_1} |\Phi_1(\xi')|\Big)\Big|\Big]. \tag{2.17}$$
Next, we apply Lemma 2.4 to Φ_2 for fixed z_1 to obtain
$$|\Phi_2'(\lambda_2)| \lesssim B\Big(\sup_{\eta'\geq\lambda_2} |\Phi_1(\eta')|\Big)\Big[1 + \Big|\log\Big(\sup_{\eta'\geq\lambda_2} |\Phi_2(\eta')|\Big)\Big|\Big]. \tag{2.18}$$
Now using (2.17), (2.18) and the definition
$$\nabla\Phi(\lambda_1, \lambda_2) := (\Phi_1'(\lambda_1), \Phi_2'(\lambda_2)),$$
we obtain the required result. $\square$

Corollary 2.6. *Let $\sigma \in \mathbb{R}^2$ be such that*
$$|\sigma| \leq B^{-1}\Big[1 + \Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1>0\\ \eta'\geq\lambda_2>0}} |\Phi(\xi', \eta')|\Big)\Big|\Big]^{-1}. \tag{2.19}$$
Then
$$\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\lambda' + i\sigma)| \leq 4 \sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\lambda')| \tag{2.20}$$
and
$$\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\nabla\Phi(\lambda' + i\sigma)| \lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\lambda')|\Big)\Big[1 + \Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\lambda')|\Big)\Big|\Big]. \tag{2.21}$$

Proof. To prove (2.20), first fix $\eta' \geq \lambda_2$ and use Corollary 2.9 in [2] and then fix $\xi' \geq \lambda_1$ and use the same corollary to get
$$\begin{aligned}\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\xi' + i\theta, \eta' + i\delta)| &\leq \sup_{\eta'\geq\lambda_2}\Big(2\sup_{\xi'\geq\lambda_1} |\Phi(\xi', \eta' + i\delta)|\Big)\\ &\leq 4 \sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\Phi(\lambda')|.\end{aligned}$$

To prove (2.21), we define $\tilde\Phi(z) = \Phi(z + i\sigma)$. Then $\tilde\Phi$ is an entire function and satisfies the hypothesis of Lemma 2.5. So, by the same Lemma we get
$$|\nabla\tilde\Phi(\lambda_1, \lambda_2)| \lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\tilde\Phi(\xi', \eta')|\Big)\Big[1 + \Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}} |\tilde\Phi(\xi', \eta')|\Big)\Big|\Big].$$

for any $\lambda_1, \lambda_2 \in \mathbb{R}^+$.

Now the definition of $\tilde{\Phi}$ and use of (2.20) imply

$$\begin{aligned} &|\nabla\Phi(\lambda_1+i\theta,\lambda_2+i\delta)| \\ &\lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi'+i\theta,\eta'+i\delta)|\Big)\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi'+i\theta,\eta'+i\delta)|\Big)\Big|\Big] \\ &\lesssim 4B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi',\eta')|\Big)\Big[1+\Big|\log\Big(4\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\xi',\eta')|\Big)\Big|\Big] \\ &\lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big[(1+\log 4)+(1+\log 4)\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big|\Big] \\ &\lesssim B\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big[1+\Big|\log\Big(\sup_{\substack{\xi'\geq\lambda_1\\ \eta'\geq\lambda_2}}|\Phi(\lambda')|\Big)\Big|\Big]. \end{aligned} \tag{2.22}$$

Which yields the desired estimate (2.21). □

Corollary 2.7. *Let $t \in I$, $\Phi(z) = \widehat{u(t)}(z)$, σ be as in Corollary 2.6 and $a(\lambda)$ be as in (2.3). Then, for $|\sigma'| \leq |\sigma|$ fixed, we have*

$$|\nabla\Phi(\lambda-\lambda'+i\sigma')| \lesssim B\big[a(\lambda)+a(\lambda-\lambda')\big]\big[1+|\log a(\lambda)|\big]. \tag{2.23}$$

Proof. Define $\tilde{\Phi}(z) := \Phi(z+i\sigma')$, $z=(z_1,z_2)=(\xi+i\theta,\eta+i\delta)$. First, we use (2.21) with $\sigma=0$ and then use (2.20) to get the required result. The details of the proof can be found in [22]. □

3. Proof of the main result

This section is devoted to supply proof of Theorem 1.2. Although the scheme of the proof is analogous to the one we employed to get similar result for the ZK equation in [22], one needs to overcome some additional technical difficulties arising from the structure of the Fourier symbol associated to the linear KP-II equation.

