EXISTENCE OF POSITIVE SOLUTIONS FOR HIGHER ORDER SINGULAR SUBLINEAR ELLIPTIC EQUATIONS

IMED BACHAR

ABSTRACT. We present existence result for the polyharmonic nonlinear problem
\begin{equation}
(-\Delta)^{\pm} u = \varphi(., u) + \psi(., u), \quad \text{in } B
\end{equation}
\begin{equation}
u > 0, \quad \text{in } B
\end{equation}
\begin{equation}
\lim_{|x| \to 1} \frac{(-\Delta)^{j} u(x)}{(1 - |x|)^{m-j-1}} = 0, \quad 0 \leq j \leq p - 1,
\end{equation}
in the sense of distributions. Here \( m, p \) are positive integers, \( B \) is the unit ball in \( \mathbb{R}^n (n \geq 2) \) and the nonlinearity is a sum of a singular and sublinear terms satisfying some appropriate conditions related to a polyharmonic Kato class of functions \( J_{m,n}^{p} \).

1. INTRODUCTION

In this paper, we investigate the existence and the asymptotic behavior of positive solutions for the following iterated polyharmonic problem involving a singular and sublinear terms:
\begin{equation}
(-\Delta)^{p} u = \varphi(., u) + \psi(., u), \quad \text{in } B
\end{equation}
\begin{equation}
u > 0, \quad \text{in } B
\end{equation}
\begin{equation}
\lim_{|x| \to 1} \frac{(-\Delta)^{j} u(x)}{(1 - |x|)^{m-j-1}} = 0, \quad 0 \leq j \leq p - 1,
\end{equation}
in the sense of distributions. Here \( B \) is the unit ball of \( \mathbb{R}^n (n \geq 2) \) and \( m, p \) are positive integers. This research is a follow up to the work done by Shi and Yao \cite{14}, who considered the problem
\begin{equation}
\Delta u + k(x)u^{-\gamma} + \lambda u^\alpha = 0, \quad \text{in } D,
\end{equation}
\begin{equation}
u > 0, \quad \text{in } D
\end{equation}
where \( D \) is a bounded \( C^{1,1} \) domain in \( \mathbb{R}^n (n \geq 2) \), \( \gamma, \alpha \) are two constants in \( (0, 1) \), \( \lambda \) is a real parameter and \( k \) is a Hölder continuous function in \( \overline{\Omega} \). They proved the existence of positive solutions. Choi, Lazer and Mckenna in \cite{8} and \cite{11} have studied a variety of singular boundary value problems of the type \( \Delta u + p(x)u^{-\gamma} \), in a regular
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domain $D$, $u = 0$ on $\partial D$, where $\gamma > 0$ and $p$ is a nonnegative function. They proved
the existence of positive solutions. This has been extended by Mâagli and Zribi [13]
to the problem $\Delta u = -f(., u)$ in $D$, $u = 0$ on $\partial D$, where $f(x, .)$ is nonnegative and
nonincreasing on $(0, \infty)$.

On the other hand, problem (1.1) with a sublinear term $\psi(., u)$ and a singular
term $\varphi(., u) = 0$, has been studied by Mâagli, Toumi and Zribi in [12] for $p = 1$ and

Thus a natural question to ask, is for more general singular and sublinear terms
combined in the nonlinearity, whether or not the problem (1.1) has a solution, which
we aim to study in this paper.

Our tools are based essentially on some inequalities satisfied by the Green func-
tion $\Gamma_{m,n}^{(p)}$ (see (2.1) below) of the polyharmonic operator $u \mapsto (-\Delta)^{pm}u$, on the
unit ball $B$ of $\mathbb{R}^n$ ($n \geq 2$) with boundary conditions $(\frac{\partial}{\partial \nu})^j(-\Delta)^{im}u|_{\partial B} = 0$, for
$0 \leq i \leq p-1$ and $0 \leq j \leq m-1$, where $\frac{\partial}{\partial \nu}$ is the outward normal derivative. Also,
we use some properties of functions belonging to the polyharmonic Kato class $J_{m,n}^{(p)}$
which is defined as follows.

