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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO n-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We establish conditions for the linear differential equation

$$y^{(n)}(t) + p(t)y(g(t)) = 0$$

to have property A. Explicit sufficient conditions for the oscillation of the the equation is obtained while dealing with the property A of the equations. A comparison theorem is obtained for the oscillation of the equation with the oscillation of a third order ordinary differential equation.

1. Introduction

This paper concerns property A of the n-th order ($n \geq 2$) delay differential equation

$$y^{(n)}(t) + p(t)y(g(t)) = 0, \tag{1.1}$$

under certain conditions on the coefficient function $p \in C([\sigma, \infty), [0, \infty)), \sigma \in R$, and $g \in C([\sigma, \infty), R)$ such that $g(t) \leq t$ and $g(t) \to \infty$ as $t \to \infty$.

It is interesting to note that we have obtained sufficient conditions for oscillation of all solutions of (1.1) while dealing with property A of the equation. These sufficient conditions are easily verifiable and different from earlier ones (See [2, 5, 6, 8, 11, 12]). Moreover, these sufficient conditions are consistent with the situation when $p(t)$ is a constant.

A continuous function $y : [g(\sigma), \infty) \to R$ is said to be a proper solution of (1.1) if it is absolutely continuous on $(t_0, \infty), t_0 \geq \sigma$ along with its derivatives up to the $(n-1)$th order and satisfies (1.1) almost everywhere on (t_0, ∞) and $\sup\{|y(s)| : s \geq t\} > 0$ for $t \geq t_0$. A proper solution of (1.1) is called oscillatory if it has a sequence of zeros tending to infinity. Otherwise, it is called non-oscillatory. Equation (1.1) with $g(t) = t$ is said to be disconjugate on $[\sigma, \infty)$ if no nontrivial solution of the equation has more than $(n-1)$ zeros, counting muntiplicities.

A vast body of literature exist on the oscillation of (1.1). One may see the monographs due to Lakshmikantham et al [12], Gyori and Ladas [8] and the references cited therein. Higher order differential equations with property A were studied by Parhi and Padhi [15] and Koplatadze [11]. We shall see that our results are different form their results. We observe that our results do not hold for the case $g(t) = t$ (See Theorems 2.1-2.4 and 2.25 and Corollaries 2.5 and 2.26).

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Let $y(t)$ be a positive solution of (1.1) for $t \geq t_0\sigma$. Then there exists a $t_1 > t_0$ such that $y(g(t)) > 0$ for $t \geq t_1$. Then $y^{(n)}(t) \leq 0$ for $t \geq t_1$,and so by a lemma due to Kiguradze [10], there exists an integer l, $0 \leq l \leq n-1$ such that $n+l$ odd and

$$\begin{gathered} y^{(i)}(t) > 0, \quad i = 0,1,2,\dots,l, \\ (-1)^{i+l} y^{(i)}(t) > 0, \quad i = l+1,\dots,n. \end{gathered} \tag{1.2}$$

for large t. Again, for $l \in \{1,2,3,\dots,n-1\}, n+l$ odd, the following inequality holds for large t, say for $t \geq t_2$.

$$|y(t)| \geq \frac{(t-t_2)^{(n-1)}}{(n-1)(n-2)\dots(n-l)} |y^{(n-1)}(2^{n-l-1}t)|, \quad t \geq t_2. \tag{1.3}$$

Let N denote the set of all nonoscillatory solutions of (1.1) and N_l denote the set of all nonoscillatory solutions of (1.1) satisfying (1.2). Then

$$N = \begin{cases} N_0 \cup N_2 \cup \dots \cup N_{n-1} & \text{if } n \text{ is odd}, \\ N_1 \cup N_3 \cup \dots \cup N_{n-1} & \text{if } n \text{ is even}. \end{cases}$$

Definition. We say that (1.1) has property A if any of its solution is oscillatory when n is even and either is oscillatory or satisfies N_0 when n is odd.

The following conjecture is given in [10, pp.29, Problem 1.14], which we state as a problem.

Problem 1.1. *Let $M_{n^*} = \max(\lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1))$. If*

$$\int^\infty t^{n-1} \big[p(t) - \frac{M_{n^*}}{t^n}\big]\, dt = \infty,$$

then (1.1) with $g(t) = t$ has property A.

Our Theorem 2.20 gives a partial answer to the above problem for the case $n = 2$ and $g(t) = t$ in (1.1).

The following lemma, due to Kiguradze [10], is needed for our use in the sequel.

Lemma 1.2. *Let for a certain $l \in \{1,2,3,\dots,n-1\}$, the inequality (1.2) hold. Then*

$$\int_{t_1}^\infty s^{n-l-1} |y^{(n)}(s)|\, ds < \infty, \tag{1.4}$$

$$y^{(i)}(t) \geq y^{(i)}(t_1) + \frac{1}{(l-i-1)!} \int_{t_1}^t (t-s)^{l-i-1} y^{(i)}(s)\, ds \tag{1.5}$$

for $t \geq t_1, i = 0,1,2,\dots,l-1$ and

$$y^{(l)}(t) \geq \frac{1}{(l-i-1)!} \int_t^\infty (s-t)^{n-l-1} |y^{(n)}(s)|\, ds \tag{1.6}$$

for $t \geq t_1$. If in addition

$$\int_{t_1}^\infty s^{n-l} |y^{(n)}(s)|\, ds = \infty, \tag{1.7}$$

then there exists $t_2 \geq t_1$ such that

$$y^{(l-1)}(t) \geq \frac{t}{(n-l)!} \int_t^\infty s^{n-l-1} |y^{(n)}(s)|\, ds \tag{1.8}$$

for $t \geq t_2$ and

$$i y^{(l-1)} \geq t y^{(l-i+1)}(t) \geq (i-1) y^{(l-i)}(t) \tag{1.9}$$

for $t \geq t_2$, $i \in \{1, 2, \dots, l\}$.

