*Electronic Journal of Differential Equations*, Vol. 2006(2006), No. 04, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

# REGULARIZATION AND ERROR ESTIMATES FOR NONHOMOGENEOUS BACKWARD HEAT PROBLEMS

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Abstract. In this article, we study the inverse time problem for the non-homogeneous heat equation which is a severely ill-posed problem. We regularize this problem using the quasi-reversibility method and then obtain error estimates on the approximate solutions. Solutions are calculated by the contraction principle and shown in numerical experiments. We obtain also rates of convergence to the exact solution.

## 1. Introduction

For a positive real number $T$, consider the problem of finding the temperature $u(x,t)$, such that

$$u_t - u_{xx} = f(x,t), \quad 0 \le x \le \pi,\ 0 < t < T, \tag{1.1}$$

$$u(0,t) = u(\pi,t) = 0, \quad 0 < t < T, \tag{1.2}$$

$$u(x,T) = g(x), \quad 0 \le x. \le \pi \tag{1.3}$$

where $g(x), f(x,t)$ are given functions. This problem is called the backward heat problem (BHP), or final-value problem. As is known, such problem is severely ill-posed; i.e., solutions do not always exist, and when they exist, they do not depend continuously on the given data. In fact, for small noise contaminating physical measurements, the corresponding solutions have large errors. This makes difficult to use numerical calculations with inexact data. Hence, a regularization is needed. When $f = 0$, we have a homogenous problem,

$$\begin{gathered} u_t + Au = 0, \quad 0 < t < T, \\ u(T) = \varphi. \end{gathered} \tag{1.4}$$

that has been considered by several authors in the previous four decades. Lattes and Lions [12], Miller [14], Payne [16], Huang and Zheng [10], and Lavrentiev [13] have approximated (1.4) by perturbing the operator $A$. This approach called the "quasi-reversibility method". The main idea of this method is that by perturbing the equation in the ill-posed problem, one may obtain a well-posed problem. Then use the solution of the well-posed problem as an approximate solutions of the ill-posed problem.

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2000 *Mathematics Subject Classification.* 35K05, 35K99, 47J06, 47H10.
*Key words and phrases.* Backward heat problem; ill-posed problem; contraction principle; quasi-reversibility methods.

Submitted November 11, 2005. Published January 11, 2006.

Lattes and Lions [12] regularized the problem by adding a "corrector" to the main equation. They considered the problem

$$\begin{gathered} u_t + Au - \epsilon A^* Au = 0, \quad 0 < t < T, \\ u(T) = \varphi. \end{gathered}$$

Alekseeva and Yurchuk [1] considered the problem

$$\begin{gathered} u_t + Au + \epsilon A u_t = 0, \quad 0 < t < T, \\ u(T) = \varphi. \end{gathered} \tag{1.5}$$

Gajewski and Zaccharias [8] consider a problem similar to (1.5). Their error estimate for the approximate solutions is

$$\|u^\epsilon(t) - u(t)\|^2 \le \frac{2}{t^2}(T-t)\|u(0)\| \tag{1.6}$$

Note that these estimate can not be used at the time $t = 0$. Showalter [17, 18] presented a different method for regularizing (1.4), which is a stability estimate better than the previous ones. Using Showalter's idea, Clark and Oppenheimer [5] used the quasi-boundary method to regularize the backward problem with

$$\begin{gathered} u_t + Au(t) = 0, \quad 0 < t < T, \\ u(T) + \epsilon u(0) = \varphi\,. \end{gathered}$$

A similar approaches known as quasiboundary method was given in [15]. Also, we have to mention that nonstandard conditions for the parabolic equation have been considered in some recent papers [2, 3]. Denche and Bessila [7] approximated this problem by perturbing the final condition (1.2) with a derivative of the same order as the equation:

$$\begin{gathered} u_t + Au(t) = 0, \quad 0 < t < T, \\ u(T) - \epsilon u'(0) = \varphi\,. \end{gathered}$$

Huang and Zheng [9] considered problem (1.5) where operator $-A$ is the generator of an analytic semigroup in a Banach space. However, they do not give error estimates and effective methods of calculation.

