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ON THE FIRST EIGENVALUE OF THE STEKLOV EIGENVALUE PROBLEM FOR THE INFINITY LAPLACIAN

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ABSTRACT. Let Λ_p^p be the best Sobolev embedding constant of $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, where Ω is a smooth bounded domain in $\mathbb{R}^N$. We prove that as $p \to \infty$ the sequence Λ_p converges to a constant independent of the shape and the volume of Ω, namely 1. Moreover, for any sequence of eigenfunctions u_p (associated with Λ_p), normalized by $\|u_p\|_{L^\infty(\partial\Omega)} = 1$, there is a subsequence converging to a limit function u_∞ which satisfies, in the viscosity sense, an ∞-Laplacian equation with a boundary condition.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. The minimum Λ_p^p of the Rayleigh quotient, among all nonzero functions in the Sobolev space $W^{1,p}(\Omega)$,

$$\frac{\int_\Omega (|\nabla u|^p + |u|^p)dx}{\int_{\partial\Omega} |u|^p dx}$$

is the first eigenvalue of the problem

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u = 0, \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and $\partial/\partial\nu$ is the outer normal derivative along $\partial\Omega$. The eigenvalue problem (1.1) is understood in the weak sense, i.e, $(u, \lambda) \in W^{1,p}(\Omega) \times \mathbb{R}^+$ is an eigenpair if

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dx = \lambda \int_{\partial\Omega} |u|^{p-2}uvds, \quad \forall v \in W^{1,p}(\Omega).$$

The first eigenvalue $\Lambda_p^p = \lambda_1$ is the best constant of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ and it satisfies

$$\Lambda_p \|u\|_{L^p(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega).$$

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In this paper we are interested in finding the limit as $p \to \infty$ of Λ_p. Alternatively, we want to look at the limit of the minimum, as $p \to \infty$, of the ratio

$$\frac{\left(\int_\Omega (|\nabla u|^p + |u|^p) dx\right)^{1/p}}{\left(\int_{\partial\Omega} |u|^p ds\right)^{1/p}}.$$

It is easy to see that for any positive numbers a and b,

$$\lim_{p\to\infty} (a^p + b^p)^{1/p} = \max\{a, b\}.$$

Thus we anticipate that

$$\lim_{p\to\infty} \frac{\left(\int_\Omega (|\nabla u|^p + |u|^p) dx\right)^{1/p}}{\left(\int_{\partial\Omega} |u|^p dx\right)^{1/p}} = \frac{\max\{\|\nabla u\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)}\}}{\|u\|_{L^\infty(\partial\Omega)}}.$$

However, the minimization problem, with minimum value equal 1,

$$\inf_{u\in W^{1,\infty}(\Omega)\setminus\{0\}} \frac{\max\{\|\nabla u\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)}\}}{\|u\|_{L^\infty(\partial\Omega)}}, \tag{1.2}$$

has too many solutions. In fact, given a minimizer, we can modify it on any ball inside the domain to obtain another one. The correct Euler-Lagrange equation turns out to be

$$\begin{gathered} \max\{u - |\nabla u|, -\Delta_\infty u\} = 0, \quad \text{in } \Omega, \\ \min\{|\nabla u| - u, \frac{\partial u}{\partial \nu}\} = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where the operator

$$\Delta_\infty u = \sum_{i,j=1}^N \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial x_i} \frac{\partial u}{\partial x_i},$$

is called the ∞-Laplacian.

It is clear that for each $p > 1$ the Sobolev embedding constant Λ_p depends on the shape and the volume of domain Ω. We show that when passing to the limit, Λ_p converges to a constant independent of the domain Ω, namely 1. We can also choose a sequence of first eigenfunctions u_p such that the sequence converges uniformly in $C^\alpha(\overline{\Omega})$ to a function u_∞ that satisfies (1.3) in the viscosity sense. In this case we say $(u_\infty, 1)$ is an eigenpair of (1.3). Our main result is:

Theorem 1.1. *For the first eigenvalue of* (1.1) *we have*

$$\lim_{p\to\infty} \Lambda_p^{1/p} = 1.$$

For each $p > 1$, let u_p positive eigenfunction be a positive eigenfunction associated with Λ_p^p such that $\|u_p\|_{L^\infty(\partial\Omega)} = 1$. Then there exists a sequence $p_i \to \infty$ such that $u_{p_i} \to u_\infty$ in $C^\alpha(\overline{\Omega})$. The limit u_∞ is a solution of (1.3) in the viscosity sense.