Proof of Theorem 1.2. If possible, suppose $u(t) \neq 0$ for some $t \in I$. Our aim is to get a contradiction by using the estimates derived in the previous section.

Let $t_1, t_2 \in I$, with t_1 as in (2.14). Using Duhamel's principle, we have

$$u(t_2) = S(t_2-t_1)u(t_1) - \frac{1}{2}\int_{t_1}^{t_2} S(t_2-t')(u^2)_x(t')\,dt'. \tag{3.1}$$

Taking Fourier transform in the space variables with $\lambda=(\xi,\eta)$ we get

$$\widehat{u(t_2)}(\lambda) = e^{i(t_2-t_1)(\xi^3-\frac{\eta^2}{\xi})}\widehat{u(t_1)}(\lambda) - \frac{i\xi}{2}\int_{t_1}^{t_2} e^{i(t_2-t')(\xi^3-\frac{\eta^2}{\xi})}\widehat{u^2(t')}(\lambda)\,dt'. \tag{3.2}$$

Let $t_2 - t_1 = \Delta t$ and then make a change of variables $s = t' - t_1$ to get

$$\widehat{u(t_2)}(\lambda) = e^{i\Delta t(\xi^3-\frac{\eta^2}{\xi})}\Big[\widehat{u(t_1)}(\lambda) - \frac{i\xi}{2}\int_0^{\Delta t} e^{-is(\xi^3-\frac{\eta^2}{\xi})}\widehat{u^2(s+t_1)}(\lambda)\,ds\Big]. \tag{3.3}$$

Since $u(t), t \in I$ has compact support, by Paley-Wiener theorem, $\widehat{u(t)}(\lambda)$ has analytic continuation in $\mathbb{C}^2$, and we have for $\sigma = (\theta, \delta)$

$$\begin{aligned}\widehat{u(t_2)}(\lambda + i\sigma) = e^{i\Delta t\{(\xi+i\theta)^3 - \frac{(\eta+i\delta)^2}{\xi+i\theta}\}}\Big[\widehat{u(t_1)}(\lambda + i\sigma) \\ - \frac{i(\xi+i\theta)}{2}\int_0^{\Delta t} e^{-is\{(\xi+i\theta)^3 - \frac{(\eta+i\delta)^2}{\xi+i\theta}\}}\widehat{u^2(s+t_1)}(\lambda+i\sigma)\,ds\Big].\end{aligned} \tag{3.4}$$

Since

$$\begin{aligned}(\xi+i\theta)^3 - \frac{(\eta+i\delta)^2}{\xi+i\theta} &= \xi^3 - 3\xi\theta^2 - \frac{1}{\xi^2+\theta^2}(\xi\eta^2 - \xi\delta^2 + 2\eta\theta\delta) \\ &\quad + i\{3\xi^2\theta - \theta^3 - \frac{1}{\xi^2+\theta^2}(2\xi\eta\delta - \theta\eta^2 + \theta\delta^2)\},\end{aligned}$$

using Lemma 2.1 we get from (3.4)

$$\begin{aligned}& ce^{\Delta t\{3\xi^2\theta - \theta^3 - \frac{2\xi\eta\delta - \theta(\eta^2-\delta^2)}{\xi^2+\theta^2}\}} \\ &\geq \big|\widehat{u(t_1)}(\lambda+i\sigma)\big| - \frac{|\xi+i\theta|}{2}\int_0^{\Delta t} e^{s\{3\xi^2\theta - \theta^3 - \frac{2\xi\eta\delta - \theta(\eta^2-\delta^2)}{\xi^2+\theta^2}\}}\big|\widehat{u^2(s+t_1)}(\lambda+i\sigma)\big|\,ds.\end{aligned} \tag{3.5}$$

Let us take $|\lambda| = \max\{|\xi|, |\eta|\}$ very large such that

$$\xi\eta > 0 \qquad \text{and} \qquad |\xi| \sim |\eta|. \tag{3.6}$$

Also, let us choose $\sigma = \sigma(\lambda)$ with $|\sigma| = \max\{|\theta|, |\delta|\} \approx 0$ with

$$\theta\Delta t < 0 \qquad \text{and} \qquad \delta\Delta t > 0. \tag{3.7}$$

Moreover, let us suppose the following conditions are satisfied

$$\frac{1}{|\xi|} \ll |\theta|,\, |\delta| \quad \text{and} \quad \frac{1}{|\eta|} \ll |\theta|,\, |\delta|. \tag{3.8}$$