2. Main Results

Theorem 2.1. *Let* $g(t) < t$ *and for every* $l \in \{1, 2, 3, \dots, n-1\}$ *such that* $n+l$ *is odd,*

$$\limsup_{t\to\infty} (t - g(t))^l \int_{g^{-1}(t)}^{\infty} (s-t)^{n-l-1} p(s)\, ds > (n-l-1)!.l1 \tag{2.1}$$

hold. Then (1.1) *has property A.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $y(t) > 0$ for $t \geq t_0 > \sigma$. Thus there exists a $T_1 \geq t_0$ such that $y(g(t)) > 0$ for $t \geq T_1$. Consequently, from (1.1), it follows that $y^{(n)}(t) \leq 0$ for $t \geq T_1$. Then, there exists a $l \in \{0, 1, 2, \dots, n-1\}$ and $n+l$ odd such that (1.2) holds for some $t \geq t_1 > T_1$. We claim that $l = 0$. If not, then $l \in \{1, 2, \dots, n-1\}$. Putting $i = 0$ in (1.5), we get

$$y(t) \geq \frac{1}{(l-1)!} \int_{t_1}^{t} (t-s)^{l-1} y^{(l)}(s)\, ds, \quad t \geq t_1. \tag{2.2}$$

We can find a $t_2 \geq t_1$ such that $g(t) > t_1$ for $t \geq t_2$. Hence, for $t \geq t_2$

$$y(t) \geq \frac{y^{(l)}(t)}{(l-1)!} \int_{g(t)}^{t} (t-s)^{l-1}\, ds \geq \frac{y^{(l)}(t)}{(l-1)!} . \frac{(t-g(t))^l}{l};$$

that is,

$$y(t) \geq \frac{(t-g(t))^l}{l!} y^{(l)}(t). \tag{2.3}$$

Using (1.6) in (2.3), we obtain

$$\begin{aligned}
y(t) &\geq \frac{(t-g(t))^l}{l!} . \frac{1}{(n-l-1)!} \int_{t}^{\infty} (s-t)^{n-l-1} |y^{(n)}(s)|\, ds \\
&\geq \frac{(t-g(t))^l}{l!} . \frac{1}{(n-l-1)!} \int_{g^{-1}(t)}^{\infty} (s-t)^{n-l-1} |y^{(n)}(s)|\, ds \\
&\geq \frac{(t-g(t))^l}{l!} . \frac{1}{(n-l-1)!} \int_{g^{-1}(t)}^{\infty} (s-t)^{n-l-1} p(s) y(g(s))\, ds \\
&\geq \frac{(t-g(t))^l}{l!} . \frac{1}{(n-l-1)!} y(t) \int_{g^{-1}(t)}^{\infty} (s-t)^{n-l-1} p(s)\, ds
\end{aligned}$$

for $t \geq t_2$, which is a contradiction to the hypothesis of the theorem. Hence (1.1) has property A. This completes the proof of the theorem. □

Theorem 2.2. *Suppose that for every* $l \in \{1, 2, 3, \dots, n-1\}$ *,* $n+l$ *is odd,,*

$$\limsup_{t\to\infty} t^{n-1} \int_{g^{-1}(t)}^{\infty} p(s)\, ds > (n-1)\dots(n-l) 2^{(n-1)(n-l)}, \tag{2.4}$$

holds. Then (1.1) *has property A.*

Proof. Let $y(t)$ be a non-oscillatory solution of (1.1). Without any loss of generality, we may assume that $y(t) > 0$ for $t \geq t_0 > \sigma$. Then there exists a $t_1 \geq t_0$ such that $y(g(t)) > 0$ for $t \geq t_1$. Consequently, it follows from (1.1) that $y^{(n)}(t) \leq 0$ for $t \geq t_1$ and (1.2) holds. If possible, suppose that (1.1) has not property A. Then

$l \in \{1,2,3,\dots,n-1\}$. Clearly (1.3) holds for some $t \geq t_2 \geq t_1$. Since $y'(t) > 0$, then for $t > t.2^{l+1-n} \geq t_2$, we have

$$y(t) \geq y(2^{l+1-n}t) \geq \frac{1}{(n-1)\dots(n-l).2^{(n-1)(n-l)}} t^{n-1} y^{(n-1)}(t). \tag{2.5}$$

On the other hand, integrating (1.1) from $t(\geq t_2)$ to ∞,we have

$$y^{(n-1)}(t) > \int_t^\infty p(s)y(g(s))\,ds > \int_{g^{-1}(t)}^\infty p(s)y(g(s))\,ds > y(t)\int_{g^{-1}(t)}^\infty p(s)\,ds.$$

Then (2.5) gives

$$1 \geq \frac{1}{(n-1)\dots(n-l).2^{(n-1)(n-l)}} t^{n-1} \int_{g^{-1}(t)}^\infty p(s)\,ds$$

for $t \geq t_2$, which contradicts (2.4). Hence (1.1) has property A. The Theorem is proved. □

Theorem 2.3. *Suppose that $g(t) < t$ and for every $l \in \{1,2,3,\dots,n-1\}$ such that $n+l$ is odd, the following inequality*

$$\limsup_{t\to\infty} \int_{g(t)}^t (t-s)^{l-1} \int_{g^{-1}(g^{-1}(s))}^\infty (u-s)^{n-l-1} p(u)\,du\,ds > (l-1)!.(n-l-1)! \tag{2.6}$$

holds. Then (1.1) has property A.

