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SINGULAR INTEGRALS OF THE TIME-HARMONIC RELATIVISTIC DIRAC EQUATION ON A PIECEWISE LIAPUNOV SURFACE

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Abstract. We give a short proof of a formula of Poincaré-Bertrand in the setting of time-harmonic solutions of the relativistic Dirac equation on a piecewise Liapunov surface, as well as for some versions of quaternionic analysis.

1. Introduction

Let Γ be a closed Liapunov curve in the complex plane and let f be a Hölder function on $\Gamma \times \Gamma$. Then, everywhere on Γ,

$$\begin{aligned} &\frac{1}{\pi i}\int_{\Gamma_\tau}\frac{d\tau}{\tau - t}\cdot\frac{1}{\pi i}\int_{\Gamma_{\tau_1}}\frac{f(\tau,\tau_1)d\tau_1}{\tau_1-\tau} \\ &= f(t,t) + \frac{1}{\pi i}\int_{\Gamma_{\tau_1}} d\tau_1 \cdot \frac{1}{\pi i}\int_{\Gamma_\tau}\frac{f(\tau,\tau_1)d\tau}{(\tau-t)(\tau_1-\tau)}, \end{aligned} \tag{1.1}$$

which is usually called the Poincaré-Bertrand formula, the integrals being understood in the sense of the Cauchy principal value. The Poincaré-Bertrand formula plays a significant role in the theory of one-dimensional singular integral equations with the Cauhy kernel and its numerous applications. Indeed, all the integrals in (1.1) contain the (singular) Cauchy kernel, and its importance for one-dimensional complex analysis is obvious.

It is known that the theory of solutions of the Dirac equation reduces, in some degenerate cases, to that of complex holomorphic functions. Hence, one may consider the former to be a generalization of the latter. At the same time, not many facts from the holomorphic function theory have their extensions onto the Dirac equation theory. In the present paper we study a number of generalization of (1.1). In realizing this study we follow the approach first presented in [3] and developed in [4], [6], [9] which are based on the intimate relation between time-harmonic bispinor fields and quaternion-valued α-hyperholomorphic functions, see the book [4]. This approach proved to be quite efficient and heuristic since it allows the exploitation of profound similarity between holomorphic functions in one variable and α-hyperholomorphic functions.

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The paper is organized as follows. In Section 2 the reader can find the Poincaré-Bertrand formula for time-harmonic Dirac bispinors, i.e, time-harmonic solutions of the relativistic Dirac equation. The proof can be found in the Section 6, and is based on the contents of Sections 3-5. In section 4 we present the Poincaré-Bertrand formula for α-hyperholomorphic quaternionic function theory on a piece-wise Liapunov surface.

Note that the Poincaré-Bertrand formula on closed piece-wise smooth manifold in $\mathbb{C}^n$ for Bochner-Martinelli type singular integrals was studied, for example, by Liangyu Lin and Chunhui Qiu [5].

2. Time-harmonic bispinor fields theory and the Cauchy-Dirac integral

Let Ω be a domain in $\mathbb{R}^3$, $\Gamma := \partial\Omega$ be its boundary. We consider the following *Dirac equation* for a free massive particle of spin $\frac{1}{2}$:

$$\mathbb{D}[\Phi] := \Big(\gamma_0\partial_t - \sum_{k=1}^{3}\gamma_k\partial_k + im\Big)[\Phi] = 0,$$

where the Dirac matrices have the standard Dirac-Pauli form

$$\gamma_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_1 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 := \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

and where $\partial_t := \frac{\partial}{\partial t}$; $\partial_k := \frac{\partial}{\partial x_k}$, $m \in \mathbb{R}$, $\Phi : \mathbb{R}^4 \to \mathbb{C}^4$. Suppose that the spinor field Φ is time-harmonic (= monochromatic):

$$\Phi(t, x) = q(x)e^{i\omega t},$$

where $\omega \in \mathbb{R}$ is the frequency and $q : \Omega \subset \mathbb{R}^3 \to \mathbb{C}^4$ is the amplitude. Then the relativistic Dirac equation is equivalent to the *time-harmonic Dirac equation*:

$$\mathbb{D}_{\omega,m}[q] := \Big(i\omega\gamma_0 - \sum_{k=1}^{3}\gamma_k\partial_k + im\Big)[q] = 0.$$

