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EXISTENCE OF SOLUTIONS FOR SEMILINEAR NONLOCAL CAUCHY PROBLEMS IN BANACH SPACES

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Abstract. In this paper, we study a semilinear differential equations with nonlocal initial conditions in Banach spaces. We derive conditions for f, $T(t)$, and g for the existence of mild solutions.

1. Introduction

In this paper we discuss the nonlocal initial value problem (IVP for short)

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in (0, b), \tag{1.1}$$

$$u(0) = g(u) + u_0, \tag{1.2}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (i.e. C_0-semigroup) $T(t)$ in Banach space X and $f : [0, b] \times X \to X$, $g : C([0, b]; X) \to X$ are given X-valued functions.

The above nonlocal IVP has been studied extensively. Byszewski and Lasmikanthem [4, 5, 6] give the existence and uniqueness of mild solution when f and g satisfying Lipschitz-type conditions. Ntougas and Tsamatos [12, 13] study the case of compactness conditions of g and $T(t)$. In [10] Lin and Liu discuss the semilinear integro-differential equations under Lipschitz-type conditions. Byszewski and Akca [7] give the existence of functional-differential equation when $T(t)$ is compact, and g is convex and compact on a given ball of $C([0, b]; X)$. In [8] Fu and Ezzinbi study the neutral functional differential equations with nonlocal initial conditions. Benchohra and Ntouyas [3] discuss the second order differential equations with nonlocal conditions under compact conditions. Aizicovici and McKibben [1] give the existence of integral solutions of nonlinear differential inclusions with nonlocal conditions.

In references authors give the conditions of Lipschitz continuous of g as f be Lipschitz continuous, and give the compactness conditions of g as $T(t)$ be compact and g be uniformly bounded. In this paper we give the existence of mild solution of IVP (1.1) and (1.2) under following conditions of g, $T(t)$ and f:

(1) g and f are compact, $T(t)$ is a C_0-semigroup
(2) g is Lipschitz continuous, f is compact and $T(t)$ is a C_0-semigroup

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(3) g is Lipschitz continuous and $T(t)$ is compact.

Also give existence results in above cases without the assumption of uniformly boundedness of g.

Let $(X, \|\cdot\|)$ be a real Banach space. Denoted by $C([0,b];X)$ the space of X-valued continuous functions on $[0,b]$ with the norm $|u| = \sup\{\|u(t)\|, t \in [0,b]\}$ and denoted by $L(0,b;X)$ the space of X-valued Bochner integrable functions on $[0,b]$ with the norm $\|u\|_1 = \int_0^b \|u(t)\|dt$.

By a *mild solution* of the nonlocal IVP (1.1) and (1.2) we mean the function $u \in C([0,b];X)$ which satisfies

$$u(t) = T(t)u_0 + T(t)g(u) + \int_0^t T(t-s)f(s,u(s))ds \tag{1.3}$$

for all $t \in [0,b]$.

A C_0-semigroup $T(t)$ is said to be *compact* if $T(t)$ is compact for any $t > 0$. If the semigroup $T(t)$ is compact then $t \mapsto T(t)x$ are equicontinuous at all $t > 0$ with respect to x in all bounded subsets of X; i.e., the semigroup $T(t)$ is *equicontinuous*.

To prove the existence results in this paper we need the following fixed point theorem by Schaefer.

Lemma 1.1 ([15]). *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F : S \to S$ be a continuous and compact map, and let the set $\{x \in S : x = \lambda Fx \text{ for some } \lambda \in (0,1)\}$ be bounded. Then F has at least one fixed point in S.*

In this paper we suppose that A generates a C_0 semigroup $T(t)$ on X. And, without loss of generality, we always suppose that $u_0 = 0$.

2. Main Results

In this section we give some existence results of the nonlocal IVP (1.1) and (1.2). Here we list the following results.

(Hg) (1) $g : C([0,b];X) \to X$ is continuous and compact.
(2) There exist $M > 0$ such that $\|g(u)\| \le M$ for $u \in C([0,b];X)$.

(Hf) (1) $f(\cdot,x)$ is measurable for $x \in X$,$f(t,\cdot)$ is continuous for a.e. $t \in [0,b]$.
(2) There exist a function $a(\cdot) \in L^1(0,b,R^+)$ and an increasing continuous function $\Omega : R^+ \to R^+$ such that $\|f(t,x)\| \le a(t)\Omega(\|x\|)$ for all $x \in X$ and a.e. $t \in [0,b]$.
(3) $f : [0,b] \times X \to X$ is compact.