Definition 1.1 ([2]). A Borel measurable function $q$ in $B$ belongs to the class $J_{m,n}^{(p)}$
if $q$ satisfies the condition

$$\lim_{\alpha \to 0} \left( \sup_{x \in B} \int_{B \cap B(x,\alpha)} \left( \frac{\delta(y)}{\delta(x)} \right)^m \Gamma_{m,n}^{(p)}(x,y) |q(y)| dy \right) = 0,$$

(1.3)

where $\delta(x) = 1 - |x|$, denotes the Euclidean distance between $x$ and $\partial B$.

Typical examples of elements in the class $J_{m,n}^{(p)}$ are functions in $L^s(B)$, with

$$s > \frac{n}{2pm} \quad \text{if } n > 2pm$$

or with

$$s > \frac{n}{2(p-1)m} \quad \text{if } 2(p-1)m < n < 2pm$$

or with

$$s \in (1, \infty] \quad \text{if } n \leq 2(p-1)m$$

or with $n = 2pm$; see [2]. Furthermore, if $q(x) = (\delta(x))^{-\lambda}$, then $q \in J_{m,n}^{(p)}$ if and
only if

$$\lambda < 2m, \quad \text{if } p = 1 \quad (\text{see } [3]) \quad \text{or}$$

$$\lambda < 2m + 1, \quad \text{if } p \geq 2 \quad (\text{see } [2]).$$

For the rest of this paper, we refer to the potential of a nonnegative measurable
function $f$, defined in $B$ by

$$V_{p}(f)(x) = \int_{B} \Gamma_{m,n}^{(p)}(x,y) f(y) dy.$$
(H3) For each $c > 0$, the function $x \mapsto \varphi(x, c(\delta(x))^m)$ is in $L^r(B)$.

(H4) $\psi$ is a nonnegative Borel measurable function on $B \times [0, \infty)$, continuous with respect to the second variable such that there exist a nontrivial nonnegative function $h \in L^1_{\text{loc}}(B)$ and a nontrivial nonnegative function $k \in \mathcal{F}_{m,n}^{(1)}$ such that

$$h(x)f(t) \leq \psi(x,t) \leq (\delta(x))^m k(x)g(t), \quad \text{for } (x,t) \in B \times (0, \infty),$$

where $f : [0, \infty) \to [0, \infty)$ is a measurable nondecreasing function satisfying

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty,$$

and $g$ is a nonnegative measurable function locally bounded on $[0, \infty)$ satisfying

$$\limsup_{t \to \infty} \frac{g(t)}{t} < \|V_p((\delta(.))^m k)\|_{\infty}.$$  \hspace{1cm} (1.6)

(H5) The function $x \mapsto (\delta(x))^m k(x)$ is in $L^r(B)$.

Using a fixed point argument, we shall prove the following existence result.

**Theorem 1.2.** Assume (H1)–(H5). Then \(1.1\) has at least one positive solution $u \in C^{2pm-1}(B)$, such that

$$a_j(\delta(x))^m \leq (-\Delta)^j u(x) \leq V_{p-j}(\varphi, a_j(\delta(.))^m)(x) + b_j V_{p-j}((\delta(.))^m k)(x),$$

for $j \in \{0, \ldots, p-1\}$. In particular,

$$a_j(\delta(x))^m \leq (-\Delta)^j u(x) \leq c_j(\delta(x))^m,$$

where $a_j, b_j, c_j$ are positive constants.

Typical examples of nonlinearities satisfying (H1)–(H5) are:

$$\varphi(x,t) = k(x)(\delta(x))^{m\gamma + m\alpha - \gamma},$$

for $\gamma \geq 0$, and

$$\psi(x,t) = k(x)(\delta(x))^{m\alpha} \log(1 + t^\beta),$$

for $\alpha, \beta \geq 0$ such that $\alpha + \beta < 1$, where $k$ is a nontrivial nonnegative functions in $L^r(B)$.