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_0 > \sigma$. Thus (1.2) holds for some $t \geq t_1 > t_0$. Suppose that $l \in \{1,2,\dots,n-1\}$. Putting $i = 0$ in (1.5), we get

$$y(t) \geq \frac{1}{(l-1)!}\int_{t_1}^t (t-s)^{l-1} y^{(l)}(s)\,ds. \tag{2.7}$$

From (1.5), we obtain

$$y^{(l)}(t) \geq \frac{1}{(n-l-1)!}\int_t^\infty (s-t)^{n-l-1} p(s)y(g(s))\,ds. \tag{2.8}$$

Then from (2.7) and (2.8), we obtain

$$y(t) \geq \frac{1}{(n-l-1)!.(l-1)!}\int_{t_1}^t (t-s)^{l-1}\int_s^\infty (u-s)^{n-l-1} p(u)y(g(u))\,du\,ds. \tag{2.9}$$

We can find a $t_2 \geq t_1$ such that $g(t) > t_1$ for $t \geq t_2$. Thus, for $t \geq t_2$

$$y(t) \geq \frac{1}{(n-l-1)!.(l-1)!}\int_{g(t)}^t (t-s)^{l-1}\int_{g^{-1}(g^{-1}(s))}^\infty (u-s)^{n-l-1} p(u)y(g(u))\,du\,ds$$

which in turn, yields

$$1 \geq \frac{1}{(n-l-1)!.(l-1)!}\int_{g(t)}^t (t-s)^{l-1}\int_{g^{-1}(g^{-1}(s))}^\infty (u-s)^{n-l-1} p(u)\,du\,ds.$$

Taking limit sup., we obtain a contradiction. Consequently, (1.1) has property A. Hence the theorem is proved. □

Theorem 2.4. *Let $g(t) < t$ and*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} (s-g(t))^{n-1} p(s)\, ds > (n-1)!. \tag{2.10}$$

Then (1.1) *has no solution satisfying the property $(-1)^i y^{(i)}(t) > 0$ for large t.*

Proof. If possible, suppose that (1.1) has a nonoscillatory solution $y(t)$ satisfying the property $(-1)^i y^{(i)}(t) > 0$ for large t. Then $l = 0$ in (1.2). Suppose that $y(g(t)) > 0$ and $y(t) > 0$ for some $t \geq t_1 > \sigma$. From Lemma 1.2 due to Kiguradze and Chanturia [10], it follows for $i = 0$, that

$$\begin{aligned} y(t) &\geq \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} p(s) y(g(s))\, ds \\ &\geq \frac{1}{(n-1)!} \int_t^{g^{-1}(t)} (s-t)^{n-1} p(s) y(g(s))\, ds \\ &\geq \frac{y(t)}{(n-1)!} \int_t^{g^{-1}(t)} (s-t)^{n-1} p(s)\, ds, \end{aligned}$$

that is,

$$(n-1)! \geq \int_t^{g^{-1}(t)} (s-t)^{n-1} p(s)\, ds,$$

for some $t \geq t_2 \geq t_1$. Then there exists a $t_3 \geq t_2$ such that $g(t) > t_2$ for $t \geq t_3$. Hence for $t \geq t_3$, we have

$$(n-1)! \geq \int_{g(t)}^{t} (s-g(t))^{n-1} p(s)\, ds.$$

Taking limit sup., we obtain a contradiction. Hence $l \neq 0$. The theorem is proved. $\square$

Corollary 2.5. *Suppose that $g(t) < t$,* (2.10) *holds and either* (2.1) *or* (2.4) *or* (2.6)*is satisfied. Then every solution of* (1.1) *oscillates.*

Example 2.6. Consider

$$y'''(t) + \frac{30}{t^3} y(t/2^{1/3}) = 0, \quad t \geq 2. \tag{2.11}$$

By Theorem 2.2, (2.11) has property A. In particular, $y(t) = 1/t^3$ is a nonoscillatory solution of (2.11).

Example 2.7. Consider

$$y'''(t) + \frac{82}{t^3} y(t/3) = 0, \quad t \geq 1. \tag{2.12}$$

Theorem 2.1 can be applied to this example where as Theorem 2.3 fails to hold. On the other hand, (2.10) is satisfied. Hence by Corollary 2.5, all solutions of (2.12) are oscillatory.