This is the equation which we are going to consider. We shall consider certain objects related to it in a bounded domain. Physical phenomena which gave rise to the Dirac equation occur usually in unbounded domains but some of them (the Casimir effect, for instance) take place in bounded domains also. For more details see, e.g. [4].

The integral

$$K_{\mathbb{D}_{\omega,m}}[g](x) := -\int_\Gamma \check{\mathcal{K}}^x_{\mathbb{D}_{\omega,m}}[\sigma_{\mathbb{D}_{\omega,m}} g(\tau)], \quad x \notin \Gamma,$$

plays the role of the Cauchy-type integral in the theory of time-harmonic bispinor fields with $g : \Gamma \to \mathbb{C}^4$(see [6]) and we shall call it the Cauchy-Dirac-type integral, where

$$\sigma_{\mathbb{D}_{\omega,m}} := \frac{1}{2}\begin{pmatrix} (n_2 - in_1) & in_3 & in_3 & (n_2 + in_1) \\ -n_3 & i(n_2 + in_1) & i(n_2 - in_1) & -n_3 \\ -in_3 & -(n_2 + in_1) & (n_2 - in_1) & in_3 \\ -i(n_2 - in_1) & n_3 & -n_3 & i(n_2 + in_1) \end{pmatrix} dS,$$

$\vec{n}(\tau) = (n_1(\tau), n_2(\tau), n_3(\tau))$ is an outward pointing normal unit vector on Γ at $\tau \in \Gamma$, and dS is an element of the surface area in $\mathbb{R}^3$. The explicit form of time-harmonic relativistic Cauchy-Dirac kernel $\check{\mathcal{K}}^x_{\mathbb{D}_{\omega,m}}$ can be seen, e.g. in reference [6].

Let $H_\mu(\Gamma, \mathbb{C}^4) := \{f \in \mathbb{C}^4 : |f(t_1) - f(t_2)| \le L_f \cdot |t_1 - t_2|^\mu; \forall\{t_1, t_2\} \subset \Gamma, L_f = \text{const}\}$ denote the class of functions satisfying the Hölder condition with the exponent $0 < \mu \le 1$. Here $|f|$ means the Euclidean norm in $\mathbb{C}^4 = \mathbb{R}^8$ while $|t|$ is the Euclidean norm in $\mathbb{R}^3$. Let Γ be a surface in $\mathbb{R}^3$ which contains a finite number of conical points and a finite number of non-intersecting edges such that none of the edges contains any of conical points. If the complement (in Γ) of the union of conical points and edges, is a Liapunov surface, then we shall refer to Γ as a piece-wise Liapunov surface in $\mathbb{R}^3$.

Theorem 2.1 (Poincaré-Bertrand formula for time-harmonic bispinor field theory on a piece-wise Liapunov surface). *Let Ω be a bounded domain in $\mathbb{R}^3$ with the piece-wise Liapunov boundary. Let $q \in H_\mu(\Gamma \times \Gamma, \mathbb{C}^4)$, $0 < \mu < 1$. Then the following equality holds, everywhere on Γ:*

