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ON A VARIATIONAL APPROACH TO EXISTENCE AND MULTIPLICITY RESULTS FOR SEMIPOSITONE PROBLEMS

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Abstract. In this paper we present a novel variational approach to the question of existence and multiplicity of positive solutions to semipositone problems in a bounded smooth domain of $\mathbb{R}^N$. We consider both the sublinear and superlinear cases.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. We are interested in presenting a variational approach to the question of finding *positive solutions* (i.e. nonnegative solutions without interior zeros in Ω) to a class of problems of the form

$$\begin{cases} -\Delta u = \lambda f(u) \quad \text{in } \Omega \\ \quad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a positive parameter and $f : [0, +\infty) \to \mathbb{R}$ is a continuous function satisfying the condition

(F0) $f(0) = -a < 0$.

Such problems are usually referred in the literature as *semipositone problems*. We refer the reader to [13], where Castro and Shivaji initially called them *nonpositone problems*, in contrast with the terminology *positone problems*, coined by Cohen and Keller in [18], when the nonlinearity f was positive and monotone. Here we will consider both the *sublinear case*, where f satisfies

(F1) $\lim_{s\to+\infty} \frac{f(s)}{s} = 0 < \lambda_1$,

(with $\lambda_1 > 0$ denoting the first eigenvalue of $-\Delta$ under Dirichlet boundary condition on Ω) and the *superlinear, subcritical case*, where f is such that

(F2) $\lim_{s\to+\infty} \frac{f(s)}{s} = +\infty, \quad |f(s)| \le C(1 + |s|^{p-2})$,

with $2 \le p < 2^* = \frac{2N}{N-2}$ if $N \ge 3$ ($2^* = +\infty$ if $N = 1, 2$). In this latter case, an assumption that is usually made to deal with compactness properties is the *Ambrosetti-Rabinowitz* condition:

($\hat{\text{F}}$2) $F(s) \le \theta f(s)s$ for all $s \ge K$ (and some $\theta \in (0, \frac{1}{2})$).

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The usual approaches to such semipositone problems are through *quadrature methods* (see e.g. [14, 11]), the method of *sub-super-solution* (e.g. [6, 10]), *degree theory* and/or *bifurcation theory* (see e.g. [2, 3]). We refer the author to the survey paper by Castro-Maya-Shivaji [12] and references therein. Let us consider the sublinear case, for example. As is well-known, in this case a super-solution can be easily found by considering the solution $\overline{u} > 0$ of the linear problem

$$-\Delta u = \lambda(\epsilon u + B)\,,$$

where $0 < \lambda\epsilon < \lambda_1$ and $B > 0$ are such that (cf. (F1))

$$f(s) \leq \epsilon s + B \quad \forall\ s > 0\,.$$

Moreover, by using the maximum principle, it follows that such a super-solution is an upper bound for any positive solution, or even sub-solution, of (1.1) (see [19]). Therefore, the main difficulty in proving the existence of a positive solution for (1.1) consists in finding a positive sub-solution. As a matter of fact, as can be easily seen, no positive sub-solution can exist if f does not assume positive values; and the fact that f has negative values for $s > 0$ small precludes the existence of any such sub-solution with small L^∞ norm. Thus the nonlinearity f must assume positive values and, as suggested by the results in [17], that should happen in such a way that

(F3) $F(\delta) > 0$ for some $\delta > 0$,

where $F(u) = \int_0^u f(s)ds$ as usual. In addition, one must also have λ bounded away from zero (see [20] and Lemma 3.1 below).