To complete the introduction let us mention some recent work on the subject. In [7, 8] the authors study eigenvalue problem for the ∞-Laplacian with Dirichlet boundary condition. In [1], Steklov eigenvalues for the ∞-Laplacian are studied when one considers the limit as $p \to \infty$ of the eigenvalue problem

$$\begin{gathered} -\Delta_p u = 0, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{p-2} u, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.4}$$

It is known that the structure of the spectrum of (1.4) is the same as that of (1.1), see [3] and [12] . However, the first eigenvalue of (1.4) is 1 for any $p > 1$ with corresponding constant eigenfunctions in $W^{1,p}(\Omega)$; thus, theorem 1.1 is trivial for problem (1.4). Some arguments and technicalities used here are adapted from [1, 7, 8].

2. Main Results

We first recall the definition of viscosity solutions. Let

$$F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N) \to \mathbb{R},$$
$$B : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R},$$

where $\mathcal{S}(N)$ denotes the set of $N \times N$ symmetric matrices.

Consider the boundary-value problem

$$\begin{gathered} F(x, u, \nabla u, D^2u) = 0, \\ B(x, u, \nabla u) = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

Definition 2.1. (i) An upper semicontinuous function u is a *viscosity subsolution* of (2.1) if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$, we have

$$\min\{F(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)), B(x_0, \phi(x_0), \nabla\phi(x_0))\} \le 0, \quad x_0 \in \partial\Omega,$$
$$F(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \le 0, \quad x_0 \in \Omega.$$

(ii) A lower semicontinuous function u is a *viscosity supersolution* of (2.1) if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$, we have

$$\max\{F(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)), B(x_0, \phi(x_0), \nabla\phi(x_0))\} \ge 0, \quad x_0 \in \partial\Omega,$$
$$F(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \ge 0, \quad x_0 \in \Omega.$$

(iii) u is a *viscosity solution* of (2.1) if it is both a supersolution and a subsolution.

In (i) and (ii) the extrema at x_0 need not be strict. We refer to [4] for the theory of viscosity solutions in general and [2] for viscosity solutions with general boundary conditions.

If u is a smooth eigenfunction of (1.1) then by differentiation we get

$$\begin{gathered} -|\nabla u|^{p-2}\Delta u - (p-2)|\nabla u|^{p-4}\Delta_\infty u + |u|^{p-2}u = 0, \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} - \lambda|u|^{p-2}u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{2.2}$$

In this case,

$$F(x, z, X, S) = -|X|^{p-2}\text{trace}(S) - (p-2)|X|^{p-4}\langle S \cdot X, X\rangle + |z|^{p-2}z, \tag{2.3}$$
$$B(x, z, X) = |X|^{p-2}\langle X, \nu(x)\rangle - \lambda|z|^{p-2}z. \tag{2.4}$$

It is known that eigenfunctions of (1.1) are in $C^{1,\alpha}(\overline{\Omega})$, see [9, 10] and references therein. Thus it makes sense to talk about viscosity solutions. The following lemma tells us that an eigenfunction is a viscosity solution.

Lemma 2.2. *A weak solution u of* (1.1) *is a viscosity solution of* (2.2).