$$\begin{aligned} &\int_{\Gamma_{\tau_1}} \int_{\Gamma_\tau} \check{\mathcal{K}}^t_{\mathbb{D}_{\omega,m}} \Big[\sigma_{\mathbb{D}_{\omega,m,\tau}} \check{\mathcal{K}}^\tau_{\mathbb{D}_{\omega,m}} [\sigma_{\mathbb{D}_{\omega,m,\tau_1}} q(\tau_1, \tau)]\Big] + \frac{1-\gamma(t)}{2} q(t,t) \\ &= \int_{\Gamma_\tau} \check{\mathcal{K}}^t_{\mathbb{D}_{\omega,m}} \Big[\sigma_{\mathbb{D}_{\omega,m,\tau}} \int_{\Gamma_{\tau_1}} \check{\mathcal{K}}^\tau_{\mathbb{D}_{\omega,m}} [\sigma_{\mathbb{D}_{\omega,m,\tau_1}} q(\tau_1, \tau)]\Big], \end{aligned} \tag{2.1}$$

where the integrals being understood in the sense of the Cauchy principal value, $\gamma(t) := \frac{\eta(t)}{4\pi}$; $\eta(t)$ is the measure of a solid angle of the tangential conical surface at the point t or is the solid measure of the tangential dihedral angle at the point t.

The proof will be presented in Section 6. Note that if Γ is a Liapunov surface, then formula (2.1) coincides with the result in paper [6].

3. Basic facts of hyperholomorphic function theory

In this section, we provide some background on quaternionic analysis needed in this paper. For more information, we refer the reader to [1], [4].

Let $\mathbb{H}(\mathbb{C})$ be the set of complex quaternions, it means that each quaternion a is represented in the form $a = \sum_{k=0}^3 a_k i_k$, with the standard basis $\{i_0 := 1, i_1, i_2, i_3\}$, where $\{a_k : k \in \mathbb{N}_3^0 := \mathbb{N}_3 \cup \{0\};\ \mathbb{N}_3 := \{1, 2, 3\}\} \subset \mathbb{C}$. We use the Euclidean norm $|a|$ in $\mathbb{H}(\mathbb{C})$, defined by $|a| := \sqrt{\sum_{k=0}^3 |a_k|^2}$.

Let $\lambda \in \mathbb{H}(\mathbb{C}) \setminus \{0\}$, and let α be its complex-quaternionic square root: $\alpha \in \mathbb{H}(\mathbb{C})$, $\alpha^2 = \lambda$. The function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{H}(\mathbb{C})$ is called *left-α-hyperholomorphic* if

$$D_\alpha f := f\alpha + i_1 \frac{\partial}{\partial x_1} f + i_2 \frac{\partial}{\partial x_2} f + i_3 \frac{\partial}{\partial x_3} f = 0.$$

Let $\alpha \in \mathbb{H}(\mathbb{C})$ and let θ_α be the fundamental solution of the Helmholtz operator $\Delta_\lambda := \Delta + I\lambda$, where $\Delta := \sum_{k=1}^3 \frac{\partial^2}{\partial x_k^2}$ and I is the identity operator. Then the fundamental solution of the operator D_α, $\mathcal{K}_\alpha$, is given by the formula (see [4]):

$$\mathcal{K}_\alpha(x) := -D_\alpha \theta_\alpha(x),$$

and its explicit form can be seen, e.g., in [10]. We shall use the notation $C^p(\Omega, \mathbb{H}(\mathbb{C}))$, $p \in \mathbb{N} \cup \{0\}$ which has the usual component-wise meaning. Denote by $\mathfrak{G}$ the set of zero divisors from $\mathbb{H}(\mathbb{C})$, i.e., $\mathfrak{G} := \{a \in \mathbb{H}(\mathbb{C}) \mid a \neq 0;\ \exists b \neq 0 :\ ab = 0\}$. Let $\sigma_\tau = \sum_{k=1}^{3} (-1)^{k-1} i_k dx_{[k]}$, where $dx_{[k]}$ denotes as usual the differential form $dx_1 \wedge dx_2 \wedge dx_3$ with the factor dx_k omitted. Let $\Omega = \Omega^+$ be a domain in $\mathbb{R}^3$ with the boundary Γ which is assumed to be a piece-wise Liapunov surface; denote $\Omega^- := \mathbb{R}^3 \setminus (\Omega^+ \cup \Gamma)$. If f is a Hölder function then its α-hyperholomorphic left Cauchy-type integral is defined (see [4, Subsection 4.16]):