With these choices, (3.5) can be written as

$$\begin{aligned}e^{\Delta t\{3\xi^2\theta - \frac{2\xi\eta\delta - \theta\eta^2}{\xi^2+\theta^2}\}} &\gtrsim \big|\widehat{u(t_1)}(\lambda+i\sigma)\big| \\ &\quad - |\xi|\int_0^{\Delta t} e^{s\{3\xi^2\theta - \frac{2\xi\eta\delta-\theta\eta^2}{\xi^2+\theta^2}\}}\big|\widehat{u^2(s+t_1)}(\lambda+i\sigma)\big|\,ds.\end{aligned} \tag{3.9}$$

Now, taking into consideration of (3.6) and (3.7), the estimate (3.9) yields

$$\begin{aligned}& e^{-|\Delta t|\{3\xi^2|\theta| + \frac{2|\xi\eta\delta| + |\theta|\eta^2}{\xi^2+\theta^2}\}} \\ &\gtrsim \big|\widehat{u(t_1)}(\lambda+i\sigma)\big| - |\xi|\int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta| + \frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}}\big|\widehat{u^2(t_1\pm s)}(\lambda+i\sigma)\big|\,ds.\end{aligned} \tag{3.10}$$

Where "+" sign corresponds to $\Delta t > 0$ and $'-'$ sign corresponds to $\Delta t < 0$. In what follows we consider the case $\Delta t > 0$ (the case $\Delta t < 0$ is similar). Since $e^{-x} < 1$ for $x > 0$, (3.10) can be written as

$$\begin{aligned}& e^{-\{3\xi^2 + \frac{\eta^2}{\xi^2+\theta^2}\}|\theta\Delta t|} \\ &\gtrsim \big|\widehat{u(t_1)}(\lambda+i\sigma)\big| - |\xi|\int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta| + \frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}}\big|\widehat{u^2(t_1+s)}(\lambda+i\sigma)\big|\,ds.\end{aligned}$$

Finally, this last estimate can be written as

$$\begin{aligned}
&e^{-\{3\xi^2+\frac{\eta^2}{\xi^2+\theta^2}\}|\theta\Delta t|}\\
&\gtrsim \big|\widehat{u(t_1)}(\lambda)\big| - |\xi|\int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}}\big|\widehat{u^2(t_1+s)}(\lambda)\big|\,ds\\
&- \big|\widehat{u(t_1)}(\lambda+i\sigma)-\widehat{u(t_1)}(\lambda)\big|\\
&- |\xi|\int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}}\big|\widehat{u^2(t_1+s)}(\lambda+i\sigma)-\widehat{u^2(t_1+s)}(\lambda)\big|\,ds\\
&:= I_1 - I_2 - I_3.
\end{aligned} \tag{3.11}$$

Now, we proceed to obtain appropriate estimates for I_1, I_2 and I_3 to arrive at a contradiction in (3.11). To obtain estimate for I_1, we use definition of $u^*(\lambda)$, i.e., (2.2) and the estimate (2.12) to get

$$\begin{aligned}
&|\xi|\int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}}\big|\widehat{u(t_1+s)}\big| * \big|\widehat{u(t_1+s)}\big|(\lambda)\,ds\\
&\le |\xi|(u^* * u^*)(\lambda)\int_0^{|\Delta t|} e^{-s\{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}}ds\\
&\le |\xi|(a*a)(\lambda)\frac{1-e^{-|\Delta t|\{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}}}{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}}\\
&\le \frac{|\xi|(a*a)(\lambda)}{3|\xi||\xi\theta|} \lesssim \frac{a(\lambda)}{3|\xi\theta|}.
\end{aligned}$$

Therefore,

$$I_1 \gtrsim a(\lambda) - \frac{a(\lambda)}{3|\xi\theta|} \ge \frac{a(\lambda)}{3}. \tag{3.12}$$

To get estimate for I_2, let us define $\Phi(z)=\widehat{u(t_1)}(z)$, $z=(z_1,z_2)\in\mathbb{C}^2$. Now, using (2.14) one can obtain

$$|\Phi(\lambda)| = |\widehat{u(t_1)}(\lambda)| = \sup_{\substack{|\xi'|\ge|\xi|\\|\eta'|\ge|\eta|}} |\Phi(\lambda')| = a(\lambda). \tag{3.13}$$