Recently Ben Othman [5] considered \(1.1\) when $p = 1$ and the functions $\varphi, \psi$ satisfy hypotheses similar to the ones stated above. Then she proved that \(1.1\) has a positive continuous solutions $u$ satisfying

$$a_0(\delta(x))^m \leq u(x) \leq V_1(\varphi, a_0(\delta(.))^m)(x) + b_0 V_1((\delta(.))^m k)(x).$$

Here we prove an existence result for the more general problem \(1.1\) and obtain estimates both on the solution $u$ and their derivatives $(-\Delta)^j u$, for all $j \in \{1, \ldots, p-1\}$.

To simplify our statements, we define some convenient notations:

(i) Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$ and let $\overline{B} = \{x \in \mathbb{R}^n : |x| \leq 1\}$, for $n \geq 2$.

(ii) $B(B)$ denotes the set of Borel measurable functions in $B$, and $B^+(B)$ the set of nonnegative ones.

(iii) $C(B)$ is the set of continuous functions in $B$.

(iv) $C^j(B)$ is the set of functions having derivatives of order $\leq j$, continuous in $B$ ($j \in \mathbb{N}$).

(v) For $x, y \in B$, $|x, y|^2 = |x - y|^2 + (1 - |x|^2)(1 - |y|^2)$. 


(vi) Let $f$ and $g$ be two positive functions on a set $S$. We call $f \preceq g$, if there is $c > 0$ such that $f(x) \leq cg(x)$, for all $x \in S$.
We call $f \sim g$, if there is $c > 0$ such that $\frac{1}{c}g(x) \leq f(x) \leq cg(x)$, for all $x \in S$.
(vii) For any $q \in B(B)$, we put
\[
\|q\|_{m,n,p} := \sup_{x \in B} \int_B \frac{\delta(y)}{\delta(x)}^m \Gamma^{(p)}_{m,n}(x,y)|q(y)|dy.
\]

2. Properties of the iterated Green function and the Kato class

Let $m \geq 1$, $p \geq 1$ be a positive integer and $\Gamma^{(p)}_{m,n}$ be the iterated Green function of the polyharmonic operator $u \mapsto (-\Delta)^m u$, on the unit ball $B$ of $\mathbb{R}^n$ ($n \geq 2$) with boundary conditions $(\frac{\partial}{\partial \nu})^j(-\Delta)^m u|_{\partial B} = 0$, for $0 \leq i \leq p - 1$ and $0 \leq j \leq m - 1$, where $\frac{\partial}{\partial \nu}$ is the outward normal derivative.

Then for $p \geq 2$ and $x, y \in B$,
\[
\Gamma^{(p)}_{m,n}(x,y) = \int_B \cdots \int_B G_{m,n}(x,z_1)G_{m,n}(z_1,z_2) \cdots G_{m,n}(z_{p-1},y)dz_1 \cdots dz_{p-1},
\]
where $G_{m,n}$ is the Green function of the polyharmonic operator $u \mapsto (-\Delta)^m u$, on $B$ with Dirichlet boundary conditions $(\frac{\partial}{\partial \nu})^j u = 0$, $0 \leq j \leq m - 1$.

Recall that Boggio in [6] gave an explicit expression for $G_{m,n}$: For each $x, y$ in $B$,
\[
G_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{\frac{|x-y|}{(v^2-1)^{m-1}}} (v^2-1)^{m-1}dv,
\]
where $k_{m,n}$ is a constant positive.

In this section we state some properties of $\Gamma^{(p)}_{m,n}$ and of functions belonging to the Kato class $\mathcal{J}_m^{(p)}$. These properties are useful for the statements of our existence result, and their proofs can be found in [2].

Proposition 2.1. On $B^2$, the following estimates hold:
\[
\Gamma^{(p)}_{m,n}(x,y) \sim \begin{cases} 
\frac{(\delta(x)\delta(y))^m}{|x-y|^{2m-n}|x,y|^m}, & \text{for } n > 2pm, \\
\frac{(\delta(x)\delta(y))^m}{|x,y|^m} \log(1 + \frac{|x,y|^2}{|x-y|^2}), & \text{for } n = 2pm \\
\frac{(\delta(x)\delta(y))^m}{|x,y|^{2(p-1)m}}, & \text{for } 2(p-1)m < n < 2pm.
\end{cases}
\]