As already mentioned, our main objective in this article is to present a variational approach to the question of existence and multiplicity of positive solutions to such semipositone problems. We will do so by looking at (1.1) as a problem with the discontinuous nonlinearity $g(s)$ defined by

$$g(s) = H(s)f(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ f(s) & \text{if } s > 0\,, \end{cases} \tag{1.2}$$

where $H(s) = 0$ for $s \leq 0$, $H(s) = 1$ for $s > 0$ denotes the Heaviside function. More precisely, we will be considering the slightly modified problem

$$\begin{cases} -\Delta u = \lambda g(u) \quad \text{in } \Omega \\ \quad u = 0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{1.3}$$

We note that the set of positive solutions of (1.1) and (1.3) do coincide. Moreover, any non-zero solution u of (1.3) is nonnegative and, in fact, if the set $A_u := \{x \in \Omega \mid u(x) = 0\}$ has measure zero then u is an *a.e. positive* solution of (1.1). We will show this to be the case for some solutions when Ω is a ball.

We should mention that our results were inspired by the works of Ambrosetti-Struwe [5] and Chang [15]. On the other hand, we are not aware of any other work where solutions of semipositone problems were obtained directly through variational techniques. However, the authors in [7] have considered existence results for problem (1.3) through approximation of the discontinuous nonlinearity by a sequence of continuous functions. Variational methods were then applied to the corresponding sequence of problems and limits were taken. We believe that our direct variational approach to such problems is rather natural and conducive to dealing with more general situations.

Our main results concerning problems (1.1) and (1.3) are the following:

Theorem 1.1. *Assume* (F0), (F1) *and* (F3). *Then, there exist* $0 < \Lambda_0 \leq \Lambda_2$ *such that* (1.3) *has no nontrivial nonnegative solution for* $0 < \lambda < \Lambda_0$, *and has at least two nontrivial nonnegative solutions* $\widehat{u}_\lambda$, $\widehat{v}_\lambda$ *for all* $\lambda > \Lambda_2$. *Moreover, when* Ω *is a ball* $B_R = B_R(0)$, *these two solutions are non-increasing, radially symmetric and, if* $N \geq 2$, *at least one of them is positive, hence a solution of* (1.1).

Theorem 1.2. *Assume* (F0), (F2), ($\hat{\text{F}}$2) *and* (F3). *Then,* (1.3) *has at least one nonnegative solution* $\widehat{v}_\lambda$ *for all* $\lambda > 0$. *If* $\Omega = B_R$ *then* $\widehat{v}_\lambda$ *is non-increasing, radially symmetric and one of the two alternatives occurs:*

(i) *There exists* $\Lambda_1 > 0$ *such that, for all* $0 < \lambda < \Lambda_1$, $\widehat{v}_\lambda$ *is a positive solution of* (1.1) *having negative normal derivative on* ∂B_R;
(ii) *For some sequence* $\mu_n \to 0$, *problem* (P_{μ_n}) *has a positive solution* $\widehat{w}_{\mu_n}$ *with zero normal derivative on* ∂B_R.

2. The Abstract Framework

We start by recalling some basic results on variational methods for locally Lipschitz functionals $I : X \to \mathbb{R}$ defined on a real Banach space X with norm $\|\cdot\|$ (cf. [16, 15]), that is, for functionals such that, for each $u \in X$, there is a neighborhood $N = N_u$ of u and a constant $K = K_u$ for which

$$|I(v) - I(w)| \leq K\|v - w\| \quad \forall v, w \in N.$$

For given $u, h \in X$, the *generalized directional derivative of* I *at* u *in the direction of* h is defined by the formula

$$I^0(u; h) = \limsup_{k\to 0,\, t\downarrow 0} \frac{1}{t}[I(u + k + th) - I(u + k)]$$

The following properties are known to hold:

(*i*) $h \mapsto I^0(u; h)$ is sub-additive, positively homogeneous, continuous, and convex;
(*ii*) $|I^0(u; h)| \leq K_u\|h\|$;
(*iii*) $I^0(u; -h) = (-I)^0(u; -h)$.