Proof. We present the details for the case of supersolutions. Let $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u(x_0) = \phi(x_0)$ and $u(x) > \phi(x)$, for $x \neq x_0$. We want to show that

$$-|\nabla\phi|^{p-2}\Delta\phi(x_0) - (p-2)|\nabla\phi|^{p-4}\Delta_\infty\phi(x_0) + |u|^{p-2}u(x_0) \geq 0.$$

Suppose that this is not the case, then by continuity there exists a radius $r > 0$ such that, for any $x \in B(x_0, r)$,

$$-|\nabla\phi|^{p-2}\Delta\phi(x) - (p-2)|\nabla\phi|^{p-4}\Delta_\infty\phi(x) + |u|^{p-2}u(x) < 0.$$

Set $m = \inf\{u(x) - \phi(x) : |x - x_0| = r\} > 0$ and let $\Phi(x) = \phi + \frac{1}{2}m$. The function Φ satisfies $\Phi < u$ on $\partial B(x_0, r)$, $\Phi(x_0) > u(x_0)$ and

$$-\operatorname{div}(|\nabla\Phi|^{p-2}\nabla\Phi(x)) + |u|^{p-2}u(x) < 0.$$

Multiplying by $(\Phi - u)^+$ extended by zero outside $B(x_0, r)$ we get

$$\int_{\{\Phi>u\}} |\nabla\Phi|^{p-2}\nabla\Phi \cdot \nabla(\Phi - u) + \int_{\{\Phi>u\}} |u|^{p-2}u(\Phi - u) < 0. \tag{2.5}$$

Since u is a weak solution, we have

$$\int_{\{\Phi>u\}} |\nabla u|^{p-2}\nabla u \cdot \nabla(\Phi - u) + \int_{\{\Phi>u\}} |u|^{p-2}u(\Phi - u) = 0. \tag{2.6}$$

Subtracting (2.6) from (2.5) we get

$$\int_{\{\Phi>u\}} \langle|\nabla\Phi|^{p-2}\nabla\Phi - |\nabla u|^{p-2}\nabla u, \nabla(\Phi - u)\rangle < 0.$$

We obtain a contradiction since the left hand side is bounded below by

$$C(N,p)\int_{\{\Phi>u\}} |\nabla\Phi - \nabla u|^p,$$

where $C(N,p)$ is a positive constant depending only on N and p.

Let λ be the eigenvalue corresponding to u in (1.1). If $x_0 \in \partial\Omega$ and ϕ is a function in $C^2(\overline{\Omega})$ such that $u(x_0) = \phi(x_0)$ and $u(x) > \phi(x)$, for $x \neq x_0$. We want to prove that either

$$-|\nabla\phi|^{p-2}\Delta\phi(x_0) - (p-2)|\nabla\phi|^{p-4}\Delta_\infty\phi(x_0) + |u|^{p-2}u(x_0) \geq 0,$$

$$\text{or} \quad |\nabla\phi|^{p-2}\frac{\partial\phi}{\partial\nu}(x_0) - \lambda|u|^{p-2}u(x_0) \geq 0.$$

Suppose that this is not the case. We repeat the previous argument to obtain

$$\begin{aligned}&\int_{\{\Phi>u\}} |\nabla\Phi|^{p-2}\nabla\Phi \cdot \nabla(\Phi - u) + \int_{\{\Phi>u\}} |u|^{p-2}u(\Phi - u)\\ &< \lambda\int_{\partial\Omega\cap\{\Phi>u\}} |u|^{p-2}u(\Phi - u),\end{aligned}$$

and

$$\begin{aligned}&\int_{\{\Phi>u\}} |\nabla u|^{p-2}\nabla u \cdot \nabla(\Phi - u) + \int_{\{\Phi>u\}} |u|^{p-2}u(\Phi - u)\\ &= \lambda\int_{\partial\Omega\cap\{\Phi>u\}} |u|^{p-2}u(\Phi - u),\end{aligned}$$

which implies

$$\int_{\{\Phi>u\}} \langle |\nabla\Phi|^{p-2}\nabla\Phi - |\nabla u|^{p-2}\nabla u, \nabla(\Phi - u)\rangle < 0.$$

A contradiction is established. This proves that u is a viscosity supersolution. As mentioned, the proof that u is a viscosity subsolution is similar. □

We are ready to pass to limit as $p \to \infty$ in the eigenvalue problem. Using the characterization

$$\Lambda_p = \min_{u\neq 0} \frac{\left(\int_\Omega (|\nabla u|^p + |u|^p)dx\right)^{1/p}}{\left(\int_{\partial\Omega} |u|^p ds\right)^{1/p}}, \tag{2.7}$$

one can show the following statement.