Proposition 2.2. With the above notation,
\[
(\delta(x)\delta(y))^m \leq \Gamma^{(p)}_{m,n}(x,y),
\]
\[
\Gamma^{(p)}_{m,n}(x,y) \leq \Gamma^{(p-1)}_{m,n}(x,y), \quad \text{for } p \geq 2.
\]
\[
\Gamma^{(p)}_{m,n}(x,y) \leq \delta(x)\delta(y)\Gamma^{(p)}_{m-1,n}(x,y), \quad \text{for } m \geq 2.
\]
In particular,
\[
\mathcal{J}_m^{(1)} \subset \mathcal{J}_m^{(2)} \cdots \subset \mathcal{J}_m^{(p)}, \quad \mathcal{J}_1^{(p)} \subset \mathcal{J}_2^{(p)} \subset \cdots \subset \mathcal{J}_m^{(p)}.
\]

Proposition 2.3. Let $q$ be a function in $\mathcal{J}_m^{(p)}$. Then
\[
The function $x \mapsto (\delta(x))^{2m}q(x)$ is in $L^1(B)$.
\]
\[
\|q\|_{m,n,p} < \infty.
\]
3. Existence result

We are concerned with the existence of positive solutions for the iterated polyharmonic nonlinear problems (1.1). For the proof, we need the next Lemma. For a given nonnegative function \( q \) in \( \mathcal{J}_{m,n}^{(p)} \), we define

\[ \mathcal{M}_q = \{ \theta \in \mathcal{B}(B), |\theta| \leq q \}. \]

Lemma 3.1. For any nonnegative function \( q \in \mathcal{J}_{m,n}^{(p)} \), the family of functions

\[
\left\{ \int_B \left( \frac{\delta(y)}{\delta(x)} \right)^m \Gamma_{m,n}^{(p)}(x,y) |\theta(y)| dy : \theta \in \mathcal{M}_q \right\}
\]

is uniformly bounded and equicontinuous in \( \mathcal{B} \) and consequently it is relatively compact in \( C(\overline{B}) \).

Proof. Let \( q \) be a nonnegative function \( q \in \mathcal{J}_{m,n}^{(p)} \) and \( L \) be the operator defined on \( \mathcal{M}_q \) by

\[ L\theta(x) = \int_B \left( \frac{\delta(y)}{\delta(x)} \right)^m \Gamma_{m,n}^{(p)}(x,y) |\theta(y)| dy. \]

By \( \mathcal{B} \), for each \( \theta \in \mathcal{M}_q \), we have

\[
\sup_{x \in \mathcal{B}} \int_B \left( \frac{\delta(y)}{\delta(x)} \right)^m \Gamma_{m,n}^{(p)}(x,y) |\theta(y)| dy \leq \| q \|_{m,n,p} < \infty.
\]

Then the family \( L(\mathcal{M}_q) \) is uniformly bounded. Next, we prove the equicontinuity of functions in \( L(\mathcal{M}_q) \) on \( \overline{B} \). Indeed, let \( x_0 \in \overline{B} \) and \( \varepsilon > 0 \). By (1.3), there exists \( \alpha > 0 \) such that for each \( x, x' \in B(x_0, \alpha) \cap B \), we have

\[ |L\theta(x) - L\theta(x')| \leq \int_B \left| \frac{\Gamma_{m,n}^{(p)}(x,y)}{(\delta(x))^m} - \frac{\Gamma_{m,n}^{(p)}(x',y)}{(\delta(x'))^m} \right| |\delta(y)| |q(y)| dy \]

\[ \leq \varepsilon + \int_{B \setminus B(x_0, 2\alpha)} \left| \frac{\Gamma_{m,n}^{(p)}(x,y)}{(\delta(x))^m} - \frac{\Gamma_{m,n}^{(p)}(x',y)}{(\delta(x'))^m} \right| |\delta(y)| |q(y)| dy \]

\[ + \int_{B \setminus B(x_0 + 2\alpha)} \left| \frac{\Gamma_{m,n}^{(p)}(x,y)}{(\delta(x))^m} - \frac{\Gamma_{m,n}^{(p)}(x',y)}{(\delta(x'))^m} \right| |\delta(y)| |q(y)| dy \]

