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A RIEMANN PROBLEM WITH SMALL VISCOSITY AND DISPERSION

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Abstract. In this paper we prove existence of global solutions to a hyperbolic system in elastodynamics, with small viscosity and dispersion terms and derive estimates uniform in the viscosity-dispersion parameters. By passing to the limit, we prove the existence of solution the Riemann problem for the hyperbolic system with arbitrary Riemann data.

1. Introduction

In this paper first we consider the boundary-value problem, for a system of nonlinear ordinary differential equations,

$$\begin{gathered} -\xi \frac{du}{d\xi} + u\frac{du}{d\xi} - \frac{d\sigma}{d\xi} = \epsilon \frac{d^2u}{d\xi^2} + \gamma\epsilon^2 \frac{d^3u}{d\xi^3}, \\ -\xi \frac{d\sigma}{d\xi} + u\frac{d\sigma}{d\xi} - k^2\frac{du}{d\xi} = \epsilon \frac{d^2\sigma}{d\xi^2} + \gamma\epsilon^2 \frac{d^3\sigma}{d\xi^3} \end{gathered} \tag{1.1}$$

for $\xi \in [a, b]$ with boundary conditions

$$\begin{gathered} u(a) = u_L, u(b) = u_R, \\ \sigma(a) = \sigma_L, \sigma(b) = \sigma_R. \end{gathered} \tag{1.2}$$

This system is the self-similar vanishing diffusion-dispersion approximation of initial value problem for the system of equations which comes in elastodynamics:

$$\begin{gathered} u_t + uu_x - \sigma_x = 0, \\ \sigma_t + u\sigma_x - k^2u_x = 0, \end{gathered} \tag{1.3}$$

with Riemann initial data

$$(u(x,0), \sigma(x,0)) = \begin{cases} (u_L, \sigma_L), & x < 0 \\ (u_R, \sigma_R) & x > 0. \end{cases} \tag{1.4}$$

Here u is the velocity, σ is the stress and $k > 0$ is the speed of propagation of the elastic waves. The system (1.3) is nonconservative, strictly hyperbolic system with characteristic speeds

$$\lambda_1(u, \sigma) = u - k, \lambda_2(u, \sigma) = u + k \tag{1.5}$$

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and Riemann invariants

$$r_1(u,\sigma) = \sigma + ku, r_2(u,\sigma) = \sigma - ku \tag{1.6}$$

and was studied by many authors [1, 2, 3, 4, 6], with initial datas under various conditions using differece schemes or diffusion approximations. In this paper we analyse diffusion-dispersion approximations for the Riemann problem (1.3) and (1.4).

First we show the existence of smooth solutions $(u^\epsilon, \sigma^\epsilon)$of the problem (1.1)-(1.2) and derive estimates in the space of bounded variation, uniformly in $\epsilon > 0$. We do not give any restrictions on the size of the initial data.

Next we study $(u^\epsilon, \sigma^\epsilon)$ as ϵ tends to 0 and show the limit function is a weak solution to (1.3) with the Riemann initial data (1.4) The nonconservative product which appear in the equation (1.3) is justified by the work of LeFloch and Tzavaras [8] on nonconservative products.

2. Self-similar vanishing diffusion-dispersion approximation

In this section, we consider the system (1.1) and (1.2) and prove the existence of smooth solutions. It is more convenient to work with Riemann invariants (1.5). Given the data $(u_L, \sigma_L), (u_R, \sigma_R)$, we define

$$\begin{aligned} r_{1L} &= \sigma_L + ku_L, r_{1R} = \sigma_R + ku_R, \\ r_{2L} &= \sigma_L - ku_L, r_{2R} = \sigma_R - ku_R. \end{aligned} \tag{2.1}$$

The characteristic speeds (1.5) in terms of the Riemann invariants take the form

$$\lambda_1(r_1, r_2) = \frac{r_1 - r_2}{2k} - k, \quad \lambda_2(r_1, r_2) = \frac{r_1 - r_2}{2k} + k.$$