Although there are many publication on the backward problem, most of them are for the homogeneous case, and the literature of the non-homogeneous case is quite scarce. In this paper, we consider backward heat problem in the non-homogeneous case. Our results generalize many results in previous papers; see for example [1, 2, 3, 4, 5, 9, 8, 17]. We use quasi-reversibility to approximate Problem (1.1)–(1.3) as the follows:

$$u^\epsilon_t - u^\epsilon_{xx} - \epsilon u^\epsilon_{xxxx} = \sum_{n=1}^{\infty} e^{-\epsilon n^4 (T-t)} f_n(t) \sin(nx), \quad 0 \le x \le \pi,\ 0 < t < T, \tag{1.7}$$

$$u^\epsilon(0,t) = u^\epsilon(\pi,t) = u^\epsilon_{xx}(0,t) = u^\epsilon_{xx}(\pi,t) = 0, \quad 0 < t < T, \tag{1.8}$$

$$u^\epsilon(x,T) = g(x), \quad 0 \le x \le \pi, \tag{1.9}$$

where $\epsilon$ is a positive parameter and

$$f_n(t) = \frac{2}{\pi}\langle f(x,t), \sin(nx)\rangle = \frac{2}{\pi}\int_0^\pi f(x,t)\sin(nx)dx\,,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(0,\pi)$. First, we shall prove that, the (unique) solution $u^\epsilon$ of (1.6)–(1.8) is

$$u^\epsilon(x,t) = \sum_{n=1}^{\infty} (e^{(T-t)(n^2-\epsilon n^4)} g_n - \int_t^T e^{(s-t)(n^2-\epsilon n^4)} e^{-\epsilon n^4(T-s)} f_n(s)ds) \sin(nx), \tag{1.10}$$

where $g_n = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)dx$.

In Section 2, we shall prove that (1.7)–(1.9) is well-posed. In Section 3, we estimate the error between an exact solution $u$ of (1.1)–(1.3)) and the approximation solution $u^\epsilon$ of (1.7)–(1.9). In fact, we shall prove that

$$\|u(.,t) - u^\epsilon(.,t)\| \le \epsilon(T-t)\sqrt{\frac{8}{t^4}\|u(.,0)\|^2 + t^2\|\frac{\partial^4 f(x,t)}{\partial x^4}\|^2_{L^2(0,T;L^2(0,\pi))}}. \tag{1.11}$$

Note that with this inequality, the error can be estimated at $t=0$. Note also that (1.11) is similar (1.6) when $f=0$. In Section 3, we obtain also some other results, including converges rates.

## 2. The well-posed Problem

In this section, we shall study the existence, uniqueness and stability of a (weak) solution to (1.7)–(1.9). In fact, one has the following result.

**Theorem 2.1.** *Let $f(x,t) \in L^2(0,T;L^2(0,\pi))$ and let $g(x) \in L^2(0,\pi)$. Then (1.7)–(1.9) has unique a weak solution $u^\epsilon(x,t)$ which is in $C([0,T];L^2(0,\pi)) \cap L^2(0,T;H_0^1(0,\pi)\cap H^2(0,\pi))$, and is given by (1.10). Furthermore, the solution depends continuously on $g$ in $C([0,T];L^2(0,\pi))$.*

*Proof.* The proof is divided into three steps. In step 1, we prove that the function $u^\epsilon(t)$ given by (1.10), is a solution of (1.7)–(1.9). In Step 2, we prove the uniqueness. Finally in Step 3, we prove the stability of the solution.

**Step 1:** Functions given by (1.10) are solutions of (1.7)–(1.9). Let $u^\epsilon(x,t)$ be given by (1.10). Then we can verify directly that $u^\epsilon(x,t) \in C([0,T];L^2(0,\pi)) \cap L^2(0,T;H_0^1(0,\pi)\cap H^2(0,\pi))$. In fact, $u^\epsilon \in C^\infty((0,T];H_0^1(0,\pi)))$. Moreover,