Example 2.8. *Inequality* (2.6) *to the equation*

$$y'''(t) + \frac{63}{t^3} y(t/2) = 0, \quad t \geq 1. \tag{2.13}$$

is satisfied, where as (2.1) *fails to hold. Hence Theorem 2.3 can be applied to* (2.13). *Further, since,* (2.10) *is satisfied, then all solutions of* (2.13) *are oscillatory, by Corollary 2.5.*

Remark: Let $p(t) = p > 0$ be a constant and $g(t) = t - \tau$, $\tau > 0$ be a constant. Then (1.1) becomes

$$y^{(n)}(t) + py(t-\tau) = 0. \tag{2.14}$$

Clearly, the conditions of (2.1),(2.4) and (2.6) are consistent with $p(t) = p$ and $g(t) = t - \tau$. Hence form Corollary 2.5, it follows that, if

$$p\tau^n > n!, \tag{2.15}$$

then (2.14) is oscillatory.

The characteristic equation associated with (2.14) is given by

$$\lambda^n + pe^{-\tau\lambda} = 0. \tag{2.16}$$

Setting $F(\lambda) = \lambda^n + pe^{-\tau\lambda}$, we see that $F(\lambda) > 0$ for $\lambda \geq 0$. Suppose that $\lambda < 0$. We claim that $F(\lambda) > 0$ for $\lambda < 0$. If possible suppose that $F(\lambda) \leq 0$ for $\lambda < 0$. Then $\lambda^n \leq -pe^{-\tau\lambda}$. Then $\lambda^n\tau^n \leq -n!.e^{-\tau\lambda}$. If n is even, then $\lambda^n\tau^n \leq 0$, a contradiction. Hence n must be odd. Let $\lambda = -\gamma, \gamma > 0$. Then $\gamma^n\tau^n \geq n!.e^{\tau\gamma}$. Setting $\tau\gamma = \beta$, we see that $\beta^n \geq n!.e^{\beta}$, a contradiction. Hence our claim holds, that is, $F(\lambda) > 0$ for $\lambda < 0$. Thus (2.15) implies that all solutions of (2.14) are oscillatory.

Remark: Although the conditions in Theorems 2.1 and 2.1 are legitimate, these are not efficient. When $g(t)$ is close to t, the conditions (2.1) and (2.6) fails to hold. This is evident from the following examples : If we replace $g(t) = \frac{t}{3}$ in (2.12) by $g(t) = \frac{3t}{4}$, then the equation becomes

$$y'''(t) + \frac{82}{t^3}y(\frac{3t}{4}) = 0, t \geq 1. \tag{2.17}$$

Condition (2.1) fails to hold and hence Theorem 2.1 cannot be applied to (2.17). Similarly,consider the equation

$$y'''(t) + \frac{46}{t^3}y(\frac{t}{2}) = 0, t \geq 1. \tag{2.18}$$

Theorem 2.3 can be applied to this example. On the other hand, if $g(t) = \frac{t}{2}$ in (2.18) is replaced by $g(t) = \frac{10t}{11}$, then (2.18) becomes

$$y'''(t) + \frac{46}{t^3}y(\frac{10t}{11}) = 0, t \geq 1, \tag{2.19}$$

then(2.6) fails and hence Theorem 2.3 cannot be applied. The following theorems provides sufficient conditions for (1.1) to have property A when $g(t)$ is close to t.

Theorem 2.9. *Assume that $g(t) < t$ and $t - g(t) \to \infty$ as $t \to \infty$.If, for every $l \in \{1, 2, \dots, n-1\}$ such that $n + l$ is odd,*

$$\limsup_{t\to\infty}(g(t))^l \int_{g^{-1}(t)}^{\infty} (s-t)^{n-l-1}p(s)\,ds > (n-l-1)!.l! \tag{2.20}$$

holds, then (1.1) has property A.

Proof. We can find a $t_2 > t_1$ such that $t - g(t) > t_1$ for $t \geq t_2$. Hence for $t \geq t_2$, (2.2) gives

$$y(t) \geq \frac{y^{(l)}(t)}{(l-1)!}\int_{t-g(t)}^{t}(t-s)^{l-1}\,ds \geq \frac{g^l(t)}{l!}y^{(l)}(t).$$

using (1.6)in the above inequality, we obtain a contradiction. The proof is complete. $\square$

Corollary 2.10. *Suppose that the conditions of Theorems 2.4 and 2.9 are satisfied. then all solutions of* (1.1) *oscillates.*

Example 2.11. By Theorem 2.9, (2.17) has property A.

Theorem 2.12. *Let* $g(t) < t$ *and* $t - g(t) \to \infty$ *as* $t \to \infty$.*If for every* $l \in \{1, 2, \dots, n-1\}$ *with* $n + l$ *odd,*

$$\limsup_{t\to\infty} \int_{t-g(t)}^{t} (t-s)^{l-1} \int_{g^{-1}(g^{-1}(s))}^{\infty} (u-s)^{n-l-1} p(u)\,du\,ds > (l-1)!.(n-l-1)! \tag{2.21}$$

holds, then (1.1) *has property A.*

Proof. Proceeding as in the proof of Theorem 2.3, we arrive at (2.9) for $t \geq t_1$. Then we can find a $t_2 \geq t_1$ such that $t - g(t) > t_1$ for $t \geq t_2$. Hence from (2.9), we obtain

$$y(t) \geq \frac{1}{(n-l-1)!.(l-1)!} \int_{t-g(t)}^{t} (t-s)^{l-1} \int_{g^{-1}(g^{-1}(s))}^{\infty} (u-s)^{n-l-1} p(u) y(g(u))\,du\,ds$$

which further yields

$$1 \geq \frac{1}{(n-l-1)!.(l-1)!} \int_{t-g(t)}^{t} (t-s)^{l-1} \int_{g^{-1}(g^{-1}(s))}^{\infty} (u-s)^{n-l-1} p(u)\,du\,ds.$$

Taking limit sup. both sides in the above inequality, we obtain a contradiction. This completes the proof of the theorem. □

Corollary 2.13. *Suppose that the conditions of Theorem 2.4 and 2.12 are satisfied. Then all solutions of* (1.1) *are oscillatory.*

Example 2.14. By Theorem 2.12, (2.19) has property A.