Lemma 2.3.

$$\limsup_{p\to\infty} \Lambda_p \le 1.$$

Proof. Let $v(x) \equiv 1$, $x \in \overline{\Omega}$. It follows from (2.7) that

$$\Lambda_p \le \frac{\left(\int_\Omega (|\nabla v|^p + |v|^p)dx\right)^{1/p}}{\left(\int_{\partial\Omega} |v|^p ds\right)^{1/p}} = \frac{|\Omega|^{1/p}}{|\partial\Omega|^{1/p}},$$

where $|\Omega|$ is the Lebesgue measure of Ω and $|\partial\Omega|$ is the boundary measure of $\partial\Omega$. Letting $p \to \infty$ we obtain the inequality. □

We recall from [9, 11] that the first eigenvalue Λ_p^p is isolated and simple. Any eigenfunction associated with Λ_p^p is either positive or negative in Ω and any other eigenfunction (not associated with Λ_p^p) has to change sign. We show that Λ_p converges to $\Lambda_\infty = 1$, the minimum value of (1.2). We also construct a minimizer of (1.2).

Proposition 2.4. *Given u_p, a positive eigenfunction of* (1.1) *associated with eigenvalue Λ_p^p, normalized by $\|u_p\|_{L^\infty(\partial\Omega)} = 1$. Then there exists a sequence $p_i \to \infty$ such that $u_{p_i} \to u_\infty$ in $C^\alpha(\overline{\Omega})$, where the limit u_∞ is a minimizer of* (1.2) *and*

$$\lim_{p\to\infty} \Lambda_p = 1.$$

Proof. Fix $q > N$. For any $p > q$, one has

$$\begin{aligned} \Big(\int_\Omega |\nabla u_p|^q + |u_p|^q\Big)^{1/q} &\le |\Omega|^{(1/q)-(1/p)} \Big[\int_\Omega (|\nabla u_p|^q + |u_p|^q)^{p/q}\Big]^{1/p} \\ &\le (2|\Omega|)^{(1/q)-(1/p)} \Big(\int_\Omega |\nabla u_p|^p + |u_p|^p\Big)^{1/p} \\ &= (2|\Omega|)^{(1/q)-(1/p)} \Lambda_p \Big(\int_{\partial\Omega} |u_p|^p\Big)^{1/p} \\ &\le \Lambda_p (2|\Omega|)^{(1/q)-(1/p)} |\partial\Omega|^{1/p}. \end{aligned} \tag{2.8}$$

In above expression, the first inequality follows from a Hölder inequality. We have used that $(a+b)^r \le 2^{r-1}(a^r + b^r)$, $r \ge 1$, for the second inequality and that $\|u_p\|_{L^\infty(\partial\Omega)} = 1$ for the last inequality. We obtain from (2.8) that $\{u_p\}$ is uniformly bounded in $W^{1,q}(\Omega)$. Thus there exists a subsequence $\{u_{p_i}\}$ converging to a function u_∞ weakly in $W^{1,q}(\Omega)$. Since $q > N$, the Sobolev compact embedding

$W^{1,q}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$ holds for any $\alpha \in (0, 1 - N/q)$. It follows that $\{u_{p_i}\}$ converges to u_∞ uniformly in $C^\alpha(\overline{\Omega})$. Moreover, as $p_i \to \infty$, (2.8) becomes

$$\Big(\int_\Omega |\nabla u_\infty|^q + |u_\infty|^q\Big)^{1/q} \le \liminf_{p_i\to\infty} \Lambda_{p_i}(2|\Omega|)^{1/q} \le (2|\Omega|)^{1/q}. \tag{2.9}$$