Therefore, the so-called *generalized gradient of* I *at* u, written $\partial I(u)$, is defined as the subdifferential of the convex function $I^0(u; h)$ at $h = 0$, that is,

$$\mu \in \partial I(u) \subset X^* \quad \Longleftrightarrow \quad \langle \mu, h\rangle \leq I^0(u; h) \quad \forall h \in X.$$

For the convenience of the reader, we list below some of the main properties of the generalized gradient $\partial I(u)$:

(1) For each $u \in X$, $\partial I(u)$ is a non-empty convex and w^*-compact subset of X^*;
(2) $\|\mu\|_{X^*} \leq K_u$ for all $\mu \in \partial I(u)$;
(3) If $I,\ J : X \to \mathbb{R}$ are locally Lipschitz functionals then
$$\partial(I + J)(u) \subset \partial I(u) + \partial J(u);$$
(4) $\partial(\lambda I)(u) = \lambda\partial I(u)$ for all $\lambda \in \mathbb{R}$;
(5) If I is a convex functional then $\partial I(u)$ coincides with the usual subdifferential of I in the sense of convex analysis;
(6) If I has a Gateaux derivative $DI(v)$ at every point v of a neighborhood N of u and $DI : N \to X^*$ is continuous, then $\partial I(u) = \{DI(u)\}$;

(7) $I^0(u;h) = \max\{\langle \mu, h\rangle \mid \mu \in \partial I(u)\}$ for all $h \in X$;
(8) If I has a local minimum (or a local maximum) at $u_0 \in X$ then $0 \in \partial I(u_0)$.

Now, by definition, one says that $u \in X$ is a *critical point* of the locally Lipschitz functional I if

$$0 \in \partial I(u).$$

In this case the real number $c = I(u)$ is called a critical value of I. Note that property (8) above says that a local minimum (or local maximum) of I is a critical point of I.

On the other hand, I is said to satisfy the Palais-Smale condition $(PS)_c$ at the level $c \in \mathbb{R}$ if, for any sequence (u_n) such that $I(u_n) \to c$ and $\lambda(u_n) := \min_{\mu\in\partial I(u_n)} \|\mu\|_{X^*} \to 0$, one can extract a convergent subsequence. Finally, we point out that many of the results of the classical critical point theory have been extended by Chang [15] to this setting of locally Lipschitz functionals. For example, one has the celebrated:

Theorem 2.1 (Mountain-Pass Theorem, Ambrosetti-Rabinowitz [4])**.** *Let X be a reflexive Banach space and $I : X \to \mathbb{R}$ be a locally Lipschitz functional satisfying $(PS)_c$ for all $c > 0$ and the following geometric conditions:*

(i) *$I(0) = 0$ and there exist $\rho, \alpha > 0$ such that $I(u) \geq \alpha$ if $\|u\| = \rho$;*
(ii) *there exists $e \in X$ such that $\|e\| > \rho$ and $I(e) \leq 0$.*

Then I has a critical value $c \geq \alpha$ given by

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = e\}$.

For the rest of this article, we denote the H_0^1-norm by $\|u\| = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$ and we often use the same letter C to represent various positive constants.

3. Proofs of the Main Results

Now, having listed some basic results on critical point theory for Lipschitz functionals, let us consider the functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \lambda \int_\Omega G(u)\,dx,$$

where $G(u) = \int_0^u g(s)\,ds$ and $g(s)$ were defined in (1.2). Clearly $G : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function and satisfies $G(s) = 0$ for $s \leq 0$. In view of [15, Theorems 2.1 and 2.2], the above formula for $I_\lambda(u)$ defines a locally Lipschitz functional on $H_0^1(\Omega)$ whose critical points are solutions of the differential inclusion

$$-\Delta u(x) \in \lambda[\underline{g}(u(x)), \overline{g}(u(x))] \quad \text{a.e. in } \Omega,$$

where $\underline{g}(s) := \min\{g(s-0), g(s+0)\}$ and $\overline{g}(s) := \max\{g(s-0), g(s+0)\}$. In our present case, it follows that $\underline{g}(s) = \overline{g}(s) = f(s)$ for $s > 0$, $\underline{g}(s) = \overline{g}(s) = 0$ for $s < 0$, and $\underline{g}(0) = -a$, $\overline{g}(0) = 0$.

We start with some preliminary lemmas.