Let $y(t)$ be a nonoscillatory solution of (1.1) such that (2.2) holds for $t \geq t_1$. Then for $t > t_2 \geq 2t_1$, (2.2) gives

$$y(t) \geq \frac{1}{(l-1)!} \int_{t/2}^{t} (t-s)^{l-1} y^{(l)}(s)\,ds, \quad t \geq t_1.$$

Using (1.6) and the above inequality, we obtain the following theorem.

Theorem 2.15. *Let* $g(t) \leq t$. *If for every* $l \in \{1, 2, \dots, n-1\}$ *such that* $n + l$ *is odd,*

$$\limsup_{t\to\infty} t^l \int_{g^{-1}(t)}^{\infty} (s-t)^{n-l-1} p(s)\,ds > (n-l-1)!.l!.2^l$$

holds, then (1.1) *has property A.*

Theorem 2.16. *Let* $g(t) \leq t$ *and for every* $l \in \{1, 2, \dots, n-1\}$ *such that* $n + l$ *is odd,*

$$\limsup_{t\to\infty} \int_{t/2}^{t} (t-s)^{l-1} \int_{g^{-1}(g^{-1}(s))}^{\infty} (u-s)^{n-l-1} p(u)\,du\,ds > (l-1)!.(n-l-1)! \tag{2.22}$$

holds, then (1.1) *has property A.*

Proof. Proceeding as in the proof of Theorem 2.3, we obtain (2.9). Then for $t \geq t_2 > 2t_1$, (2.9) yields a contradiction. This completes the proof of the theorem. □

We note that when $g(t) = t/2$, then Theorem 2.3, 2.12 and 2.16 give same sufficient conditions to have property A of (1.1).

Corollary 2.17. *Suppose that the conditions of Theorem 2.4 are satisfied. If either of the conditions of Theorem 2.15 or 2.16 hold, then all solutions of* (1.1) *are oscillatory.*

Example 2.18. Consider

$$y'''(t) + \frac{44}{t^3} y(\frac{3t}{5}) = 0, t \geq 1.$$

Theorem 2.1 and Theorem 2.9 can be applied to this example, whereas Theorem 2.15 cannot be applied to this example.

Example 2.19. Consider

$$y'''(t) + \frac{160}{t^3} y(\frac{t}{3}) = 0, t \geq 1.$$

By Theorem 2.15 this equation has property A, whereas Theorem 2.9 fails.

Theorem 2.20. *Let $g'(t) > 0$. If for every $l \in \{1, 2, 3, \dots, n-1\}$ such that $n + l$ is odd,*

$$\int^{\infty} H_l(t)\, dt = \infty, \tag{2.23}$$

then then for n even every solution of (1.1) *oscillates and for n odd every solution of* (1.1) *is either oscillates or tend to zero as $t \to \infty$,in particular,* (1.1) *has property A, where*

$$H_{n-1}(t) = t^{n-1} p(t) - \frac{(n-1)!.(n-1)2^{n-4} t^{n-3}}{g'(t) g^{n-2}(t)} \tag{2.24}$$

and

$$H_l(t) = \frac{t^l}{(n-l-2)!} \int_t^{\infty} (s-t)^{n-l-2} p(s)\, ds - \frac{l!.l.2^{l-3} t^{l-2}}{g'(t) g^{l-1}(t)}, \tag{2.25}$$

for $l = 1, 2, 3, \dots, n-2$.

Remark: Let $g(t) = t$ and $n = 2$. From Theorem 2.20, it follows that, if

$$\int^{\infty} [tp(t) - \frac{1}{4t}]\, dt = \infty, \tag{2.26}$$

then

$$y'' + p(t) y = 0 \tag{2.27}$$

is oscillatory. This gives a partial answer to Problem 1.1. Further, our result improves the results due to Kneser [16, pp.45] and Hille and Kneser [16, Theorem 2.41]. We note that Theorem 2.20 holds for (1.1) with $g(t) = t$ for $n = 2$ and $n = 3$. however, the theorem cannot be applied to higher order ordinary differential equations, viz., (1.1) with $g(t) = t$ and $n \geq 4$, because of the conditions (2.23) and (2.25). Now, suppose that $n = 3$ and $g(t) = t$. then Theorem 2.20 yields that, if

$$\int^{\infty} [t^2 p(t) - \frac{2}{t}]\, dt = \infty,$$

then

$$y''' + p(t) y = 0 \tag{2.28}$$

has property A. On the other hand, from Hanan [9, Theorem 5.7], and Kiguradze and Chanturia [10, Theorem 1.1], it follows that (2.28) has property A if

$$\int^{\infty}[t^2p(t)-\frac{2}{3\sqrt{3}t}]\,dt=\infty. \tag{2.29}$$

hence Theorem 2.20 is yet to be improved.