On the other hand,

$$\max\{\|\nabla u_\infty\|_{L^q(\Omega)}, \|u_\infty\|_{L^q(\Omega}\} \le \Big(\int_\Omega |\nabla u_\infty|^q + |u_\infty|^q\Big)^{1/q} \tag{2.10}$$

Letting $q \to \infty$, (2.9) and (2.10) imply that

$$\max\{\|\nabla u_\infty\|_{L^\infty(\Omega)}, \|u_\infty\|_{L^\infty(\Omega)}\} \le 1.$$

The uniform convergence of $\{u_{p_i}\}$ in $C^\alpha(\overline{\Omega})$ gives $\|u_\infty\|_{L^\infty(\partial\Omega)} = 1$. Hence

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \le 1 \text{ and } \|u_\infty\|_{L^\infty(\Omega)} = \|u_\infty\|_{L^\infty(\partial\Omega))} = 1.$$

Clearly, u_∞ is the minimizer of (1.2). Furthermore,

$$1 = \|u_\infty\|_{L^\infty(\Omega)} \le \lim_{q\to\infty} \Big(\int_\Omega |\nabla u_\infty|^q + |u_\infty|^q\Big)^{1/q} \le \liminf_{p_i\to\infty} \Lambda_{p_i},$$

which together with the lemma 2.3 gives $\lim_{p_i\to\infty} \Lambda_{p_i} = 1$. Since the limit holds for any subsequence, we conclude that $\lim_{p\to\infty} \Lambda_p = 1$. The proof is complete. □

Let us verify that the limit of (2.2) as $p \to \infty$ is (1.3) in the viscosity sense. We obtain from proposition 2.4 that there is a sequence of positive eigenfuntions $\{u_{p_i}\}$ converging to u_∞ uniformly in $\overline{\Omega}$ as $p_i \to \infty$. Consequently, $u_\infty \ge 0$ in $\overline{\Omega}$.

Lemma 2.5. *u_∞ is a viscosity solution of (1.3), i.e.,*

$$\begin{gathered} \max\{u_\infty - |\nabla u_\infty|, -\Delta_\infty u_\infty\} = 0, \quad \text{in } \Omega, \\ \min\{|\nabla u_\infty| - u, \frac{\partial u_\infty}{\partial \nu}\} = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{2.11}$$

Proof. First let us check

$$\max\{u_\infty - |\nabla u_\infty|, -\Delta_\infty u_\infty\} = 0 \quad \text{in } \Omega. \tag{2.12}$$

Fix $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and $u(x) < \phi(x)$, for $x \ne x_0$. Also fix $R > 0$ such that $B(x_0, 2R) \subset \Omega$. For $0 < r < R$ we have

$$\sup\{u_\infty(x) - \phi(x) : x \in B(x_0, R) \setminus B(x_0, r)\} < 0.$$

As $u_{p_i} \to u_\infty$ uniformly in $\overline{B(x_0, R)}$, for i large enough we conclude that

$$\sup\{u_{p_i}(x) - \phi(x) : x \in B(x_0, R) \setminus B(x_0, r)\} < u_{p_i}(x_0) - \phi(x_0).$$

Therefore for such indices i, $u_{p_i} - \phi$ attains its maximum at $x_i \in B(x_0, r)$. By letting $r \to 0$ we obtain $x_i \to x_0$ as $i \to \infty$. We relabel and denote by $\{x_i\}$ and $\{p_i\}$ the subsequences $\{x_{i_r}\}$ and $\{p_{i_r}\}$. Since u_{p_i} is a subsolution of (2.2) and $x_0 \in \Omega$,

$$-|\nabla\phi|^{p_i-2}\Delta\phi(x_i) - (p_i - 2)|\nabla\phi|^{p_i-4}\Delta_\infty\phi(x_i) + |u_{p_i}|^{p_i-2}u_{p_i}(x_i) \le 0. \tag{2.13}$$