$$K_\alpha[f](x) := -\int_\Gamma \check{\mathcal{K}}_\alpha^x[\sigma_\tau f(\tau)],\ x \in \Omega^\pm,$$

where

(1) If $\alpha = \alpha_0 \in \mathbb{C}$, then

$$\check{\mathcal{K}}_\alpha^x[f](\tau) := \mathcal{K}_{\alpha_0}(x - \tau) f(\tau).$$

(2) If $\alpha \notin \mathfrak{G}$, $\vec{\alpha}^2 \neq 0$, then

$$\check{\mathcal{K}}_\alpha^x[f](\tau) := \frac{1}{2\sqrt{\vec{\alpha}^2}} \mathcal{K}_{\xi_+}(x) f(\tau)(\sqrt{\vec{\alpha}^2} + \vec{\alpha}) + \frac{1}{2\sqrt{\vec{\alpha}^2}} \mathcal{K}_{\xi_-}(x) f(\tau)(\sqrt{\vec{\alpha}^2} - \vec{\alpha}). \tag{3.1}$$

(3) If $\alpha \notin \mathfrak{G}$, $\vec{\alpha}^2 = 0$, then

$$\check{\mathcal{K}}_\alpha^x[f](\tau) := \mathcal{K}_{\alpha_0}(x) f(\tau) + \frac{\partial}{\partial \alpha_0}[\mathcal{K}_{\alpha_0}](x) f(\tau)\vec{\alpha}. \tag{3.2}$$

(4) If $\alpha \in \mathfrak{G}$, $\alpha_0 \neq 0$, then

$$\check{\mathcal{K}}_\alpha^x[f](\tau) := \frac{1}{2\alpha_0} \mathcal{K}_{2\alpha_0}(x) f(\tau)\alpha + \frac{1}{2\alpha_0} \mathcal{K}_0(x) f(\tau)\overline{\alpha}. \tag{3.3}$$

(5) If $\alpha \in \mathfrak{G}$, $\alpha_0 = 0$, then

$$\check{\mathcal{K}}_\alpha^x[f](\tau) := \mathcal{K}_0(x) f(\tau) + \theta_0(x) f(\tau)\alpha. \tag{3.4}$$

For more information about α-hyperholomorphic functions, we refer the reader to [1], [4], [7].

4. The Poincaré-Bertrand formula for α-hyperholomorphic function theory on a piece-wise Liapunov surface

Theorem 4.1 (Poincaré-Bertrand formula for α-hyperholomorphic function theory on a piece-wise Liapunov surface)**.** *Let Ω be a bounded domain in $\mathbb{R}^3$ with piece-wise Liapunov boundary and let $f \in H_\mu(\Gamma \times \Gamma, \mathbb{H}(\mathbb{C}))$. Then the following equality holds everywhere on Γ:*

$$\begin{aligned} &\int_{\Gamma_{\tau_1}} \int_{\Gamma_\tau} \check{\mathcal{K}}_\alpha^t[\sigma_\tau \check{\mathcal{K}}_\alpha^\tau[\sigma_{\tau_1} f(\tau_1, \tau)]] + \frac{1 - \gamma(t)}{2} f(t,t) \\ &= \int_{\Gamma_\tau} \check{\mathcal{K}}_\alpha^t\Big[\sigma_\tau \int_{\Gamma_{\tau_1}} \check{\mathcal{K}}_\alpha^\tau[\sigma_{\tau_1} f(\tau_1, \tau)]\Big]. \end{aligned} \tag{4.1}$$