Theorem 2.1. *If (Hg) and (Hf) are satisfied, then there is at least one mild solution for the IVP* (1.1) *and* (1.2) *provided that*

$$\int_0^b a(s)ds < \int_{NM}^{+\infty} \frac{ds}{N\Omega(s)}, \tag{2.1}$$

where $N = \sup\{\|T(t)\|, t \in [0,b]\}$.

Next, we give an existence result when g is Lipschitz:

(Hg') There exist a constant $k < 1/N$ such that $\|g(u) - g(v)\| \le k|u - v|$ for $u, v \in C([0,b];X)$.

Theorem 2.2. *If (Hg'), (Hg)(2), and (Hf) are satisfied, then there is at least one mild solution for the IVP* (1.1) *and* (1.2) *when* (2.1) *holds.*

Above we suppose that g is uniformly bounded. Next, we give existence results without the hypothesis (Hg)(2).

Theorem 2.3. *If (Hg)(1) and (Hf) are satisfied, then there is at least one mild solution for the IVP* (1.1) *and* (1.2) *provided that*

$$\int_0^b a(s)ds < \liminf_{T\to\infty} \frac{T - N\alpha(T)}{N\Omega(T)}, \tag{2.2}$$

where $\alpha(T) = \sup\{\|g(u)\|; |u| \le T\}$.

Theorem 2.4. *If (Hg') and (Hf) are satisfied, then there is at least one mild solution for the IVP* (1.1) *and* (1.2) *provided that*

$$\int_0^b a(s)ds < \liminf_{T\to\infty} \frac{T - NkT}{N\Omega(T)}. \tag{2.3}$$

Next, we give an existence result when g is Lipschitz and the semigroup $T(t)$ is compact.

Theorem 2.5. *Assume that (Hg'), (Hf)(1), (Hf)(2) are satisfied, and assume that* $T(t)$ *is compact. Then there is at least one mild solution for the IVP* (1.1) *and* (1.2) *provided that*

$$\int_0^b a(s)ds < \liminf_{T\to\infty} \frac{T - NkT}{N\Omega(T)}. \tag{2.4}$$

At last we would like to discuss the IVP (1.1) and (1.2) under the following growth conditions of f and g.

(Hf)(2') There exist $m(\cdot), h(\cdot) \in L^1(0,b;R^+)$ such that

$$\|f(t,x)\| \le m(t)\|x\| + h(t),$$

for a.e. $t \in [0,b]$ and $x \in X$.

(Hg)(2') There exist constant c, d such that for $u \in C([0,b];X)$, $\|g(u)\| \le c|u| + d$.

Clearly (Hf)(2') is the special case of *H(f)(2)* with $a(t) = max\{m(t), h(t)\}$ and $\Omega(s) = s + 1$.

Theorem 2.6. *Assume (Hg)(1), (Hg)(2'), (Hf)(1), (Hf)(2'), and assume (Hf)(3) is true, or* $T(t)$ *is compact. Then there is at least one mild solution for the IVP* (1.1) *and* (1.2) *provided that*

$$Nce^{N\|m\|_1} < 1, \tag{2.5}$$

where $\|\cdot\|_1$ *means the* $L^1(0,b)$ *norm.*

Theorem 2.7. *Assume (Hg'), (Hf)(1), (Hf)(2'), and assume (Hf)(3) is true or* $T(t)$ *is compact. Then there is at least one mild solution for the IVP* (1.1) *and* (1.2) *provided that*

$$Nke^{N\|m\|_1} < 1. \tag{2.6}$$

3. Proofs of Main Results

We define $K : C([0,b];X) \to C([0,b];X)$ by

$$(Ku)(t) = \int_0^t T(t-s)f(s,u(s))ds \tag{3.1}$$

for $t \in [0,b]$. To prove the existence results we need following lemmas.

Lemma 3.1. *If (Hf) holds, then K is continuous and compact; i.e. K is completely continuous.*

Proof. The continuity of K is proved as follows. Let $u_n \to u$ in $C([0,b];X)$. Then

$$|Ku_n - Ku| \le N \int_0^b \|f(s,u_n(s)) - f(s,u(s))\| ds.$$

So $Ku_n \to Ku$ in $C([0,b];X)$ by the Lebesgue's convergence theorem.