$$\begin{aligned} u_t^\epsilon(x,t) = \sum_{n=1}^{\infty} \Big( &(-n^2+\epsilon n^4)(e^{(T-t)(n^2-\epsilon n^4)} g_n \\ &- \int_t^T e^{(s-t)(n^2-\epsilon n^4)} e^{-\epsilon n^4(T-s)} f_n(s)ds)\sin(nx)\Big) \\ &+ \sum_{n=1}^{\infty}\int_t^T e^{-\epsilon n^4(T-t)} f_n(s)ds \sin(nx), \end{aligned}$$

$$\begin{aligned} &u_{xx}^\epsilon(x,t) \\ &= \sum_{n=1}^{\infty}(-n^2)(e^{(T-t)(n^2-\epsilon n^4)} g_n - \int_t^T e^{(s-t)(n^2-\epsilon n^4)} e^{-\epsilon n^4(T-s)} f_n(s)ds)\sin(nx), \\ &u_{xxxx}^\epsilon(x,t) \\ &= \sum_{n=1}^{\infty} n^4(e^{(T-t)(n^2-\epsilon n^4)} g_n - \int_t^T e^{(s-t)(n^2-\epsilon n^4)} e^{-\epsilon n^4(T-s)} f_n(s)ds)\sin(nx). \end{aligned}$$

Hence
$$u_t^\epsilon(x,t) - u_{xx}^\epsilon(x,t) - \epsilon u_{xxxx}^\epsilon(x,t) = \sum_{n=1}^\infty e^{-\epsilon n^4(T-t)} f_n(t)\sin(nx).$$
We also have
$$u^\epsilon(x,T) = \sum_{n=1}^\infty g_n \sin(nx) = g(x).$$
**Step 2:** Problem (1.7)–(1.9) has unique solution. Suppose the there are two solution $u(x,t)$ and $v(x,t)$. Then we need to show that $u(x,t) = v(x,t)$. Let $w(x,t) = u(x,t) - v(x,t)$. Then $w(x,t)$ satisfies the system
$$\begin{gathered} w_t(x,t) - w_{xx}(x,t) - \epsilon w_{xxxx}(x,t) = 0, \quad (x,t) \in (0,\pi)\times(0,T), \\ w(x,T) = 0, \quad x \in (0,\pi), \\ w(0,t) = w(\pi,t) = w_{xx}(0,t) = w_{xx}(\pi,t) = 0. \end{gathered} \tag{2.1}$$
For $k > 0$, we define $\psi(x,t) = e^{k(t-T)}w(x,t)$. Note that $\psi(x,t)$ satisfies
$$\begin{gathered} \psi_t(x,t) - \psi_{xx}(x,t) - \epsilon\psi_{xxxx}(x,t) - k\psi(x,t) = 0, \quad (x,t) \in (0,\pi)\times(0,T), \\ \psi(x,T) = 0, \quad x \in (0,\pi), \\ \psi(0,t) = \psi(\pi,t) = \psi_{xx}(0,t) = \psi_{xx}(\pi,t) = 0. \end{gathered} \tag{2.2}$$
Multiplying (2.2) by $\psi(x,t)$ and integrating on $x$ from 0 to $\pi$, we obtain
$$\begin{gathered} \int_0^\pi \frac{d}{dt}\psi(x,t)\psi(x,t)dx - \int_0^\pi \psi_{xx}(x,t)\psi(x,t)dx \\ - \int_0^\pi \psi_{xxxx}(x,t)\psi(x,t)dx - \int_0^\pi k\psi(x,t)\psi(x,t)dx = 0. \end{gathered}$$
Applying the Green formula, we have
$$\int_0^\pi \psi_{xx}(x,t)\psi(x,t)dx = -\int_0^\pi \psi_x(x,t)\psi_x(x,t)dx = -\|\nabla\psi(x,t)\|^2,$$
$$\begin{aligned} \int_0^\pi \psi_{xxxx}(x,t)\psi(x,t)dx &= -\int_0^\pi \psi_{xxx}(x,t)\psi_x(x,t)dx \\ &= \int_0^\pi \psi_{xx}(x,t)\psi_{xx}(x,t)dx = \|\Delta\psi(x,t)\|^2. \end{aligned}$$
It follows that
$$\frac{d}{dt}\|\psi(x,t)\|^2 + \|\nabla\psi(x,t)\|^2 - \epsilon\|\Delta\psi(x,t)\|^2 - k\|\psi(x,t)\|^2 = 0\,.$$
Using Schwartz inequality, we have
$$\begin{aligned} \|\nabla\psi(x,t)\|^2 &= \int_0^\pi -\psi_{xx}(x,t)\psi(x,t)dx \\ &= \langle -\Delta\psi(x,t), \psi(x,t)\rangle \\ &\le \epsilon\|\Delta\psi(x,t)\|^2 + \frac{1}{4\epsilon}\|\psi(x,t)\|^2. \end{aligned}$$
Therefore,
$$\frac{d}{dt}\|\psi(x,t)\|^2 \ge (k - \frac{1}{4\epsilon})\|\psi(x,t)\|^2,$$