Consider the rectangle

$$D = [\min(r_{1L}, r_{1R}), \max(r_{1L}, r_{1R})] \times [\min(r_{2L}, r_{2R}), \max(r_{2L}, r_{2R})],$$

and consider the minimum and maximum of the eigenvalues on this square

$$\lambda_j^m = \min_D \lambda_j(r_1, r_2), \quad \lambda_j^M = \max_D \lambda_j(r_1, r_2), \quad j = 1, 2. \tag{2.2}$$

We choose $\gamma > 0$, small and the boundary points a, b such that

$$0 < \gamma < \frac{1}{4(\lambda_2^M - \lambda_1^m)}, \quad \lambda_1^m - \frac{1}{\gamma} < a < \lambda_1^m < \lambda_2^M < b \tag{2.3}$$

The choice of a, b, in this fashion is to ensure that there is no boundary effect in the limit, that is, for $\xi < \lambda_1^m$ and $\xi > \lambda_2^M$ the limiting values of (u, σ) are (u_L, σ_L) (u_R, σ_R) respectively.

Theorem 2.1. *Under the assumptions* (2.3)*, for each fixed $\epsilon > 0$ there exits a smooth solution $(u^\epsilon(\xi), \sigma^\epsilon(\xi))$ for* (1.1) *and* (1.2) *satisfying the estimates*

$$|u^\epsilon(\xi)| + |\sigma^\epsilon(\xi)| \le C, \quad \int_a^b |\frac{du^\epsilon}{d\xi}| d\xi + \int_a^b |\frac{d\sigma^\epsilon}{d\xi}(\xi)| d\xi \le C, \tag{2.4}$$

$$|u^\epsilon(\xi) - u_L| + |\sigma^\epsilon(\xi) - \sigma_L| \le \frac{C}{\delta} e^{\frac{-(\xi - \lambda_1^m)^2}{2\epsilon}}, \quad a \le \xi \le \lambda_1^m - \delta \tag{2.5}$$

$$|u^\epsilon(\xi) - u_R| + |\sigma^\epsilon(\xi) - \sigma_R| \le \frac{C}{\delta} e^{\frac{-(\xi - \lambda_2^M)^2}{2\epsilon}}, \quad \lambda_2^M + \delta \le \xi \le b, \tag{2.6}$$

for some constant $C > 0$ independent of $\epsilon > 0$ and for $\delta > 0$, small.

Proof. For convenience of notation, in the rest of this section we drop the dependence of ϵ and write u, σ, r_1, r_2 ect. In terms of the Riemann invariants (1.5), the problem (1.1) and (1.2) takes the form

$$\begin{aligned} -\xi\frac{dr_1}{d\xi} + \lambda_1(r_1,r_2)\frac{dr}{d\xi} &= \epsilon^2\frac{d^2r_1}{d\xi^2} + \gamma\epsilon\frac{d^3r_1}{d\xi^3}, \\ -\xi\frac{dr_2}{d\xi} + \lambda_2(r_1,r_2)\frac{dr_2}{d\xi} &= \epsilon\frac{d^2r_2}{d\xi^2} + \gamma\epsilon^2\frac{d^3r_2}{d\xi^3} \end{aligned} \tag{2.7}$$

on $[a,b]$ with boundary conditions

$$r_1(a) = r_{1L}, \quad r_1(b) = r_{1R}, \quad r_2(a) = r_{2L}, \quad r_2(b) = r_{2R} \tag{2.8}$$

where r_{1L}, r_{1R}, r_{2L} and r_{2R} are given by (2.1).