Lemma 3.1. *Assume* (F0), (F1) *and* (F3). *Then there exists $\Lambda_0 > 0$ such that* (1.3) *has no nontrivial solution $0 \leq u \in H_0^1(\Omega)$ for $0 < \lambda < \Lambda_0$.*

Proof. If $u \geq 0$ is a solution of (1.3) then, multiplying the equation by u and integrating over Ω yields

$$\frac{1}{2}\|u\|^2 = \lambda \int_\Omega g(u)u\,dx = \lambda\Big(\int_{[u\leq\delta_0]} ug(u)\,dx + \int_{[u\geq\delta_0]} ug(u)\,dx\Big)\,,$$

hence

$$\frac{1}{2}\|u\|^2 \leq \lambda \int_{[u\geq\delta_0]} ug(u)\,dx\,, \tag{3.1}$$

where we have chosen $\delta_0 > 0$ so that $g(s) \leq 0$ for $0 \leq s \leq \delta_0$ (such a δ_0 exists in view of (F0)). Now, since (F1) implies the existence of $C > 0$ such that

$$sg(s) \leq C(1 + s^2)$$

for all $s \geq 0$, we obtain from (3.1) that

$$\frac{1}{2}\|u\|^2 \leq \lambda C \int_{[u\geq\delta_0]} (1+u^2)\,dx \leq \lambda C(\frac{1}{\delta_0^2}+1)\int_{[u\geq\delta_0]} u^2\,dx \leq \lambda C \int_\Omega u^2\,dx\,,$$

so that

$$\frac{1}{2}\|u\|^2 \leq \lambda C\|u\|^2\,,$$

where this last constant $C > 0$ is independent of both u and λ. Therefore we must have

$$\lambda \geq \frac{1}{2C} := \Lambda_0 > 0\,.$$

□

Lemma 3.2. *Assume* (F0) *and either* (F1) *or* (F2). *Then* $u = 0$ *is a strict local minimum of the functional* I_λ.

Proof. In fact, we only need to assume (F0) and the condition

$$G(s) \leq C(1 + |s|^{2^*}) \text{ for all } s \in \mathbb{R},$$

which is implied by either (F1) or (F2). Recall also that $G(s) = 0$ for $s \leq 0$. Then, with $\delta_0 > 0$ as in the proof of Lemma 3.1 and noticing that $G(s) \leq 0$ for all $-\infty < s \leq \delta_0$, we can write for an arbitrary $u \in H_0^1(\Omega)$,

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \lambda\int_\Omega G(u)\,dx \\
&\geq \frac{1}{2}\|u\|^2 - \lambda\int_{[u\geq\delta_0]} G(u)\,dx \\
&\geq \frac{1}{2}\|u\|^2 - \lambda C\int_{[u\geq\delta_0]} (1+u^{2^*})\,dx \\
&\geq \frac{1}{2}\|u\|^2 - \lambda C(\frac{1}{\delta_0^{2^*}}+1)\int_{[u\geq\delta_0]} u^{2^*}\,dx\,,
\end{aligned}$$

so that, using Sobolev embedding theorem in the last inequality, and with a constant $C > 0$ independent of u and Ω, we obtain

$$I_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \lambda C\|u\|^{2^*} = \frac{1}{2}\|u\|^2(1 - 2\lambda C\|u\|^{2^*-2}).$$

Therefore, for each $0 < \rho < \rho_\lambda := 1/(2\lambda C)^{2^*-2}$, it follows that $I_\lambda(u) \geq \alpha_\rho > 0$ if $\|u\| = \rho$. This shows that $u = 0$ is a strict local minimum of I_λ. □

Remark 3.3. We note that both $\rho_\lambda > 0$ and $\alpha_\rho > 0$ obtained in the above proof do not depend on the domain Ω.

Lemma 3.4. *Under the same assumptions as in Lemma 3.2, let $\widehat{u} \in H_0^1(\Omega)$ be a critical point of I_λ. Then, $\widehat{u} \in C^{1,\epsilon}(\overline{\Omega})$ and $\widehat{u}$ is a nonnegative solution of* (1.3).