Proof of Theorem 2.20. If possible, suppose that (1.1) dose not have property A. Then (1.1) admits a nonoscillatory solution $y(t)$ such that $y\in N_l$ where $l\in\{1,2,3,\dots,n-1\}$. We may assume, without any loss of generality, that $y(t)>0$ and $y(g(t))>0$ for $t\geq t_1>\sigma$. Clearly, (1.2) holds, where $l\in\{0,1,2,3,\dots,n-1\}$ and $n+l$ odd.

Let $l=n-1$. Set $z(t)=\frac{t^{n-1}y^{(n-1)}(t)}{y(g(t))}$. Then

$$z'(t)=-t^{n-1}p(t)+\frac{n-1}{t}z(t)-g'(t)\frac{y'(g(t))}{y(g(t))}z(t). \tag{2.30}$$

Putting $i=1$, $l=n-1$ in (1.5), we obtain, for $t\geq t_1$

$$y'(t)\geq\frac{1}{(n-2)!}(t-t_1)^{n-2}y^{(n-1)}(t).$$

Hence for $t\geq 2t_1$, we get

$$y'(t)\geq\frac{t^{n-2}}{(n-2)!.2^{n-2}}y^{(n-1)}(t).$$

Thus, for $t\geq t_2>2t_1$,

$$y'(g(t))\geq\frac{(g(t))^{n-2}}{(n-2)!.2^{n-2}}y^{(n-1)}(t)$$

Using the above inequality, (2.30) yields

$$z'(t)\leq -F_{n-1}(t), \tag{2.31}$$

where

$$F_{n-1}(t)=t^{n-1}p(t)-\frac{n-1}{t}z(t)+\frac{g'(t)(g(t))^{n-2}}{(n-2)!2^{n-2}t^{n-1}}z^2(t),$$

which as a function of z, attains the minimum $H_{n-1}(t)$ given in (2.24). Now, the integration of (2.31) from t_2 to t yields $z(t)<0$ for large t, a contradiction. Next, suppose that $l\in\{1,2,3,\dots,n-2\}$. Setting $z_1(t)=\frac{t^l y^{(l)}(t)}{y(g(t))}$, $t\geq t_1$, we see that $z_1(t)>0$ for $t\geq t_1$ and

$$z_1'(t)=\frac{t^l y^{(l+1)}(t)}{y(g(t))}+\frac{l}{t}z_1(t)-g'(t)\frac{y'(g(t))}{y(g(t))}z_1(t). \tag{2.32}$$

Putting $i=1$ in (1.5), we get

$$y'(t)\geq\frac{1}{(l-1)!}(t-t_1)^{l-1}y^{(l)}(t).$$

Thus, for $t\geq t_2\geq 2t_1$,

$$y'(t)\geq\frac{1}{(l-1)!.2^{l-1}}t^{l-1}.y^{(l)}(t).$$

We can find a $t_3 > t_2$ such that $g(t) > t_2$ for $t \geq t_3$. Hence

$$y'(g(t)) \geq \frac{1}{(l-1)!.2^{l-1}}(g(t))^{l-1}y^{(l)}(g(t)) > \frac{1}{(l-1)!.2^{l-1}}(g(t))^{l-1}y^{(l)}(t) \tag{2.33}$$

for $t \geq t_3$. Putting $i = l+1, k = n$ and $s > t \geq t_3$ in the inequality

$$y^{(i)}(t) = \sum_{j=i}^{k-1} \frac{(t-s)^{j-i}}{(j-i)!}y^{(j)}(s) + \frac{1}{(k-i-1)!}\int_s^t (t-u)^{k-i-1}y^{(k)}(u)\,du, \tag{2.34}$$

and letting $s \to \infty$, we obtain

$$y^{(l+1)}(t) \leq -\frac{y(g(t))}{(n-l-2)!}\int_t^\infty (s-t)^{n-l-2}p(s)\,ds. \tag{2.35}$$

Making the use of (2.33) and (2.35) in (2.32), we have

$$z_1'(t) \leq -F_l(t), \tag{2.36}$$

where

$$F_l(t) = \frac{g'(t).g^{l-1}(t)}{(l-1)!.2^{l-1}.t^l}z_1^2(t) - \frac{l}{t}z_1(t) + \frac{t^l}{(n-l-2)!}\int_t^\infty (s-t)^{n-l-2}p(s)\,ds,$$

which as a function of z_1, attains the minimum $H_l(t)$ given in (2.25). In view of the conditions (2.23) and (2.25), integration of (2.36) yields a contradiction. Hence (1.1) has property A, that is $l = 0$ for $t \geq t_2 \geq t_1$.. Thus the theorem is proved when n is even. Now $l = 0$ implies that n is odd. Our theorem will be proved if we can show that $y(t) \to 0$ as $t \to \infty$. Since $l = 0$ then $\lim y(t) = \lambda, 0 \leq \lambda < \infty$ exists. We claim that $\lambda = 0$. If not, them for $0 < \epsilon < \lambda$, there exists a $t_3 \geq t_2$ such that $y(g(t)) > \lambda - \epsilon$ for $t \geq t_3$. Now putting $i = 0, k = n$ and $s > t = t_3$ and letting $s \to \infty$ in (2.34), we obtain

$$y(t_3) > (\lambda - \epsilon)\int_{t_3}^\infty (u-t_3)^{n-1}p(u)\,du$$

which further gives

$$\int_{t_3}^\infty (u-t_3)^{n-1}p(u)\,du < \infty. \tag{2.37}$$

On the other hand, the condition (2.23) with $l = n-1$ yields that $\int_{t_3}^\infty t^{n-1}p(t)\,dt = \infty$ which contradicts to (2.37). Hence $\lambda = 0$. This completes the proof of the theorem. $\square$