From (1.6), the definition of $r_1(u,\sigma), r_2(u,\sigma)$, we have

$$u = \frac{r_1(u,\sigma) - r_2(u,\sigma)}{2k}, \quad \sigma = \frac{r_1(u,\sigma) + r_2(u,\sigma)}{2}.$$

Then to prove the theorem, it is sufficient to prove the the existence of r_1, r_2 solution of (2.7) and (2.8), with following estimates

$$\begin{aligned} r_1(\xi) &\in [\min(r_{1L}, r_{1R}),\ \max(r_{1L}, r_{1R})], \quad \xi \in [a,b], \\ r_2(\xi) &\in [\min(r_{2L}, r_{2R}),\ \max(r_{2L}, r_{2R})], \quad \xi \in [a,b]; \end{aligned} \tag{2.9}$$

$$\begin{aligned} |r_1(\xi) - r_{1L}| &\le \frac{C}{\delta}\exp\big(\frac{-(\xi-\lambda_1^m)^2}{2\epsilon}\big), \quad a \le \xi \le \lambda_1^m - \delta, \\ |r_2(\xi) - r_{2L}| &\le \frac{C}{\delta}\exp\big(\frac{-(\xi-\lambda_2^m)^2}{2\epsilon}\big), \quad a \le \xi \le \lambda_2^m - \delta; \end{aligned} \tag{2.10}$$

$$\begin{aligned} |r_1(\xi) - r_{1R}| &\le \frac{C}{\delta}\exp\big(\frac{-(\xi-\lambda_1^M)^2}{2\epsilon}\big), \quad \lambda_1^M + \delta \le \xi \le b, \\ |r_2(\xi) - r_{2R}| &\le \frac{C}{\delta}\exp\big(\frac{-(\xi-\lambda_2^M)^2}{2\epsilon}\big), \quad \lambda_2^M + \delta \le \xi \le b; \end{aligned} \tag{2.11}$$

$$\int_a^b |\frac{dr_1}{d\xi}| d\xi \le |r_{1R} - r_{1L}|, \quad \int_a^b |\frac{dr_2}{d\xi}| d\xi \le |r_{2R} - r_{2L}|. \tag{2.12}$$

To prove this we reduce (2.7) and (2.8) to an integral equation using some ideas of LeFloch and Rohde [7] and Joseph and LeFloch [5] and use a fixed point argument. First note that (2.7) can be written in the form

$$\begin{aligned} \gamma\epsilon^2\frac{d^3r_1}{d\xi^3} + \epsilon\frac{d^2r_1}{dx^2} - (\lambda_1(r_1,r_2) - \xi)\frac{dr_1}{d\xi} &= 0, \\ \gamma\epsilon^2\frac{d^2r_2}{d\xi^2} + \epsilon\frac{d^2r_2}{dx^2} - (\lambda_2(r_1,r_2) - \xi)\frac{dr_2}{d\xi} &= 0. \end{aligned} \tag{2.13}$$

For $j = 1, 2$, let

$$\varphi_1(\xi) = \frac{dr_1}{d\xi}, \quad \varphi_2(x) = \frac{dr_2}{d\xi} \tag{2.14}$$

Then from (2.13) we get

$$\gamma\epsilon^2\varphi_i'' + \epsilon\varphi_i' - (\lambda_i(r_1(\xi), r_2(\xi)) - \xi)\,\varphi_i = 0. \tag{2.15}$$

Suppose we are given r_1, r_2 smooth functions, taking values in the rectangle D and of finite total variation independent of ϵ, (2.15) is a second order linear ordinary differential equation for φ_i. Under the transformation,

$$H_i = e^{\frac{-1}{2\epsilon\gamma}\xi}\varphi_i$$

(2.15) reduces to

$$H_i'' = \frac{\mu_i(y)}{\gamma\epsilon^2}H \tag{2.16}$$

where

$$\mu_i(\xi) = \frac{1}{4\gamma} + (\lambda_i(r_1(\xi), r_2(\xi)) - \xi). \tag{2.17}$$