Now since for \( y \in B^{c}(x, 2\alpha) \cap B \), from Proposition 2.1 we have

\[ \Gamma_{m,n}^{(p)}(x,y) \leq (\delta(x)\delta(y))^m. \]

We deduce that

\[ \int_{B \setminus B(x_0, 2\alpha)} \left| \frac{\Gamma_{m,n}^{(p)}(x,y)}{(\delta(x))^m} - \frac{\Gamma_{m,n}^{(p)}(x',y)}{(\delta(x'))^m} \right| |\delta(y)| |q(y)| dy \]

\[ \leq \int_{B \setminus B(x_0, 2\alpha)} (\delta(y))^2 |q(y)| dy, \]

which tends by (2.4) to zero as \( \alpha \to 0 \).

Since for \( y \in B^{c}(x_0, 2\alpha) \cap B \), the function \( x \mapsto \left( \frac{\delta(y)}{\delta(x)} \right)^m \Gamma_{m,n}^{(p)}(x,y) \) is continuous on \( B(x_0, \alpha) \cap B \), by (2.4) and by the dominated convergence theorem, we have

\[ \int_{B \setminus B^{c}(x_0, 2\alpha) \cap B^{c}(x, 2\alpha)} \left| \frac{\Gamma_{m,n}^{(p)}(x,y)}{(\delta(x))^m} - \frac{\Gamma_{m,n}^{(p)}(x',y)}{(\delta(x'))^m} \right| |\delta(y)| |q(y)| dy \to 0 \]
as \(|x - x'| \to 0\). This proves that the family \(L(M_q)\) is equicontinuous in \(\mathcal{B}\). It follows by Ascoli’s theorem, that \(L(M_q)\) is relatively compact in \(C(\mathcal{B})\).

The next remark will be used to obtain regularity of the solution.

**Remark 3.2.** Let \(r > n\) and \(f\) be a nonnegative measurable function in \(L^r(\mathcal{B})\). Then \(V_p f \in C^{2pm-1}(\mathcal{B})\).

Indeed, by using the regularity theory of [1] (see also [10, Theorem 5.1], and [7, Theorem IX.32]), we obtain that \(V_p f \in W^{2pm,r}(\mathcal{B})\). Furthermore, since \(r > n\), then one finds that \(V_p f \in C^{2pm-1}(\mathcal{B})\) (see [9, Chap. 7, p.158], or [7, Corollary IX.15]).

**Proof of Theorem 1.2.** Let \(K\) be compact in \(B\) such that \(\gamma := \int_K h(y)dy > 0\) and define \(r_0 := \min_{y \in K} (\delta(y))^m > 0\).

By (1.5) we can find \(a > 0\) such that for each \(t \geq 0\),

\[
c(\delta(x)\delta(y))^m \leq \Gamma^{[p]}_{m,n}(x,y).
\]

By (1.5) we can find \(a > 0\) such that \(c r_0 \gamma f(ar_0) \geq a\).

By (H4) and (2.3), the function \(k \in J^{[1]}_{m,n} \subset J^{[p]}_{m,n}\); then it follows from (2.5) that

\[
\delta := \|V_p((\delta(.))^m k)\|_\infty \leq \|k\|_{m,n,p} < \infty.
\]

Let \(0 < \alpha < \frac{1}{3}\), then using (1.6) we can find \(\eta > 0\) such that for each \(t \geq \eta\),

\[
g(t) \leq \alpha t. \quad \text{Put} \quad \beta := \sup_{0 \leq t \leq \eta} g(t). \quad \text{Then we have}
\]

\[
0 \leq g(t) \leq \alpha t + \beta, \quad \text{for} \quad t \geq 0.
\]

On the other hand, using (3.2) and (2.4), there exists a constant \(c_0 > 0\) such that

\[
V_p((\delta(.))^m k)(x) \geq c_0 (\delta(x))^m.
\]

From (H2), (2.5) and (2.3) we derive that

\[
\nu := \|V_p(\varphi(\cdot, a(\delta(.))^m))\|_\infty < \infty.
\]