Proof. We will follow some of the arguments in [5, 15]. As mentioned earlier, if $\widehat{u}$ is a critical point of I_λ then it is shown in [15] that $\widehat{u}$ is a solution of the differential inclusion

$$-\Delta u \in \lambda[\underline{g}(u), \overline{g}(u)] \quad \text{a.e. in}\Omega\,, \tag{3.2}$$

where $\underline{g}(s) = \min\{g(s-0), g(s+0)\}$ and $\overline{g}(s) = \max\{g(s-0), g(s+0)\}$. Since g is only discontinuous at $s = 0$, the above differential inclusion reduces to an equality, except possibly on the subset $A \subset \Omega$ where $\widehat{u} = 0$. And, in this latter case, $-\Delta\widehat{u}$ takes on values in the bounded interval $[-a, 0]$. Therefore, by standard elliptic regularity, it follows that $\widehat{u} \in H_0^1 \cap W^{2,p}(\Omega)$ for all $p \geq 2$. In particular, $\widehat{u}$ is of class $C^{1,\epsilon}$, $0 < \epsilon < 1$.

Next, in view of a well-known result of Stampacchia, we have that $-\Delta\widehat{u} = 0$ a.e. in A. Therefore, since we defined $g(0) = 0$, it follows that

$$-\Delta\widehat{u} = g(\widehat{u}) \quad \text{a.e. in } \Omega.$$

Replacing the inclusion (3.2) on $\widehat{u}$, we conclude that $\widehat{u} \in C^{1,\epsilon}(\overline{\Omega})$ is a solution of (1.3). Finally, recalling that $g(s) = 0$ for $s \leq 0$, it is clear that $\widehat{u} \geq 0$. The proof of Lemma 3.4 is complete. □

Lemma 3.5. *Assume either* (F1) *or* (F2), ($\hat{\text{F}}$2). *Then I_λ satisfies the Palais-Smale condition $(PS)_c$ at every $c \in \mathbb{R}$.*

Proof. The proof in either case is a direct consequence of Theorem 4.3 and Theorem 4.4, respectively, in Chang [15]. In the superlinear case, it only suffices to notice that ($\hat{\text{F}}$2) implies the corresponding condition in Theorem 4.4,

$$J(u) \leq \theta \min_{\mu \in \partial J(u)} \langle \mu, u\rangle + M \quad \forall u \in H_0^1(\Omega)\,, \tag{3.3}$$

where $J(u) = \int_\Omega G(u)\,dx$, $u \in H_0^1(\Omega)$, in our present case. But this follows immediately by observing that we can identify $\mu \in (H_0^1)^*$ with a function $w \in H_0^1$ and that the inclusion

$$\partial J(u) \subset [\underline{g}(u), \overline{g}(u)]$$

says that given $w \in \partial J(u)$ then $w(x) = g(u(x))$ if $u(x) \neq 0$, $w(x) \in [-a, 0]$ if $u(x) = 0$. Therefore,

$$\langle w, u\rangle = \int_\Omega g(u)u\,dx \quad \text{for all } w \in \partial J(u),$$

so that ($\hat{\text{F}}$2) clearly implies ((3.3)) □

Lemma 3.6. *Under assumptions* (F0) *and* (F1), *let $\Omega = B_R \subset \mathbb{R}^N$ with $N \geq 2$, and let $u \in C^1(\overline{B}_R)$ be a radially symmetric, non-increasing function such that $u \geq 0$ and u is a minimizer of I_λ with $I_\lambda(u) = m < 0$. Then, u does not vanish in B_R, that is, $u(x) > 0$ for all $x \in B_R$.*

Proof. Since g is discontinuous at zero, we note that the conclusion does not follow directly from uniqueness of solution for the Cauchy problem with data at $r = R$ (In fact, writing $u = u(r)$, $r = |x|$, we may have $u(R) = u'(R) = 0$ and $u \not\equiv 0$).