By taking $\gamma > 0$ small, we have $\mu_i(y) > 0$ and we can use the theorem of Olver [9] to construct solutions to (2.15). Indeed LeFloch and Rohde [7] showed the existence of $\varphi_i(\xi), i = 1, 2$ satisfying the following properties:

$$0 < \varphi_i(\xi) \leq C/\epsilon, \int_a^b \varphi_i(\xi)d\xi = 1 \tag{2.18}$$

and

$$\varphi_i(\xi) \leq \begin{cases} \frac{C}{\epsilon}\exp\big(\frac{-c(x-\lambda_1^m)^2}{\epsilon}\big), & a \leq \xi \leq \lambda_1^m, \\ \frac{C}{\epsilon}\exp\big(\frac{-c(x-\lambda_1^m)^2}{\epsilon}\big), & \lambda_2^M \leq \xi \leq b, \end{cases} \tag{2.19}$$

Integrating once (2.14) and using (2.18) and the boundary conditions, we get,

$$\begin{aligned} r_1^\epsilon(\xi) &= r_{1L} + (r_{1R} - r_{1L})\int_a^\xi \varphi_1(y)dy, \\ r_2^\epsilon(\xi) &= r_{2L} + (r_{2R} - r_{2L})\int_a^\xi \varphi_2(y)dy. \end{aligned} \tag{2.20}$$

It follows that to solve (2.7) and (2.8) with estimates (2.4)–(2.6), it is enough to solve (2.20). To solve (2.20), we use the Schauder fixed point theorem applied to the function

$$F(r_1, r_2)(\xi) = (F_1(r_1, r_2)(\xi), F_2(r_1, r_2)(\xi))$$

where

$$\begin{aligned} F_1(r_1, r_2)(\xi) &= r_{1L} + (r_{1R} - r_{1L})\int_a^\xi \varphi_1(y)dy, \\ F_2(r_1, r_2)(\xi) &= r_{2L} + (r_{2R} - r_{2L})\int_a^\xi \varphi_2(y)dy. \end{aligned} \tag{2.21}$$

From (2.18) and (2.21), it is clear that $F_1(r, s)$ is a convex combination of r_{1L} and r_{1R} and $F_2(r, s)$ is a convex combination of r_{2L} and r_{2R}. So the estimate

$$\begin{aligned} F_1(r_1, r_2)(\xi) &\in [\min(r_{1L}, r_{1R}), \max(r_{1L}, r_{1R})], \\ F_2(r_1, r_2)(\xi) &\in [\min(r_{2L}, r_{2R}), \max(r_{2L}, r_{2R})] \end{aligned} \tag{2.22}$$

easily follows. Also from (2.18) and (2.21), we get for $j = 1, 2$

$$|\frac{dF_j(r, s)}{d\xi}(\xi)| \leq \frac{C}{\epsilon}. \tag{2.23}$$

Further, from (2.19), we get:

$$|F_1(r, s)(\xi) - r_{1L}| \leq \frac{C}{\epsilon}\int_a^\xi \exp\big(\frac{-(s-\lambda_1^m)^2}{2\epsilon}\big)ds = \frac{C\sqrt{2\epsilon}}{\epsilon}\int_{\frac{a-\lambda_1^m}{\sqrt{2\epsilon}}}^{\frac{(\xi-\lambda_1^m)}{\sqrt{2\epsilon}}} e^{-s^2}ds,$$

for $\quad a \leq \xi \leq \lambda_1^m$;

$$|F_2(r,s)(\xi) - r_{2L}| \leq \frac{C}{\epsilon}\int_0^{\xi} \exp\big(\frac{-(s-\lambda_2^m)^2}{2\epsilon}\big)ds = \frac{C\sqrt{2\epsilon}}{\epsilon}\int_{\frac{a-\lambda_2^m}{\sqrt{2\epsilon}}}^{\frac{(\xi-\lambda_2^m)}{\sqrt{2\epsilon}}} e^{-s^2}ds,$$

for $a \leq \xi \leq \lambda_2^m$;

$$|F_1(r,s)(\xi) - r_{1R}| \leq \frac{C}{\epsilon}\int_{\xi}^{b} \exp\big(\frac{-(s-\lambda_k^M)^2}{2\epsilon}\big)ds = \frac{C\sqrt{2\epsilon}}{\epsilon}\int_{\frac{(\xi-\lambda_1^M)}{\sqrt{2\epsilon}}}^{\frac{b-\lambda_1^M}{\sqrt{2\epsilon}}} e^{-s^2}ds,$$