Example 2.21. Consider

$$y'''(t) + \frac{24(t-1)^2}{t^5}y(t-1) = 0, \quad t \geq 2. \tag{2.38}$$

All the conditions of Theorem 2.20 are satisfied. Hence (2.38) has property A. In particular, $y(t) = 1/t^2$ is a nonoscillatory solution of (2.38).

Corollary 2.22. *Suppose that the conditions of Theorems 2.4 and 2.20 are satisfied. Then all solutions of* (1.1) *are oscillatory.*

Now, we consider the following ordinary differential equations associated with the delay differential equations (2.11), (2.12), (2.13), and (2.38).

$$y''' + \frac{30}{t^3} y = 0, \quad t \geq 2. \tag{2.39}$$

$$y''' + \frac{82}{t^3} y = 0, \quad t \geq 1. \tag{2.40}$$

$$y''' + \frac{63}{t^3} y = 0, \quad t \geq 1. \tag{2.41}$$

$$y''' + \frac{24(t-1)^2}{t^3} y = 0, \quad t \geq 2. \tag{2.42}$$

From Hanan [9, Theorem 5.7], it follows that (2.39)-(2.42) are oscillatory. We note that a third order ordinary differential equation is said to be oscillatory if it has an oscillatory solution ; otherwise,it is called nonoscillatory. However, all solutions of (2.39)-(2.42)are not oscillatory. This is because, (2.39)-(2.42) are of Class I or C_I and hence admits a nonoscillatory solution (see Lemma 2.2 and Theorem 3.1 in [14]). We may note that Eq.(2.28) is said to be of Class I or C_I if any of its solution $y(t)$ for which $y(t_0) = y'(t_0) = 0$ and $y''(t_0) > 0, (\sigma < t_0 < \infty)$ satisfies $y(t) > 0$ for $t \in [\sigma, t_0)$. It seems that the presence of delay in (2.12) and (2.13) is responsible for the change in the qualitative behaviour of solutions of the equations. It is easy to construct an example of a third order delay differential equation all solutions of which are oscillatory but it is not difficult to construct such an example of a third order ordinary differential equation. It is evident from the following examples due to Dolan [3] and Parhi and the author [13] respectively.

Example 2.23. *Dolan* [3]*]* All solutions of

$$\{[z' - \frac{r'(t)}{r(t)} z] + r(t) z\}' = 0$$

are oscillatory, where $r(t) = [1 + \sqrt{2}\epsilon \sin(t + \frac{\pi}{4})]^{-1} > 0$, $t \geq 0, 0 < \epsilon < \frac{1}{\sqrt{2}}$.

To the best of the authors knowledge, the following is the only explicit example of which all solutions are oscillatory.

Example 2.24 (Parhi and Padhi [13])**.** All solutions of

$$y''' - y'' + \big(\frac{1}{1.0000004} + \frac{1}{t}\big) y' - \frac{k}{t^2} y = 0, \quad t \geq 2$$

are oscillatory, where k is a constant.

Theorem 2.25. *Let $n \geq 3$. Suppose that for any $\mu \in (0, 1/2)$, each of the the third order ordinary differential equation*

$$u''' + G_l(t) u = 0, \quad i \in \{1, 2, \dots, n-1\}, \ n + l \text{ odd} \tag{2.43}$$

admits an oscillatory solution, where

$$G_{n-1}(t) = \frac{\mu}{(n-3)!} (g(t) - g(g(t)))^{n-3} \big(\frac{g(t)}{t}\big)^2 p(t) \tag{2.44}$$

and

$$\begin{aligned} G_l(t) = &\frac{\mu}{(n-l-2)!.(l-1)!} \Big(\int_t^{g^{-1}(t)} (s-t)^{n-l-2} p(s)\, ds \Big) \\ &\times \big(g(t) - g(g(t))\big)^{l-2} \big(\frac{g(g(t))}{t}\big)^2 \end{aligned} \tag{2.45}$$

for $l \in \{1, 2, 3, \dots, n-2\}$. *Then* (1.1) *has property A.*

Proof. If (1.1) has not property A, then it admits a non-oscillatory solution $y(t)$ such that (1.2) is satisfied for $l \in \{1, 2, 3, \dots, n-1\}$. We may assume, without any loss of generality, that $y(t) > 0$ and $y(g(t)) > 0$ for some $t \geq t_1 > t_0 > \sigma$. Let $l = n-1$. Setting $x(t) = y^{(n-3)}(t)$, we see that $x(t) > 0, x'(t) > 0, x''(t) > 0$ and $x'''(t) < 0$ for $t \geq t_2 \geq t_1$. For any $\mu \in (0, 1/2)$, there exists a $T\_\mu \geq> t_2$ such that