Choosing $k = 1/4\epsilon$, we have

$$\|\psi(.,T)\|^2 - \|\psi(.,t)\|^2 \geq \int_t^T (k - \frac{1}{4\epsilon})||\psi(.,s)||^2 ds = 0.$$

Since $w(.,T) = 0$ it follows that $w(.,t) = 0$ and $\psi(.,t) = 0$ therefore, $u(x,t) = v(x,t)$.

**Step 3:** The solution of (1.7)–(1.9) depends continuously on $g \in L^2(0,\pi)$. Let $u$ and $v$ be two solution of (1.7)–(1.9) corresponding to the final values $g$ and $h$, respectively. By (1.10),

$$u(x,t) = \sum_{n=1}^{\infty} (e^{(T-t)(n^2-\epsilon n^4)} g_n - \int_t^T e^{(s-t)(n^2-\epsilon n^4)} e^{-\epsilon n^4 (T-t)} f_n(s) ds) \sin(nx),$$
$$v(x,t) = \sum_{n=1}^{\infty} (e^{(T-t)(n^2-\epsilon n^4)} h_n - \int_t^T e^{(s-t)(n^2-\epsilon n^4)} e^{-\epsilon n^4 (T-t)} f_n(s) ds) \sin(nx),$$

where

$$g_n = \frac{2}{\pi}\langle g(x), \sin(nx)\rangle, \quad h_n = \frac{2}{\pi}\langle h(x), \sin(nx)\rangle$$

It follows that

$$\|u(.,t) - v(.,t)\|_H^2 = \frac{\pi}{2}\sum_{n=1}^{\infty} e^{2(n^2-\epsilon n^4)(T-t)}(g_n - h_n)^2.$$

In view of the inequality $n^2 - \epsilon n^4 \leq 1/(4\epsilon)$, we have

$$\begin{aligned}\|u(.,t) - v(.,t)\|^2 &\leq \frac{\pi}{2}\sum_{n=1}^{\infty} e^{(T-t)/2\epsilon}(g_n - h_n)^2 \\ &= \frac{\pi}{2} e^{(T-t)/2\epsilon}\sum_{n=1}^{\infty}(g_n - h_n)^2 = e^{(T-t)/2\epsilon}\|g - h\|^2.\end{aligned}$$

Hence

$$\|u(.,t) - v(.,t)\| \leq e^{(T-t)/4\epsilon}\|g - h\|.$$

This completes the proof of Step 3 and the proof of the theorem. □

## 3. Regularization of Problem (1.1)–(1.3)

We first have a uniqueness result.

**Theorem 3.1.** *Let $f(x,t) \in L^2(0,T; L^2(0,\pi))$. Then* (1.1)–(1.3) *has at most one (weak) solution in $C([0,T]; L^2(0,\pi)) \cap L^2(0,T; H_0^1(0,\pi) \cap H^2(0,\pi))$.*

The proof of the above lemma can be found in [11]. Despite the uniqueness, Problem (1.1)–(1.3) is still ill-posed. Hence, a regularization has to be used.