Proof. Let Γ be as above and $f \in H_\mu(\Gamma \times \Gamma, \mathbb{H}(\mathbb{C}))$. We begin with $\alpha = \alpha_0 \in \mathbb{C}$. Then the formulas (4.1) takes the form

$$\begin{aligned}
&\int_{\Gamma_{\tau_1}} \int_{\Gamma_\tau} \mathcal{K}_{\alpha_0}(t-\tau)\sigma_\tau \mathcal{K}_{\alpha_0}(\tau-\tau_1)\sigma_{\tau_1} f(\tau_1,\tau) + \frac{1-\gamma(t)}{2} f(t,t) \\
&= \int_{\Gamma_\tau} \int_{\Gamma_{\tau_1}} \mathcal{K}_{\alpha_0}(t-\tau)\sigma_\tau \mathcal{K}_{\alpha_0}(\tau-\tau_1)\sigma_{\tau_1} f(\tau_1,\tau),
\end{aligned}$$

and it was proved in [8]. Thus the case $\alpha = \alpha_0 \in \mathbb{C}$ is covered. For other possible situations, the argument is similar to the proof of [6, Theorem 3.1] □

5. Function theory for the quaternionic Dirac operator

We start this Section with a brief description of the relations between the time-harmonic spinor fields theory and the theory of α-hyperholomorphic functions. One can find more about this in [4], [6]. The standard Dirac matrices have the well-known properties:

$$\begin{gathered}
\gamma_0^2 = E_4, \quad \gamma_k^2 = -E_4, \quad k \in \mathbb{N}_3 := \{1,2,3\}, \\
\gamma_j\gamma_k + \gamma_k\gamma_j = 0, \quad j,k \in \mathbb{N}_3^0 := \mathbb{N}_3 \cup \{0\}, \quad j \neq k,
\end{gathered}$$

where E_4 is the 4×4 identity matrix. The products of the Dirac matrices

$$\hat{i}_0 := E_4, \quad \hat{i}_1 := \gamma_3\gamma_2, \quad \hat{i}_2 := \gamma_1\gamma_3, \quad \hat{i}_3 := \gamma_1\gamma_2, \quad \hat{i} := \gamma_0\gamma_1\gamma_2\gamma_3,$$

have the following properties:

$$\begin{gathered}
\hat{i}_0^2 = \hat{i}_0 = -\hat{i}_k^2, \quad \hat{i}_0\hat{i}_k = \hat{i}_k\hat{i}_0 = \hat{i}_k, \quad k \in \mathbb{N}_3, \\
\hat{i}_1\hat{i}_2 = -\hat{i}_2\hat{i}_1 = \hat{i}_3, \quad \hat{i}_2\hat{i}_3 = -\hat{i}_3\hat{i}_2 = \hat{i}_1, \quad \hat{i}_3\hat{i}_1 = -\hat{i}_1\hat{i}_3 = \hat{i}_2, \\
\hat{i} \cdot \hat{i}_k = \hat{i}_k \cdot \hat{i}, \quad k \in \mathbb{N}_3^0.
\end{gathered}$$

For $b \in \mathbb{H}(\mathbb{C})$, set

$$B_l(b) := \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & -b_3 & b_2 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{pmatrix}.$$

Matrix subalgebra $\mathcal{B}_l(\mathbb{C}) := \{B_l(b) : b \in \mathbb{H}(\mathbb{C})\}$ and $\mathbb{H}(\mathbb{C})$ are isomorphic as complex algebras. Abusing a little we shall not distinguish, sometimes, between $B_l(b)$, the column $\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and the quaternion b. Set

$$\mathcal{D} := i\omega\gamma_0 - E_4\partial_1 - \gamma_1\partial_2 - \gamma_3\partial_3 + im.$$

We shall consider $\mathcal{D}$ on the set $C^1(\Omega, \mathcal{B}_l(\mathbb{C}))$ of corresponding matrices. Hence for us

$$\mathcal{D} : C^1(\Omega, \mathcal{B}_l(\mathbb{C})) \to C^0(\Omega, \mathcal{B}_l(\mathbb{C})).$$