$$\frac{x(g(t))}{x(t)} \geq \mu\Big(\frac{g(t)}{t}\Big)^2 \tag{2.46}$$

for $t \geq T_\mu$ (See Theorem 2.2 in [5]). Setting $z(t) = x'(t)/x(t)$ for $t \geq T_\mu$, we get

$$z'(t) = \frac{x''(t)}{x(t)} - z^2(t). \tag{2.47}$$

Further, assuming $u(t) = \exp\big(\int_{T_\mu}^t z(s)\,ds\big)$ and using (2.46), (2.47) and the inequality

$$y(t) \geq \frac{y^{(n-3)}(t)}{(n-3)!}(t - g(t))^{n-3},$$

we obtain

$$u'''(t) + \frac{\mu}{(n-3)!}(g(t) - g(g(t)))^{n-3}\Big(\frac{g(t)}{t}\Big)^2 p(t)u(t) \leq 0$$

for $t \geq T_\mu$. From Lemma 4 in [7], it follows that (2.43) with $l = n-1$ is disconjugate on $[T_\mu, \infty)$, a contradiction.

Next let $l \in \{1, 2, 3, \dots, n-2\}$. Putting $i = l+1, k = n$ and $s = g^{-1}(t) > t_1$ in (2.34), we get

$$y^{(l+1)}(t) + \frac{1}{(n-l-2)!}\Big(\int_t^{g^{-1}(t)} (s-t)^{n-l-2} p(s)\,ds\Big) y(g(t)) \leq 0.$$

for $t \geq T \geq t_1$, which further gives, for $t \geq T$

$$\begin{aligned} y^{(l+1)}(t) + \frac{1}{(n-l-2)!.(l-1)!}&\Big(\int_t^{g^{-1}(t)} (s-t)^{n-l-2} p(s)\,ds\Big) \\ &\times (g(t) - g(g(t)))^{l-2} y^{(l-2)}(w(t)) \leq 0 \end{aligned} \tag{2.48}$$

where $g(g(t)) = w(t)$. Let $x_1(t) = y^{(l-2)}(t)$. Then $x_1(t) > 0$, $x_1'(t) > 0$, $x_1''(t) > 0$ and $x_1'''(t) < 0$ for $t \geq T$ and hence we can find a $t \geq T_\mu > T$ such that

$$\frac{x_1(w(t))}{x_1(t)} \geq \mu\Big(\frac{w(t)}{t}\Big)^2;$$

that is,

$$\frac{y^{(l-2)}(w(t))}{y^{(l-2)}(t)} \geq \mu\Big(\frac{w(t)}{t}\Big)^2, \tag{2.49}$$

for $t \geq T_{\mu'}$. Then $z_1'(t) = \frac{x_1''(t)}{x_1(t)} - z_1^2(t)$. Further, setting $v(t) = e^{\left(\int_{T_{\mu'}}^t z_1(s)\,ds\right)}$ and using (2.49),(2.48) gives

$$\begin{aligned} v'''(t) + \frac{\mu}{(n-l-2)!.(l-1)!}&\Big(\int_t^{g^{-1}(t)} (s-t)^{n-l-2} p(s)\,ds\Big) \\ &\times (g(t) - g(g(t)))^{l-2}\Big(\frac{w(t)}{t}\Big)^2 v(t) \leq 0 \end{aligned}$$

for $t \geq T_{\mu'}$.This in turn implies that (2.43) is disconjugate, by in [7, Lemma 4], a contradiction to the hypothesis of the theorem for the case $l \in \{1, 2, 3, \dots, n-2\}$. Hence (1.1) has property A. This completes the proof of the theorem. □

Corollary 2.26. *Suppose that $g(t) < t$, $n \geq 3$. If all the conditions of Theorems 2.4 and 2.25 are satisfied, then all solutions of* (1.1) *are oscillatory.*

Example 2.27. Consider

$$y'''(t) + e^{-1} y(t-1) = 0, \quad t \geq 2. \tag{2.50}$$

As $\liminf_{t\to\infty} \mu e^{-1} t(t-1)^2 > \frac{2}{3\sqrt{3}}$, then, for every $\mu \in (0, 1/2)$, the equation

$$u''' + \mu e^{-1} \big(\frac{t-1}{t}\big)^2 u = 0, \quad t \geq 2$$

admits an oscillatory solution by Theorem 5.7 of [9]. from Theorem 2.25, it follows that (2.50) has property A. In particular, $y(t) = e^{-t}$ is a solution of (2.50) for $t \geq 2$.

Remark: Consider Equations (2.12) and (2.13). For $0 < \mu < \frac{\sqrt{3}}{82}$, it follows that $\lim_{t\to\infty} t^3 \mu \frac{82}{9t^3} < \frac{2}{\sqrt{3}}$ and hence $u''' + \mu \frac{82}{9t^3} u = 0$ is nonoscillatory, by [9, Theorem 5.7]. Similarly, for $0 < \mu < \frac{4}{189\sqrt{3}}$, the equation $u''' + \mu \frac{63}{4t^3} u = 0$is nonoscillatory. Hence Corollary 2.26 cannot be applied to (2.12) and (2.13).

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