Now, since $u \not\equiv 0$ by assumption, $R_0 := \inf\{r \le R \mid u(s) = 0 \text{ for } r \le s \le R\}$ satisfies $0 < R_0 \le R$. If $R_0 = R$ there is nothing to prove in view of the fact that u is non-increasing. On the other hand, if $R_0 < R$ then $u'(R_0) = 0$ and $u(r) > 0$ for $r \in [0, R_0)$. It is not hard to prove that this contradicts that u is a minimizer of I_λ. Indeed, if $R_0 < R$ then

$$I_\lambda(u) = \frac{1}{2}\int_{B_{R_0}} |\nabla u|^2\,dx - \lambda \int_{B_{R_0}} G(u)\,dx = m < 0\,.$$

A simple calculation shows that the re-scaled function $v(r) = u(\frac{R_0 r}{R}) \in H_0^1(B_R) \cap C^1(\overline{B_R})$ satisfies

$$I_\lambda(v) = \delta^{2-N}\Big[\frac{1}{2}\int_{B_{R_0}} |\nabla u|^2\,dx - \delta^{-2}\lambda \int_{B_{R_0}} G(u)\,dx\Big],$$

where $\delta := R_0/R$ is less than 1. Therefore, since we are assuming $N \ge 2$, we would reach the contradiction $I_\lambda(v) < m$. $\square$

Proof of Theorem 1.1. We observe that the functional I_λ is weakly lower semi-continuous on $H_0^1(\Omega)$. Moreover, the sublinearity assumption (F1) on $g(u)$ implies that I_λ is coercive. Therefore, the infimum of I_λ is attained at some $\widehat{u}_\lambda$:

$$\inf_{u \in H_0^1} I_\lambda(u) = I_\lambda(\widehat{u}_\lambda).$$

And, in view of Lemma 3.4, $\widehat{u}_\lambda \in C^{1,\epsilon}(\overline{\Omega})$ is a nonnegative solution of (1.3). We now claim that $\widehat{u}_\lambda$ is nonzero for all $\lambda > 0$ large.

Claim: There exists $\Lambda > 0$ such that $I_\lambda(\widehat{u}_\lambda) < 0$ for all $\lambda \ge \Lambda$.

In order to prove the claim it suffices to exhibit an element $\widehat{w} \in H_0^1(\Omega)$ such that $I_\lambda(\widehat{w}) < 0$ for all $\lambda > 0$ large. This is quite standard considering that $G(\delta) > 0$ by (F3). Indeed, letting $\Omega_\epsilon := \{x \in \Omega \mid dist(x, \partial\Omega) > \epsilon\}$ for $\epsilon > 0$ small, define $\widehat{w}$ so that $\widehat{w}(x) = \delta$ for $x \in \Omega_\epsilon$ and $0 \le \widehat{w}(x) \le \delta$ for $x \in \Omega\backslash\Omega_\epsilon$. Then

$$\begin{aligned} I_\lambda(\widehat{w}) &= \frac{1}{2}\|\widehat{w}\|^2 - \lambda\Big(\int_{\Omega_\epsilon} G(\widehat{w})\,dx + \int_{\Omega\backslash\Omega_\epsilon} G(\widehat{w})\,dx\Big) \\ &\le \frac{1}{2}\|\widehat{w}\|^2 - \lambda\big(G(\delta) meas(\Omega_\epsilon) - C(1+\delta^2) meas(\Omega\backslash\Omega_\epsilon)\big)\,, \end{aligned}$$

where we note that the expression in the above parenthesis is positive if we choose $\epsilon > 0$ sufficiently small. Therefore, there exists $\Lambda > 0$ such that $I_\lambda(\widehat{w}) < 0$ for all $\lambda \ge \Lambda$, which proves the claim.