Let $B_r = \{u \in C([0,b];X); |u| \le r\}$. Form the Ascoli-Arzela theorem, to prove the compactness of K, we should prove that $KB_r \subset C([0,b];X)$ is equi-continuous and $KB_r(t) \subset X$ is pre-compact for $t \in [0,b]$ for any $r > 0$. For any $u \in B_r$ we know

$$\begin{aligned}
&\|Ku(t+h) - Ku(t)\| \\
&\le N \int_t^{t+h} \|f(s,u(s))\| ds + \int_0^t \|[T(t+h-s) - T(t-s)]f(s,u(s))\| ds \\
&\le N \int_t^{t+h} a(s)\Omega(r) ds + N \int_0^t \|[T(h) - I]f(s,u(s))\| ds.
\end{aligned}$$

Since f is compact, $\|[T(h) - I]f(s,u(s))\| \to 0$ (as $h \to 0$) uniformly for $s \in [0,b]$ and $u \in B_r$. This implies that for any $\epsilon > 0$ there existing $\delta > 0$ such that $\|[T(h) - I]f(s,u(s))\| \le \epsilon$ for $0 \le h < \delta$ and all $u \in B_r$. We know that:

$$\|Ku(t+h) - Ku(t)\| \le N\Omega(r) \int_t^{t+h} a(s) ds + N\epsilon$$

for $0 \le h < \delta$ and all $u \in B_r$. So $KB_r \subset C([0,b];X)$ is equicontinuous. The set $\{T(t-s)f(s,u(s)); t,s \in [0,b], u \in B_r\}$ is pre-compact as f is compact and $T(\cdot)$ is a C_0 semigroup.So $KB_r(t) \subset X$ is pre-compact as

$$KB_r(t) \subset t\ \overline{\text{conv}}\{T(t-s)f(s,u(s)); s \in [0,t], u \in B_r\}$$

for all $t \in [0,b]$. □

Define $J : C([0,b];X) \to C([0,b];X)$ by $(Ju)(t) = T(t)g(u)$. So u is the mild solution of IVP (1.1)and (1.2) if and only if u is the fixed point of $J + K$. We can prove the following lemma easily.

Lemma 3.2. *If (Hg)(1) is true then J is continuous and compact.*

Proof of theorem 2.1. From above we know that $J+K$ is continuous and compact. To prove the existence, we should only prove that the set of fixed points of $\lambda(J+K)$ is uniformly bounded for $\lambda \in (0,1)$ by the Schaefer's fixed point theorem (Lemma 1.1). Let $u = \lambda(J+K)u$,i.e.,for $t \in [0,b]$

$$u(t) = \lambda T(t)g(u) + \lambda \int_0^t T(t-s)f(s,u(s)) ds.$$

We have

$$\|u(t)\| \le NM + N \int_0^t a(s)\Omega(\|u(s)\| ds.$$

Denoting by $x(t)$ the right-hand side of the above inequality, we know that $x(0) = NM$ and $\|u(t)\| \le x(t)$ for $t \in [0,b]$, and

$$x'(t) = Na(t)\Omega(\|u(t)\|) \le Na(t)\Omega(x(t))$$

for a.e. $t \in [0,b]$. This implies

$$\int_{NM}^{x(t)} \frac{ds}{N\Omega(s)} \le \int_0^t a(s)ds < \int_{NM}^{\infty} \frac{ds}{N\Omega(s)},$$

for $t \in [0,b]$. This implies that there is a constant $r > 0$ such that $x(t) \le r$, where r is independent of λ. We complete the proof as $\|u(t)\| \le r$ for $u \in \{u; u = \lambda(J+K)u$ for some $\lambda \in (0,1)\}$. □

For the next lemma, let $L : C([0,b];X) \to C([0,b];X)$ be defined as $(Lu)(t) = u(t) - T(t)g(u)$.

Lemma 3.3. *If (Hg') holds then L is bijective and L^{-1} is Lipschitz continuous with constant $1/(1-Nk)$.*

Proof. For any $v \in C([0,b];X)$, by using the Banach's fixed point theorem, we know that there is unique $u \in C([0,b];X)$ satisfying $Lu = v$. It implies that L is bijective. For any $v_1, v_2 \in C([0,b];X)$,

$$\begin{aligned}\|L^{-1}v_1(t) - L^{-1}v_2(t)\| &\le \|T(t)g(L^{-1}v_1) - T(t)g(L^{-1}v_1)\| + \|v_1(t) - v_2(t)\| \\ &\le Nk|L^{-1}v_1 - L^{-1}v_2| + \|v_1(t) - v_2(t)\|\end{aligned}$$

for $t \in [0,b]$. This implies

$$|L^{-1}v_1 - L^{-1}v_2| \le \frac{1}{1-Nk}|v_1 - v_2|.$$

which completes the proof. □

Proof of Theorem 2.2. Clearly u is the mild solution of IVP and (1.2) if and only if u is the fixed point of $L^{-1}K$. Similarly with Theorem 2.1 we should only prove that the set $\{u; \lambda u = (L^{-1}K)u$ for some $\lambda > 1\}$ is bounded as $L^{-1}K$ be continuous and compact due to the fixed point theorem of Schaefer.If $\lambda u = L^{-1}Ku$. Then for any $t \in [0,b]$