Put \(b = \max\left\{\frac{a}{c_0} \alpha r_0 \frac{\alpha + \beta}{1 - \alpha} \right\}\) and let \(\Lambda\) be the convex set given by

\[
\Lambda = \left\{ u \in C(\mathcal{B}) : a(\delta(x))^m \leq u(x) \leq V_p(\varphi(\cdot, a(\delta(.))^m)(x) + bV_p((\delta(.))^m k)(x) \right\}.
\]

and \(T\) be the operator defined on \(\Lambda\) by

\[
Tu(x) = \int_B \Gamma^{[p]}_{m,n}(x,y)|\varphi(y,u(y)) + \psi(y,u(y))|dy.
\]

From (3.4), \(\Lambda \neq \emptyset\). We will prove that \(T\) has a fixed point in \(\Lambda\). Indeed, for \(u \in \Lambda\), we have by (1.4), (3.2) and the monotonicity of \(f\) that

\[
Tu(x) \geq \int_B \Gamma^{[p]}_{m,n}(x,y)\psi(y,u(y))dy
\]

\[
\geq c(\delta(x))^m \int_B (\delta(y))^m h(y)f(u(y))dy
\]

\[
\geq c(\delta(x))^m f(ar_0)r_0 \int_K h(y)dy
\]

\[
\geq a(\delta(x))^m.
\]
On the other hand, using (H1), (1.4) and (3.3), we deduce that
\[ Tu(x) \leq V_p(\varphi(\cdot, a(\delta(\cdot))^m) + \int_B \Gamma^{(p)}_{m,n}(x,y)(\delta(\cdot))^m k(y)g(u(y)) dy \]
\[ \leq V_p(\varphi(\cdot, a(\delta(\cdot))^m) + \int_B \Gamma^{(p)}_{m,n}(x,y)(\delta(\cdot))^m k(y)(\alpha u(y) + \beta) dy \]
\[ \leq V_p(\varphi(\cdot, a(\delta(\cdot))^m) + (\alpha(v + b\delta) + \beta)V_p(\delta(\cdot))^m k(x) \]
\[ \leq V_p(\varphi(\cdot, a(\delta(\cdot))^m) + bV_p(\delta(\cdot))^m k(x). \]

Let \( v(x) = \varphi(x, a(\delta(x))^m/(\delta(x))^m) \). Then using similar arguments as above, we deduce that for each \( u \in \Lambda \)
\[ \varphi(\cdot, u) \leq \varphi(\cdot, a(\delta(\cdot))^m) = (\delta(\cdot))^m v, \]
\[ \psi(\cdot, u) \leq g(u)(\delta(\cdot))^m k \leq b(\delta(\cdot))^m k. \] (3.5)

That is, \( \varphi(\cdot, u) + \psi(\cdot, u) \in \mathcal{M}_{(v + bk)(\delta(\cdot))^m}. \) Now since by (H2) and (H4), the function \( (v + bk)(\delta(\cdot))^m \in \mathcal{J}_{m,n}^{(1)} \subset \mathcal{J}_{m,n}^{(p)} \), we deduce from Lemma 3.1 that \( T(\Lambda) \) is relatively compact in \( C(\mathcal{B}) \). In particular, for all \( u \in \Lambda \), \( Tu \in C(\mathcal{B}) \) and so \( T(\Lambda) \subset \mathcal{B} \). Next, let us prove the continuity of \( T \) in \( \Lambda \). We consider a sequence \( (u_j)_{j \in \mathbb{N}} \) in \( \Lambda \) which converges uniformly to a function \( u \in \Lambda \). Then we have
\[ |Tu_j(x) - Tu(x)| \leq V_p[|\varphi(\cdot, u_j(\cdot)) - \varphi(\cdot, u(\cdot))| + |\psi(\cdot, u_j(\cdot)) - \psi(\cdot, u(\cdot))|. \]