**Theorem 3.2.** *Let $f \in L^2(0,T; L^2(0,\pi))$ be such that $\frac{\partial^4 f(x,t)}{\partial x^4} \in L^2(0,T; L^2(0,\pi))$. Suppose that Problem* (1.1)–(1.3) *has a weak solution $u$ in $C([0,T]; L^2(0,\pi)) \cap L^2(0,T; H_0^1(0,\pi) \cap H^2(0,\pi))$. Then*

$$\|u(.,t) - u^\epsilon(.,t)\| \leq \epsilon(T-t)\sqrt{\frac{8}{t^4}\|u(.,0)\|^2 + t^2\|\frac{\partial^4 f(x,t)}{\partial x^4}\|^2_{L^2(0,T;L^2(0,\pi))}},$$

*for every $t \in (0,T]$, where $u^\epsilon$ is the unique solution of* (1.7)–(1.9).

*Proof.* Suppose $u$ is the exact solution of (1.1)–(1.3). Then, as shown in [6],

$$u(x,t)=\sum_{n=1}^{\infty}\Big(e^{-tn^2}u_n(0)+\int_0^t e^{(s-t)n^2}f_n(s)ds\Big)\sin(nx). \tag{3.1}$$

where $u_n(0)=\frac{2}{\pi}\langle u(x,0),\sin(nx)\rangle$. Then

$$\begin{aligned}
g(x)&=u(x,T)\\
&=\sum_{n=1}^{\infty}\Big(e^{-Tn^2}u_n(0)+\int_0^T e^{(s-T)n^2}f_n(s)ds\Big)\sin(nx),\\
&=\sum_{n=1}^{\infty}\varphi_n\sin(nx).
\end{aligned}$$

Hence $g_n=e^{-Tn^2}u_n(0)+\int_0^T e^{(s-T)n^2}f_n(s)ds$ and

$$\begin{aligned}
u_n^\epsilon(t)&=e^{(T-t)(n^2-\epsilon n^4)}g_n-\int_t^T e^{(s-t)(n^2-\epsilon n^4)}e^{-\epsilon n^4(T-s)}f_n(s)ds,\\
&=e^{(T-t)(n^2-\epsilon n^4)}(e^{-Tn^2}u_n(0)+\int_0^T e^{(s-T)n^2}f_n(s)ds)\\
&\quad-\int_t^T e^{(s-t)(n^2-\epsilon n^4)}e^{-\epsilon n^4(T-s)}f_n(s)ds,\\
&=e^{-tn^2}e^{-\epsilon(T-t)n^4}u_n(0)+\int_0^t e^{(T-t)(n^2-\epsilon n^4)}e^{(s-T)n^2}f_n(s)ds)\\
&\quad+\int_t^T e^{(T-t)(n^2-\epsilon n^4)}e^{(s-T)n^2}f_n(s)ds\\
&\quad-\int_t^T e^{(s-t)(n^2-\epsilon n^4)}e^{-\epsilon n^4(T-s)}f_n(s)\,ds.
\end{aligned}$$

It follows that

$$u_n^\epsilon(t)=e^{-tn^2}e^{-\epsilon(T-t)n^4}u_n(0)+\int_0^t e^{(s-t)n^2}e^{-\epsilon(T-t)n^4}f_n(s)ds. \tag{3.2}$$

From (1.10), (2.1), (2.2) and using the inequality $1-e^{-x}\le x$ for $x>0$, we have

$$\begin{aligned}
&|u_n(t)-u_n^\epsilon(t)|\\
&\le e^{-tn^2}(1-e^{-\epsilon n^4(T-t)})|u_n(0)|+|\int_0^t e^{(s-t)n^2}(1-e^{-\epsilon n^4(T-t)})f_n(s)ds|\\
&\le e^{-tn^2}(1-e^{-\epsilon n^4(T-t)})|u_n(0)|+\int_0^t e^{(s-t)n^2}(1-e^{-\epsilon n^4(T-t)})|f_n(s)|ds\\
&\le e^{-tn^2}\epsilon n^4(T-t)|u_n(0)|+\int_0^t e^{(s-t)n^2}\epsilon n^4(T-t)|f_n(s)|ds\\
&=\frac{\epsilon}{t^2}e^{-tn^2}(tn^2)^2(T-t)|u_n(0)|+\epsilon(T-t)\int_0^t e^{(s-t)n^2}n^4|f_n(s)|ds\\
&\le\frac{2\epsilon}{t^2}(T-t)|u_n(0)|+\epsilon(T-t)\int_0^t n^4|f_n(s)|ds.
\end{aligned} \tag{3.3}$$