Let us choose $|\sigma|$ satisfying

$$|\sigma| \lesssim B^{-1}\big[1+|\log a(\lambda)|\big]^{-1}, \tag{3.14}$$

and use Corollary 2.6 to obtain

$$I_2 \lesssim |\sigma| \sup_{\substack{|\xi'|\ge|\xi|\\|\eta'|\ge|\eta|}} |\nabla\widehat{u(t_1)}(\lambda'+i\sigma)| \lesssim |\sigma|B\,a(\lambda)\big[1+|\log a(\lambda)|\big] \lesssim a(\lambda) \lesssim \frac{1}{15}a(\lambda). \tag{3.15}$$

To obtain estimate for I_3, we use Proposition 2.3, Corollary 2.7 and $|\sigma|$ as in (3.14) to get

$$\begin{aligned}
&\big|\widehat{u^2(t_1+s)}(\lambda+i\sigma)-\widehat{u^2(t_1+s)}(\lambda)\big|\\
&\le \int_{\mathbb{R}^2}\big|\widehat{u(t_1+s)}(\lambda-\lambda'+i\sigma)-\widehat{u(t_1+s)}(\lambda-\lambda')\big|\big|\widehat{u(t_1+s)}(\lambda')\big|\,d\lambda'\\
&\le|\sigma|\int_{\mathbb{R}^2}\sup_{|\sigma'|\le|\sigma|}\big|\nabla\widehat{u(t_1+s)}(\lambda-\lambda'+i\sigma')\big|\,a(\lambda')\,d\lambda'\\
&\le\int_{\mathbb{R}^2}[a(\lambda)+a(\lambda-\lambda')]\,a(\lambda')\,d\lambda'\\
&\le a(\lambda)c_2+(a*a)(\lambda)\\
&\le a(\lambda)(c_2+c^{-1})\lesssim a(\lambda).
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_3&\lesssim|\xi|a(\lambda)\int_0^{|\Delta t|}e^{-s\{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}}ds\\
&=|\xi|a(\lambda)\,\frac{1-e^{-|\Delta t|\{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}\}}}{3\xi^2|\theta|+\frac{2|\xi\eta\delta|+|\theta|\eta^2}{\xi^2+\theta^2}}\\
&\le\frac{|\xi|a(\lambda)}{3|\xi^2\theta|}\\
&\lesssim\frac{a(\lambda)}{|\xi\theta|}<\frac{a(\lambda)}{15}.
\end{aligned}\tag{3.16}$$

Now inserting (3.12), (3.15) and (3.16) in (3.11) and using the estimate (2.13) we get

$$e^{-\{3\xi^2+\frac{\eta^2}{\xi^2+\theta^2}\}|\theta\Delta t|}\gtrsim\frac{a(\lambda)}{3}-\frac{a(\lambda)}{15}-\frac{a(\lambda)}{15}=\frac{1}{3}a(\lambda)\gtrsim e^{-\frac{|\lambda|}{Q}}.\tag{3.17}$$

On the other hand, using (3.6) and (3.8) one can easily deduce

$$e^{-\{3\xi^2+\frac{\eta^2}{\xi^2+\theta^2}\}|\theta||\Delta t|}\le e^{-|\lambda||\Delta t|}.\tag{3.18}$$

Hence, using (3.18) in (3.17), we arrive at

$$e^{-|\lambda||\Delta t|}\gtrsim e^{-\frac{|\lambda|}{Q}},$$

which is a contradiction for $|\lambda|$ large, if we choose Q large enough such that $\frac{1}{Q}<|\Delta t|$. $\square$

Remark 3.1. Note that the Fourier symbol associated with the linear KP-I operator is $\xi^3+\frac{\eta^2}{\xi}$. In this case we cannot make choice as in (3.6) and (3.7) to obtain estimate like in (3.10) with term of the form $e^{-|\Delta t|\gamma}$, for some $\gamma>0$. As seen in the proof of Theorem 1.2, existence of such term in the RHS of (3.10) is very essential in the argument we employed. It would be interesting to obtain UCP for the KP-I equation employing some other argument or modification.

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Mahendra Panthee
Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal; and
Department of Mathematics, CMAGDS, IST, 1049–001, Av. Rovisco Pais, Lisbon, Portugal
E-mail address: mpanthee@math.ist.utl.pt