On the other hand, when $\Omega = B_R$, let $\widehat{u}_\lambda^*$ denote the *Schwarz Symmetrization* of $\widehat{u}_\lambda$, namely, $\widehat{u}_\lambda^*$ is the unique radially symmetric, non-increasing, nonnegative function in $H_0^1(B_R)$ which is equi-measurable with $\widehat{u}_\lambda$. As is well known,

$$\int_{B_R} G(\widehat{u}_\lambda^*)\,dx = \int_{B_R} G(\widehat{u}_\lambda)\,dx$$

and $\|\widehat{u}_\lambda^*\| \le \|\widehat{u}_\lambda\|$, so that $I_\lambda(\widehat{u}_\lambda^*) \le I_\lambda(\widehat{u}_\lambda)$. Therefore, we necessarily have $I_\lambda(\widehat{u}_\lambda^*) = I_\lambda(\widehat{u}_\lambda)$ and may assume that $\widehat{u}_\lambda = \widehat{u}_\lambda^*$. Moreover, $\widehat{u}_\lambda > 0$ in Ω by Lemma 3.6. Therefore, $\widehat{u}_\lambda$ is a positive solution of both (1.1) and (1.3)

Next, we recall that $u = 0$ is a strict local minimum of I_λ by Lemma 3.1. Therefore, since I_λ satisfies the Palais-Smale condition by Lemma 3.5, we can use the minima $u = 0$ and $u = \widehat{u}_\lambda$ of I_λ to apply the Mountain Pass Theorem and conclude that there exists a second nontrivial critical point $\widehat{v}_\lambda$ with $I_\lambda(\widehat{v}_\lambda) > 0$. Again, $\widehat{v}_\lambda$ is a nonnegative solution of (1.3) in view of Lemma 3.4. In addition, when $\Omega = B_R$, arguments similar to those in [8, Theorem 3.4] (see pp. 403-405) show that we may assume $\widehat{v}_\lambda = \widehat{v}_\lambda^*$. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. As is well-known, the superlinearity condition (F2) readily implies the existence of an element $e_\lambda \in H_0^1(\Omega)$ such that $I_\lambda(e_\lambda) \leq 0$. In fact, the weaker condition $\lim_{s\to+\infty} F(s)/s^2 = +\infty$ suffices for that purpose. On the other hand, Lemma 3.2 says that $u = 0$ is a (strict) local minimum of I_λ and Lemma 3.5 says that I_λ satisfies $(PS)_c$ for every $c \in \mathbb{R}$. Therefore, an application of the Mountain-Pass Theorem stated in section 2 yields the existence of a critical point $\widehat{v}_\lambda$ such that

$$I_\lambda(\widehat{v}_\lambda) > 0.$$

In particular, $\widehat{v}_\lambda \neq 0$, and it follows that $\widehat{v}_\lambda$ is a nonnegative solution of (1.3) by Lemma 3.4. As in the proof of Theorem 1.1, we may assume that $\widehat{v}_\lambda = \widehat{v}_\lambda^*$ in the case of a ball $\Omega = B_R$.

Finally, still in the case of a ball $\Omega = B_R$, we claim that there exists $\Lambda_1 > 0$ such that, for all $0 < \lambda < \Lambda_1$, $\widehat{v}_\lambda = \widehat{v}_\lambda^*$ is a positive solution of (1.3) (hence of (1.1)) having negative normal derivative on ∂B_R.

Indeed, if that is not the case then, for any given $\lambda > 0$, we can find $0 < \mu = \mu(\lambda) < \lambda$ such that the nonnegative solution $\widehat{v}_\mu = \widehat{v}_\mu^*$ of $(\widehat{P}_\mu)$ obtained above satisfies

$$\widehat{v}_\mu(r) > 0 \quad \text{for } r \in [0, R_0), \quad \widehat{v}_\mu'(R_0) = 0 \quad \text{and} \quad \widehat{v}_\mu(r) = 0 \quad \text{for } r \in [R_0, R],$$

for some $0 < R_0 \leq R$. Therefore, the re-scaled function $\widehat{w}_\mu(r) := \widehat{v}_\mu(\frac{R_0 r}{R})$ is a positive solution of (P_{μ_0}) (again in the ball B_R), with $\mu_0 := \mu R_0^2/R^2 \leq \mu$. This shows that we can always construct a decreasing sequence $\mu_n > 0$ satisfying alternative (ii) of Theorem 1.2, in case alternative (i) does not hold. □

4. Final Remarks

As we shall explain, the results in both Theorem 1.1 and Theorem 1.2 concerning the semipositone problem (1.1) in a ball are optimal in a sense to be made clear in what follows.