In view of $(a+b)^2 \le 2(a^2+b^2)$ and using Holder inequality, we obtain

$$
\begin{aligned}
|u_n(t)-u_n^\epsilon(t)|^2 &\le 2[\frac{4\epsilon^2}{t^4}(T-t)^2|u_n(0)|^2+\epsilon^2(T-t)^2(\int_0^t n^4|f_n(s)|ds)^2] \\
&\le \frac{8\epsilon^2}{t^4}(T-t)^2|u_n(0)|^2+\epsilon^2(T-t)^2t^2\int_0^t n^8|f_n(s)|^2ds.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|u(.,t)-u^\epsilon(.,t)\|^2 \\
&= \frac{\pi}{2}\sum_{n=1}^\infty |u_n(t)-u_n^\epsilon(t)|^2 \\
&\le \frac{\pi}{2}\frac{8\epsilon^2}{t^4}(T-t)^2\sum_{n=1}^\infty |u_n(0)|^2+\frac{\pi}{2}\epsilon^2(T-t)^2t^2\int_0^t\sum_{n=1}^\infty n^8|f_n(s)|^2ds \\
&= \frac{8\epsilon^2}{t^4}(T-t)^2\|u(.,0)\|^2+\epsilon^2(T-t)^2t^2\int_0^t \|\frac{\partial^4 f(x,s)}{\partial x^4}\|^2ds.
\end{aligned}
$$

This completes the proof. □

**Theorem 3.3.** *Let $u$ be a solution of (1.1)–(1.3) with $u \in L^\infty(0,T;L^2(0,\pi)) \cap L^2(0,T;H_0^1(0,\pi))$ and such that $\|\Delta^2 u(x,t)\| < \infty$ for all $t$ in $[0,T]$. Then*

$$
\|u(.,t)-u^\epsilon(.,t)\| \le \epsilon T\|\Delta^2 u(.,t)\|
$$

*Proof.* From (3.2), we have

$$
\begin{aligned}
u_n(t)-u_n^\epsilon(t) &= e^{-tn^2}(1-e^{-\epsilon n^4(T-t)})u_n(0)+\int_0^t e^{(s-t)n^2}(1-e^{-\epsilon n^4(T-t)})f_n(s)ds \\
&= (1-e^{-\epsilon n^4 T})u_n(t).
\end{aligned}
$$

Hence

$$
\|u(.,t)-u^\epsilon(.,t)\|^2 = \frac{\pi}{2}\sum_{n=1}^\infty |u_n(t)-u_n^\epsilon(t)|^2 \le \frac{\pi}{2}\epsilon^2T^2\sum_{n=1}^\infty n^8u_n^2(t) = \epsilon^2T^2\|\Delta^2 u(.,t)\|^2.
$$

This completes Proof. □

**Theorem 3.4.** *Let Problem (1.1)–(1.3) have exact solution $u \in C([0,T];L^2(0,\pi)) \cap L^2(0,T;H_0^1(0,\pi)\cap H^2(0,\pi))$, corresponding to $g$. Assume that*

$$
\frac{\partial^4 f(x,t)}{\partial x^4}, \frac{\partial u}{\partial t} \in L^2(0,T;L^2(0,\pi)), \quad \|\Delta^2 u(x,t)\| < \infty \quad \forall t \in [0,T]\,.
$$

*Let $g_\epsilon$ be the measured data such that $\|g_\epsilon - g\| \le \epsilon$. Then there exist a function $u^{\beta(\epsilon)}$ satisfying*

$$
\begin{gathered}
\|u^{\beta(\epsilon)}(.,t)-u(.,t)\| \le \frac{K}{\ln(1/\epsilon)}+\epsilon^{t/T}, \quad \forall t \in (0,T], \\
\|u^{\beta(\epsilon)}(.,0)-u(.,0)\| \le (1+C)\sqrt{\frac{T}{\ln(1/\epsilon)}}+C\frac{T}{4\ln(1/\epsilon)},
\end{gathered}
$$