- Case 1: $\phi(x_0) = u(x_0) > 0$. Then $u_{p_i}(x_i) > 0$ for large i, which implies that $|\nabla\phi(x_i)| \ne 0$ due to (2.13). Dividing by $(p_i - 2)|\nabla\phi(x_i)|^{p_i-4}$ we get

$$-\frac{|\nabla\phi|^2\Delta\phi(x_i)}{p_i - 2} - \Delta_\infty\phi(x_i) \le -\Big(\frac{u_{p_i}(x_i)}{|\nabla\phi(x_i)|}\Big)^{p_i-4}\frac{u_{p_i}^3(x_i)}{p_i - 2}. \tag{2.14}$$

Letting $p_i \to \infty$ we obtain from (2.14) that

$$\frac{\phi(x_0)}{|\nabla\phi(x_0)|} \le 1 \text{ and } -\Delta_\infty \phi(x_i) \le 0.$$

Therefore,

$$\max\{\phi(x_0) - |\nabla\phi(x_0)|, -\Delta_\infty \phi(x_0)\} \le 0. \tag{2.15}$$

• Case 2: $\phi(x_0) = u(x_0) = 0$. If $|\nabla\phi(x_0)| = 0$, then $\Delta_\infty \phi(x_0) = 0$ and thus (2.15) holds. If $|\nabla\phi(x_0)| \neq 0$, then $|\nabla\phi(x_i)| \neq 0$ for i large. We then obtain (2.14). The right-hand side of (2.14) tends to zero as $p_i \to \infty$, since

$$\lim_{p_i\to\infty} \Big(\frac{u_{p_i}(x_i)}{|\nabla\phi(x_i)|}\Big)^{p_i-4} = 0.$$

Thus $-\Delta_\infty\phi(x_i) \le 0$ and (2.15) holds in this case. From both cases we conclude that u_∞ is a viscosity subsolution of (2.12).

Next we claim that u_∞ is a viscosity supersolution of (2.12) in Ω. Fix a point $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and $u_\infty(x) > \phi(x)$, for $x \neq x_0$. We will show that

$$\max\{\phi(x_0) - |\nabla\phi(x_0)|, -\Delta_\infty \phi(x_0)\} \ge 0. \tag{2.16}$$

If $|\nabla\phi(x_0)| = 0$, there is nothing to prove. It suffices to show that if $|\nabla\phi(x_0)| \neq 0$ and $\phi(x_0) - |\nabla\phi(x_0)| < 0$, then $-\Delta_\infty\phi(x_0) \ge 0$. We follow the arguments made in the subsolution case. An analogue of (2.14) is

$$-\frac{|\nabla\phi|^2\Delta\phi(x_i)}{p_i-2} - \Delta_\infty\phi(x_i) \ge -\Big(\frac{u_{p_i}(x_i)}{|\nabla\phi(x_i)|}\Big)^{p_i-4}\frac{u^3_{p_i}(x_i)}{p_i-2}. \tag{2.17}$$

Since $\phi(x_0) - |\nabla\phi(x_0)| < 0$, $\frac{\phi(x_0)}{|\nabla\phi(x_0)|} \le 1$. Letting $p_i \to \infty$ it follows from (2.17) that $-\Delta_\infty\phi(x_0) \ge 0$ as claimed. Therefore u_∞ is a viscosity solution of (2.12).

We next need to check on the boundary using definition 2.1. Fix $x_0 \in \partial\Omega$ and a function $\phi \in C^2(\overline{\Omega})$ such that $u_\infty(x_0) = \phi(x_0)$ and $u_\infty(x) < \phi(x)$, for $x \neq x_0$. Using the uniform convergence of u_{p_i} to u_∞ we obtain that $u_{p_i} - \phi$ attains a maximum at $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If (2.13) holds for infinitely many x_i, we use the argument before to obtain (2.15). Thus we may assume that, for infinitely many $x_i \in \partial\Omega$,

$$|\nabla\phi(x_i)|^{p_i-2}\frac{\partial\phi}{\partial\nu}(x_i) \le \Lambda_{p_i}^{p_i}|u_{p_i}|^{p_i-2}u_{p_i}(x_i).$$