In [4, Section 12] (see also [2, page 7563]) there was introduced the map $\mathcal{UA}$ which transforms a function $q : \tilde{\Omega} \subset \mathbb{R}^3 \to \mathbb{C}^4$ into the function $\rho : \Omega \subset \mathbb{R}^3 \to \mathbb{H}(\mathbb{C})$ by the rule:

$$\rho = \mathcal{UA}[q] := \frac{1}{2}[-(\tilde{q}_1 - \tilde{q}_2)i_0 + i(\tilde{q}_0 - \tilde{q}_3)i_1 - (\tilde{q}_0 + \tilde{q}_3)i_2 + i(\tilde{q}_1 + \tilde{q}_2)i_3],$$

where $\tilde{q}(x) := q(x_1, x_2, -x_3)$, the domain $\tilde{\Omega}$ is obtained from $\Omega \subset \mathbb{R}^3$ by the reflection $x_3 \to -x_3$. The corresponding inverse transform is given as follows:

$$(\mathcal{UA})^{-1}[\rho] = \mathcal{A}^{-1}\mathcal{U}^{-1}[\rho] := (-i\tilde{\rho}_1 - \tilde{\rho}_2, -\tilde{\rho}_0 - i\tilde{\rho}_3, \tilde{\rho}_0 - i\rho_3, i\tilde{\rho}_1 - \tilde{\rho}_2).$$

The maps $\mathcal{UA}$ and $(\mathcal{UA})^{-1}$ may be represented in a matrix form (see [4, Subsection 12.13]):

$$\rho = \mathcal{UA}[q] := \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & i & i & 0 \end{pmatrix}\begin{pmatrix} \tilde{q}_0 \\ \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \end{pmatrix},$$

$$q = (\mathcal{UA})^{-1}[\rho] := \begin{pmatrix} 0 & -i & -1 & 0 \\ -1 & 0 & 0 & -i \\ 1 & 0 & 0 & -i \\ 0 & i & -1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{\rho}_0 \\ \tilde{\rho}_1 \\ \tilde{\rho}_2 \\ \tilde{\rho}_3 \end{pmatrix}.$$

Direct computation leads to the equality

$$\mathbb{D}_{\omega,m} = -\gamma_0 \hat{i}(\mathcal{UA})^{-1}\mathcal{D}\hat{i}_2(\mathcal{UA}), \tag{5.1}$$

on $C^1(\Omega, \mathbb{C}^4)$. Also we get $\mathcal{D}\hat{i}_2 = D_\alpha$, on $\mathcal{B}_l(\mathbb{C})$, where $\alpha := -(i\omega i_1 + m i_2)$. By these reasons $\mathcal{D}$ is termed "the quaternionic relativistic Dirac operator". Thus,

$$\ker \mathcal{D} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ker D_\alpha.$$

There exists a one-to-one correspondence between elements of $\ker \mathcal{D}$ (which are matrices) and matrices of the form $B_l(q)$, with $q = q_0 i_0 + q_1 i_1 + q_2 i_2 + q_3 i_3$ being α-hyperholomorphic function.

The "quaternionic relativistic Cauchy-Dirac kernel", i.e., the fundamental solution of $\mathcal{D}$, is given by

$$\mathcal{K}_{\mathcal{D},\alpha} := \hat{i}_2 \mathcal{K}_\alpha.$$

The integral

$$K_{\mathcal{D},\alpha}[f](x) := -\int_\Gamma \check{\mathcal{K}}^x_{\mathcal{D},\alpha}[\sigma_{\mathcal{D},\tau} f(\tau)],\ x \in \Omega^\pm,$$

plays the role of the Cauchy-type integral, the one with the quaternionic relativistic Cauchy-Dirac kernel (see [6], [4]); with $f : \Gamma \to \mathcal{B}_l(\mathbb{C})$ and

$$\sigma_{\mathcal{D},\tau} := \begin{pmatrix} -n_1(\tau) & 0 & n_3(\tau) & n_2(\tau) \\ 0 & -n_1(\tau) & n_2(\tau) & -n_3(\tau) \\ -n_3(\tau) & -n_2(\tau) & -n_1(\tau) & 0 \\ -n_2(\tau) & n_3(\tau) & 0 & -n_1(\tau) \end{pmatrix} dS.$$

We shall call also $K_{\mathcal{D},\alpha}[f]$ the quaternionic relativistic Cauchy-Dirac-type integral.