*where* $\beta(\epsilon) = \frac{T}{4\ln(1/\epsilon)}$ *and*

$$K = \frac{1}{4}T(T-t)\sqrt{\frac{8}{t^4}\|u(.,0)\|^2 + t^2\|\frac{\partial^4 f(x,t)}{\partial x^4}\|^2_{L^2(0,T;L^2(0,\pi))}},$$
$$M = \max\big\{\sup_{0\le t\le T}\|u_t(x,t)\|, T\sup_{0\le t\le T}\|\Delta^2 u(x,t)\|\big\}.$$

*Proof.* Let $v^{\beta(\epsilon)}(.,t)$ be a solution of (1.7)–(1.9) corresponding $g$, and $w^{\beta(\epsilon)}$ be solution of (1.7)–(1.9) corresponding $g_\epsilon$. We consider the function $h(t) = \frac{\ln t}{t} - \frac{\ln\epsilon}{T}$ for $\epsilon \in (0,T)$. We have $h(T) > 0$ and $\lim_{t\to 0} h(t) = -\infty$ then $h(t) = 0$ has solution in $(0,T)$. We call $t_\epsilon$ is smallest solution of it. Apply inequality $\ln t > -\frac{1}{t}$ we get $t_\epsilon < \sqrt{\frac{T}{\ln(1/\epsilon)}}$. Using Lagrange Theorem for $u(.,t)$ and $u^\epsilon(.,t)$ in $(0,t_\epsilon)$ we have

$$\|u(0) - u(t_\epsilon)\| \le t_\epsilon\|u'(\alpha)\| \le Ct_\epsilon, \quad \forall\alpha \in (0,t_\epsilon).$$

Using Theorem 3.3 we get

$$\begin{aligned}\|v^{\beta(\epsilon)}(t_\epsilon) - u(0)\| &\le \|v^{\beta(\epsilon)}(t_\epsilon - u(t_\epsilon)\| + \|u(0) - u(t_\epsilon)\|\\ &\le \beta(\epsilon)(T-t_\epsilon)\|\Delta^2 u(t_\epsilon)\| + Ct_\epsilon\\ &\le C\ (\sqrt{\frac{T}{\ln(1/\epsilon)}} + \frac{T}{4\ln(1/\epsilon)})\end{aligned}$$

We put

$$u^{\beta(\epsilon)}(t) = \begin{cases} w^{\beta(\epsilon)}(t), & 0 < t \le T,\\ w^{\beta(\epsilon)}(t_\epsilon), & t = 0.\end{cases}$$

By Step 3 of Theorem 2.1,

$$\|v^{\beta(\epsilon)}(.,t) - w^{\beta(\epsilon)}(.,t)\| \le e^{\frac{T-t}{4\beta(\epsilon)}}\|g^\epsilon - g\| = \epsilon^{t/T}\,.$$

By Theorem 3.2 and applying the triangle inequality, we have

$$\begin{aligned}\|u^{\beta(\epsilon)}(.,t) - u(.,t)\| &\le \|v^{\beta(\epsilon)}(.,t) - w^{\beta(\epsilon)}(.,t)\| + \|v^{\beta(\epsilon)}(.,t) - u(.,t)\|\\ &\le \frac{K}{\ln(1/\epsilon)} + \epsilon^{t/T}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|u^{\beta(\epsilon)}(.,0) - u(.,0)\| &\le \|v^{\beta(\epsilon)}(.,t_\epsilon) - w^{\beta(\epsilon)}(.,t_\epsilon)\| + \|v^{\beta(\epsilon)}(.,t_\epsilon) - u(.,0)\|\\ &\le (1+C)\sqrt{\frac{T}{\ln(1/\epsilon)}} + C\frac{T}{4\ln(1/\epsilon)}.\end{aligned}$$