$$\lambda u(t) = T(t)g(\lambda u) + \int_0^t T(t-s)f(s,u(s))ds.$$

We have

$$\begin{aligned}\|u(t)\| &\le \frac{1}{\lambda}NM + \frac{1}{\lambda}N\int_0^t a(s)\Omega(\|u(s)\|)ds \\ &\le NM + N\int_0^t a(s)\Omega(\|u(s)\|)ds.\end{aligned}$$

Just as proved in Theorem 2.1 we know there is a constant r which is independent of λ, such that $|u| \le r$ for all $u \in \{u; \lambda u = (L^{-1}K)u$ for some $\lambda > 1\}$. So we proved this theorem. □

Proof of Theorem 2.3. By lemma 3.1 and Lemma 3.2 we know that $J+K$ is continuous and compact. From (2.2) there exists a constant $r > 0$ such that

$$\int_0^b a(s)ds \le \frac{r - N\alpha(r)}{N\Omega(r)}. \tag{3.2}$$

For any $u \in B_r$ and $v = Ju + Ku$,we get

$$\|v(t)\| \le N\alpha(r) + N\int_0^t a(s)\Omega(r)ds \le r,$$

for $t \in [0,b]$. It implies that$(J+K)B_r \subset B_r$. By Schauder's fixed point theorem, we know that there is at least one fixed point $u \in B_r$ of the completely continuous map $J+K$, and u is a mild solution. □

Proof of Theorem 2.4. By Lemma 3.1 and Lemma 3.3 we know that $L^{-1}K$ is continuous and compact. From (2.3) there exists a constant number $r > 0$ such that

$$\int_0^b a(s)ds \le \frac{r - Nkr - N\|g(0)\|}{N\Omega(r)}. \tag{3.3}$$

For any $u \in B_r$ and $v = L^{-1}Ku$, we get

$$\|v(t)\| \le Nk|v| + N\|g(0)\| + N\int_0^t a(s)\Omega(r)ds,$$

for $t \in [0,b]$. It implies that $|v| \le r$, i.e., $L^{-1}KB_r \subset B_r$. By Schauder's fixed point theorem, there is at least one fixed point $u \in B_r$ of the completely continuous map $L^{-1}K$, and u is a mild solution. □

Proof of Theorem 2.5. By the proof of [12, Theorem 2.1] we know that K is completely continuous under (Hf)(1), (Hf)(2) and condition of compactness of semigroup $T(t)$. So $L^{-1}K$ is completely continuous. Similarly with the proof of Theorem 2.4, we complete the the proof of this theorem. □

Proof of Theorem 2.6. From [12, Theorem 2.1], Lemma 3.1 and Lemma 3.2 we know that the map $J+K$ is completely continuous. By Lemma 1.1 ,we should only prove that the set $\{u; u = \lambda(J+K)u \text{ for some } \lambda \in (0,1)\}$ is bounded. For any $u \in \{u; u = \lambda(J+K)u \text{ for some } \lambda \in (0,1)\}$, we have

$$\begin{aligned}\|u(t)\| &\le \lambda(Nc|u| + Nd) + \lambda N\int_0^t m(s)\|u(s)\|ds + \lambda N\int_0^t h(s)ds\\ &\le Nc|u| + N\int_0^t m(s)\|u(s)\|ds + N(d + \|h\|_1).\end{aligned}$$

This implies that for $t \in [0,b]$

$$|u| \le \frac{N(d + \|h\|_1)\exp(N\|m\|_1)}{1 - Nc\exp(N\|m\|_1)},$$

by Gronwall's inequality. □

Proof of Theorem 2.7. From [12, Theorem 2.1], Lemma 3.2 and Lemma 3.3 we know that the map $L^{-1}K$ is completely continuous. By Schaefer's fixed point theorem (Lemma 1.1), we should only prove that the set $\{u; u = \lambda(L^{-1}K)u \text{ for some } \lambda \in (0,1)\}$ is bounded. For any $u \in \{u; u = \lambda(L^{-1}K)u \text{ for some } \lambda \in (0,1)\}$, similarly with the estimation above, we know that

$$|u| \le \frac{N(\|g(0)\| + \|h\|_1)\exp(N\|m\|_1)}{1 - Nk\exp(N\|m\|_1)}.$$

The proof is complete. □

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