If $|\nabla\phi(x_0)| = 0$, then $\frac{\partial\phi}{\partial\nu}(x_0) = 0$. If $|\nabla\phi(x_0)| \neq 0$ we obtain

$$\frac{\partial\phi}{\partial\nu}(x_i) \le \Big(\frac{\Lambda_{p_i}|u_{p_i}(x_i)|}{|\nabla\phi(x_i)|}\Big)^{p_i-2}\Lambda_{p_i}^2 u_{p_i}(x_i).$$

Since $\Lambda_{p_i} \to 1$ as $p_i \to \infty$, we conclude that either

$$\frac{\phi(x_0)}{|\nabla\phi(x_0)|} \ge 1 \quad \text{or} \quad \frac{\partial\phi}{\partial\nu}(x_0) \le 0,$$

which implies that $\min\{|\nabla\phi(x_0)| - \phi(x_0), \frac{\partial\phi}{\partial\nu}(x_0)\} \le 0$. Therefore, at x_0,

$$\min\Big\{\max\{\phi - |\nabla\phi|, -\Delta_\infty\phi\}, \min\{|\nabla\phi| - \phi, \frac{\partial\phi}{\partial\nu}\}\Big\} \le 0. \tag{2.18}$$

Fix $x_0 \in \partial\Omega$ and a function $\phi \in C^2(\overline{\Omega})$ such that $u_\infty(x_0) = \phi(x_0)$ and $u_\infty(x) > \phi(x)$, for $x \neq x_0$. Using the uniform convergence of u_{p_i} to u_∞ we obtain that $u_{p_i} - \phi$ attains a minimum at $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If

$$-|\nabla\phi|^{p_i-2}\Delta\phi(x_i) - (p_i-2)|\nabla\phi|^{p_i-4}\Delta_\infty\phi(x_i) + |u_{p_i}|^{p_i-2}u_{p_i}(x_i) \geq 0$$

holds for infinitely many x_i, we use the argument before to obtain (2.16). Thus we may assume that, for infinitely many $x_i \in \partial\Omega$,

$$|\nabla\phi(x_i)|^{p_i-2}\frac{\partial\phi}{\partial\nu}(x_i) \geq \Lambda_{p_i}^{p_i}|u_{p_i}|^{p_i-2}u_{p_i}(x_i).$$

If $|\nabla\phi(x_0)| = 0$,

$$\max\{\phi(x_0) - |\nabla\phi|(x_0), -\Delta_\infty\phi(x_0)\} \geq 0$$

If $|\nabla\phi(x_0)| \neq 0$ we obtain

$$\frac{\partial\phi}{\partial\nu}(x_i) \geq \Big(\frac{\Lambda_{p_i}|u_{p_i}(x_i)|}{|\nabla\phi(x_i)|}\Big)^{p_i-2}\Lambda_{p_i}^2 u_{p_i}(x_i).$$

We conclude that

$$\frac{\phi(x_0)}{|\nabla\phi(x_0)|} \geq 1 \quad \text{and} \quad \frac{\partial\phi}{\partial\nu}(x_0) \geq 0,$$

which implies that $\min\{|\nabla\phi(x_0)| - \phi(x_0), \frac{\partial\phi}{\partial\nu}(x_0)\} \geq 0$. Therefore, at x_0,

$$\max\Big\{\max\{\phi - |\nabla\phi|, -\Delta_\infty\phi\}, \min\{|\nabla\phi| - \phi, \frac{\partial\phi}{\partial\nu}\}\Big\} \geq 0. \tag{2.19}$$

Inequalities (2.18) and (2.19) prove that u_∞ satisfies in the viscosity sense the boundary condition of (1.3). □

Theorem 1.1 follows from proposition 2.4 and lemma 2.5.

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