This completes the proof. □

## 4. Numerical experiments

Consider the problem

$$\begin{gathered} u_t - u_{xx} = 2e^t\sin(x),\\ u(x,1) = g(x) = e\sin(x)\end{gathered} \tag{4.1}$$

whose exact solution is $u(x,t) = e^t\sin(x)$. Note that $u(x,1/2) = \sqrt{e}\sin(x) \approx 1.648721271\sin(x)$. Let $g_n$ be the measured final data

$$g_n(x) = e\sin(x) + \frac{1}{n}\sin(nx).$$

So that the data error, at the final time, is

$$F(n) = \|g_n - g\|_{L^2(0,\pi)} = \sqrt{\int_0^\pi \frac{1}{n^2} \sin^2 nx dx} = \frac{1}{n}\sqrt{\frac{\pi}{2}}.$$

The solution of (4.1), corresponding the final value $g_n$, is

$$u^n(x,t) = e^t \sin(x) + \frac{1}{n} e^{n^2(1-t)} \sin(nx),$$

The error at the original time is

$$O(n) := ||u^n(.,0) - u(.,0)||_{L^2(0,\pi)} = \sqrt{\int_0^\pi \frac{e^{2n^2}}{n^2} \sin^2(nx)\, dx} = \frac{e^{n^2}}{n}\sqrt{\frac{\pi}{2}}.$$

Then, we notice that

$$\lim_{n\to\infty} F(n) = \lim_{n\to\infty} ||\varphi_n - \varphi_0||_{L^2(0,\pi)} = \lim_{n\to\infty} \frac{1}{n}\sqrt{\frac{\pi}{2}} = 0,$$

$$\lim_{n\to\infty} O(n) = \lim_{n\to\infty} \|u^n(.,0) - u(.,0)\|_{L^2(0,\pi)} = \lim_{n\to\infty} \frac{e^{n^2}}{n}\sqrt{\frac{\pi}{2}} = \infty.$$

From the two equalities above, we see that (4.1) is an ill-posed problem. Approximating the problem as in (1.7)–(1.9), the regularized solution is

$$\begin{aligned} u^\epsilon(x,t) &= \sum_{m=1}^{\infty} \Big( e^{(T-t)(m^2-\epsilon m^4)} g_m \\ &\quad - \int_t^T e^{(s-t)(m^2-\epsilon m^4)} e^{-\epsilon m^4(T-s)} f_m(s) ds \Big) \sin(mx), \\ u^\epsilon(x,t) &= e^{(1-t)(1-\epsilon)+1} \sin(x) \\ &\quad - 2\Big( \int_t^1 e^{(s-t)(1-\epsilon)} e^{-\epsilon(1-s)+1} ds \Big) \sin(x) + \frac{1}{n} e^{(1-t)(n^2-\epsilon n^4)} \sin(nx). \end{aligned}$$

Hence

$$u^\epsilon(x, \frac{1}{2}) = \big[ e^{\frac{3-\epsilon}{2}} - 2\int_{1/2}^1 e^{2s-1/2-\epsilon/2} ds \big] \sin(x) + \frac{1}{n} e^{\frac{1}{2}(n^2-\epsilon n^4)} \sin(nx).$$

Table 1. Approximations and error estimates for several values of $\epsilon$

| $\epsilon$ | $u_\epsilon$ | $\|u - u_\epsilon\|$ |
|---|---|---|
| $10^{-2}\sqrt{\frac{\pi}{2}}$ | $1.643563444 \sin(x) + 0.8243606355 \sin 200x$ | 0.1462051256 |
| $10^{-4}\sqrt{\frac{\pi}{2}}$ | $1.648617955 \sin(x) + 0.1648721271 \sin 10000x$ | 0.02066391506 |
| $10^{-10}\sqrt{\frac{\pi}{2}}$ | $1.648721271\big(\sin(x) + 10^{-10}\sin(10^{10}x)\big)$ | 0.00002066365678 |
| $10^{-16}\sqrt{\frac{\pi}{2}}$ | $1.648721271\big(\sin(x) + 10^{-16}\sin(10^{16}x)\big)$ | $2.066365678 \times 10^{-8}$ |
| $10^{-30}\sqrt{\frac{\pi}{2}}$ | $1.648721271\big(\sin(x) + 10^{-30}\sin(10^{30}x)\big)$ | $2.066365678 \times 10^{-15}$ |

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