4.1. **The Sublinear Case.** In view of the paper [11] we know that, in case Ω is a bounded domain with a convex outer boundary, problem (1.1) has a unique nonnegative solution for all $\lambda > 0$ large provided that, in addition to (F0), one assumes

- (i) $\lim_{s\to\infty} f(s) = \infty$,
- (ii) $\lim_{s\to\infty} f(s)/s = 0$,
- (iii) f is increasing and concave.

Furthermore, it is shown in [11] that this unique nonnegative solution is in fact positive in Ω. Therefore, under these hypotheses, we conclude that at least one of the two solutions obtained in Theorem 1.1 has to have a large zero-set in the sense that meas$\{x \in \Omega \mid u(x) = 0\} > 0$ (since, as we mentioned in the Introduction, a nontrivial solution of (1.3) with $meas\{x \in \Omega \mid u(x) = 0\} = 0$ is a nonnegative solution of (1.1)).

Moreover, in the specific case of a ball $\Omega = B_R$, we know from Theorem 1.1 that both nonnegative solutions of (1.3) are radially symmetric and non-increasing, with one of them, say $\widehat{u}_\lambda$, being in fact the unique positive solution of (1.1) for $\lambda > 0$ large. Therefore, the second solution $\widehat{v}_\lambda$ must necessarily satisfy $\widehat{v}_\lambda(r) > 0$ for $0 \le r < R_0$, $\widehat{v}_\lambda(r) = 0$ for $R_0 \le r \le R$, and $\widehat{v}'_\lambda(R_0) = 0$, for some $0 < R_0 < R$ (recall that, by Lemma 2.4, we have $\widehat{v}_\lambda \in C^{1,\epsilon}(\overline{\Omega})$). Therefore, the natural extension of $\widehat{v}_\lambda$ to $\mathbb{R}^N$, by letting $\widehat{v}_\lambda = 0$ outside B_R, yields a *bump (compactly supported) solution* of

$$-\Delta u = \lambda g(u) \quad \text{in } \mathbb{R}^N.$$

4.2. **The Superlinear Case.** In view of the paper [9], it is known that when Ω is a ball B_R, problem (1.1) has no nonnegative radially symmetric solution for all $\lambda > 0$ sufficiently large provided that, in addition to (F0), one assumes

(i) $\liminf_{s \to \infty} \frac{f(s)}{s^\alpha} > 0$ (for some $\alpha > 1$),
(ii) f is increasing.

Therefore, for $\lambda > 0$ large, the nonnegative solution $\widehat{v}_\lambda$ obtained in Theorem 1.2 for the case of the ball B_R must have a *large* zero-set. It follows, similarly to the previous case, that the natural extension of $\widehat{v}_\lambda$ to $\mathbb{R}^N$ yields again a *bump solution* of

$$-\Delta u = \lambda g(u) \quad \text{in } \mathbb{R}^N.$$

Secondly, a result in [14] yields existence of a positive radially symmetric solution for (1.1) when $\lambda > 0$ is small and, in addition to (F0), one assumes suitable technical conditions on the superlinearity f. Moreover, such a solution is shown to have negative normal derivative on ∂B_R. We thus see that, for appropriate classes of f's, alternative (i) of Theorem 1.2 must hold true. On the other hand, under still further conditions on f, it is shown in [1] that the above positive solution obtained in [14] is unique, thus precluding alternative (ii) of Theorem 1.2. It would be interesting to find out whether alternative (ii) can indeed occur in some other superlinear situations.

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