Theorem 5.1 (Poincaré-Bertrand formula for the quaternionic relativistic Cauchy-Dirac integral on a piece-wise Liapunov surface). *Let Ω be a bounded domain in $\mathbb{R}^3$ with the piece-wise Liapunov boundary and let $f \in H_\mu(\Gamma \times \Gamma, \mathcal{B}_l(\mathbb{C})), 0 < \mu < 1$.*

The following equality holds everywhere on Γ*:*

$$\begin{aligned} &\int_{\Gamma_{\tau_1}}\int_{\Gamma_\tau} \check{\mathcal{K}}^t_{\mathcal{D},\alpha}[\sigma_{\mathcal{D},\tau}\check{\mathcal{K}}^\tau_{\mathcal{D},\alpha}[\sigma_{\mathcal{D},\tau_1} f(\tau_1,\tau)]] + \frac{1-\gamma(t)}{2} f(t,t) \\ &= \int_{\Gamma_\tau} \check{\mathcal{K}}^t_{\mathcal{D},\alpha}\Big[\sigma_{\mathcal{D},\tau}\int_{\Gamma_{\tau_1}} \check{\mathcal{K}}^\tau_{\mathcal{D},\alpha}[\sigma_{\mathcal{D},\tau_1} f(\tau_1,\tau)]\Big]. \end{aligned} \tag{5.2}$$

Proof. Let $f \in H_\mu(\Gamma\times\Gamma, \mathcal{B}_l(\mathbb{C}))$, consider $\check{\mathcal{K}}^x_{\mathcal{D},\alpha}$. It was proved that

$$\check{\mathcal{K}}^x_{\mathcal{D},\alpha}[\sigma_{\mathcal{D}} f] = \hat{i}_2 \check{\mathcal{K}}^x_\alpha\big[\sigma\big(-\hat{i}_2\big) f\big].$$

Hence using formula (4.1) and after not complicated computation we obtain (5.2). $\square$

6. Proof of the Theorem 2.1

In this Section we use results form Section 4. For the reader's convenience, recall some information from [6]:

$$\begin{gathered} -\gamma_0 \hat{i}(\mathcal{U}\mathcal{A})^{-1} = -\gamma_1(\mathcal{U}\mathcal{A})^{-1}\big(\hat{i}_2\big)^{-1}, \\ \mathcal{K}_{\mathbb{D}_{\omega,m}} = (\mathcal{U}\mathcal{A})^{-1}\big(\hat{i}_2\big)^{-1}\mathcal{K}_{\mathcal{D},\alpha} = (\mathcal{U}\mathcal{A})^{-1}\mathcal{K}_\alpha, \\ \sigma_{\mathbb{D}_{\omega,m}} = \sigma_{\mathcal{D}}\hat{i}_2(\mathcal{U}\mathcal{A}) = \sigma(\mathcal{U}\mathcal{A}), \\ \check{\mathcal{K}}^x_{\mathbb{D}_{\omega,m}} = (\mathcal{U}\mathcal{A})^{-1}\big(\hat{i}_2\big)^{-1}\check{\mathcal{K}}^x_{\mathcal{D},\alpha}. \end{gathered}$$

The proof of Theorem 2.1 follows from Theorem 5.1 taking into account the above relation between the class of the time-harmonic spinor fields and α-hyperholomorphic functions.

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