Now, by (3.5), we have
\[ |\varphi(\cdot, u_j(\cdot)) - \varphi(\cdot, u(\cdot))| + |\psi(\cdot, u_j(\cdot)) - \psi(\cdot, u(\cdot))| \leq 2(1 + b)(\delta(\cdot))^m(v + k) \]
and since \( \varphi, \psi \) are continuous with respect on the second variable, we deduce by (2.5) and the dominated convergence theorem that
\[ \forall x \in B, Tu_j(x) \to Tu(x) \quad \text{as } j \to \infty \]
Since \( T\Lambda \) is relatively compact in \( C(\mathcal{B}) \), we have the uniform convergence, namely
\[ \|Tu_j - Tu\|_\infty \to 0 \quad \text{as } j \to \infty. \]
Thus we have proved that \( T \) is a compact mapping from \( \Lambda \) to itself. Hence by the Schauder fixed point theorem, there exists \( u \in \Lambda \) such that
\[ u(x) = \int_B \Gamma^{(p)}_{m,n}(x,y)[\varphi(y, u(y)) + \psi(y, u(y))] dy. \] (3.6)
Using (3.5), (H3) and (H5), for each \( y \in B \),
\[ \varphi(y, u(y)) + \psi(y, u(y)) \leq \varphi(y, a(\delta(y))^m) + b(\delta(y))^m k(y) \in \mathcal{L}^r(B). \] (3.7)
So it is clear that \( u \) satisfies (in the sense of distributions) the elliptic differential equation
\[ (-\Delta)^m u = \varphi(\cdot, u) + \psi(\cdot, u), \quad \text{in } B. \]
Furthermore, by (3.6), (3.7) and Remark 3.2, we deduce that \( u \in C^{2p-1}(B) \). Therefore, using again (3.6) and (2.1) we obtain for \( j \in \{0, \ldots, p-1\} \),
\[ (-\Delta)^m u(x) = \int_B \Gamma^{(p-j)}_{m,n}(x,y)[\varphi(y, u(y)) + \psi(y, u(y))] dy. \] (3.8)
Using similar arguments as above, we obtain for all \( j \in \{0, \ldots, p-1\} \),
\[ a_j(\delta(x))^m \leq (-\Delta)^m u(x) \leq V_{p-j}(\varphi(\cdot, a_j(\delta(\cdot))^m))(x) + b_jV_{p-j}(\delta(\cdot))^m k(x). \] (3.9)
where $a_j, b_j$ are positive constants. Finally, for $j \in \{0, \ldots, p-1\}$, from (3.9), (2.3) and (2.5), we have

$$a_j(\delta(x))^m \leq (-\Delta)^m u(x)$$

$$\leq (\delta(x))^m (\|\varphi\|_{0, a_j(\delta(x))^m} \|_{m,n,p-j} + b_j \|_m \|_{m,n,p-j})$$

$$\leq (\delta(x))^m.$$ 

So $u$ is the required solution. □

Example 3.3. Let $r > n$, $\lambda < m + \frac{1}{r}$, $\gamma \geq 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$. Let $\rho_1, \rho_2$ be a nontrivial nonnegative Borel measurable functions on $B$ satisfying $\rho_1(x) \leq (\delta(x))^{m(1+\gamma)-\lambda}$ and $\rho_2(x) \leq (\delta(x))^{m-\lambda}$. Then the problem

$$(-\Delta)^m u = \rho_1(x) u^{-\gamma} + \rho_2(x) u^\alpha \log(1 + u^\beta), \quad \text{in } B$$

$$u > 0 \quad \text{in } B$$

$$\lim_{|x| \to 1} (-\Delta)^m u(x) \left( \frac{1}{1 - |x|} \right)^{m-1} = 0, \quad \text{for } 0 \leq j \leq p-1,$$

has at least one positive solution, $u \in C^{2pm-1}(B)$, satisfying

$$(-\Delta)^m u(x) \sim (\delta(x))^m, \quad \forall j \in \{0, \ldots, p-1\}.$$

Remark 3.4. If $m = 1$ and $p \geq 1$, one can obtain similar existence result for (1.1) on a bounded domain $D \subset \mathbb{R}^n$ ($n \geq 2$) of class $C^{2p, \alpha}$ with $\alpha \in (0, 1]$.

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References


Imed Bachar
Département de mathématiques, Faculté des sciences de Tunis, campus universitaire, 2092 Tunis, Tunisia
E-mail address: Imed.Bachar@ipeit.rnu.tn