for $\lambda_1^M \leq \xi \leq b$;

$$|F_2(r_1,r_2)(\xi) - r_{2R}| \leq \frac{C}{\epsilon}\int_{\xi}^{b} \exp\big(\frac{-(s-\lambda_k^M)^2}{2\epsilon}\big)ds = \frac{C\sqrt{2\epsilon}}{\epsilon}\int_{\frac{(\xi-\lambda_2^M)}{\sqrt{2\epsilon}}}^{\frac{b-\lambda_2^M}{\sqrt{2\epsilon}}} e^{-s^2}ds,$$

for $\lambda_2^M \leq \xi \leq b$.

Now using the asymptotic expansion

$$\int_y^{\infty} e^{-y^2}dy = (\frac{1}{2y} - O(\frac{1}{y^2}))e^{-y^2}, \quad y \to \infty$$

in the above inequalities, we get

$$\begin{aligned} |F_1(r_1,r_2)(\xi) - r_{1L}| &\leq \frac{C}{\delta}\exp\big(\frac{-(\xi-\lambda_1^m)^2}{2\epsilon}\big), \quad a \leq \xi \leq \lambda_1^m - \delta,\\ |F_2(r_1,r_2)(\xi) - r_{2L}| &\leq \frac{C}{\delta}\exp\big(\frac{-(\xi-\lambda_2^m)^2}{2\epsilon}\big), \quad a \leq \xi \leq \lambda_2^m - \delta; \end{aligned} \tag{2.24}$$

$$\begin{aligned} |F_1(r_1,r_2)(\xi) - r_{1R}| &\leq \frac{C}{\delta}\exp\big(\frac{-(\xi-\lambda_1^M)^2}{2\epsilon}\big), \quad \lambda_1^M + \delta \leq \xi \leq b,\\ |F_2(r_1,r_2)(\xi) - r_{2R}| &\leq \frac{C}{\delta}\exp\big(\frac{-(\xi-\lambda_2^M)^2}{2\epsilon}\big), \quad \lambda_2^M + \delta \leq \xi \leq b. \end{aligned} \tag{2.25}$$

The estimates (2.22) and (2.23) show that $F = (F_1, F_2)$ is compact and maps the convex set $\{(r_1,r_2) \in C[a,b] \times C[a,b] : (r_1(\xi), r_2(\xi)) \in D\}$ into itself, where D is the rectangle $D = [\min(r_B,r_R), \max(r_B,r_R)] \times [\min(s_B,s_R), \max(s_B,s_R)]$, and $C[a,b]$ is the space of continuous functions with uniform norm. So by Schauder fixed point theorem there exists (r_1,r_2) such that $F(r_1,r_2) = (r_1,r_2)$, satisfies (2.20) and hence is a smooth solution to (2.7) with boundary conditions (2.8). Further the estimates (2.9)-(2.12) follows from (2.20), (2.22), (2.24), (2.25) and the fact that $F(r_1,r_2) = (r_1,r_2)$. The proof of the theorem is complete. □

3. Passage to the limit as $\epsilon \to 0$; the Riemann problem.

Here we construct solution of the Riemann problem

$$\begin{aligned} u_t + uu_x - \sigma_x &= 0,\\ \sigma_t + u\sigma_x - k^2u_x &= 0, \end{aligned} \tag{3.1}$$

with Riemann type initial data

$$(u(x,0), \sigma(x,0)) = \begin{cases} (u_L, \sigma_L) & x < 0\\ (u_R, \sigma_R) & x > 0. \end{cases} \tag{3.2}$$

Since the Riemann problem is invariant under scaling, solution is sought in the form $(u(\xi,\sigma(\xi))$ with $\xi = x/t$. Then (3.1) and (3.2) takes the form

$$\begin{aligned} -\xi\frac{du}{d\xi} + u\frac{du}{d\xi} - \frac{d\sigma}{d\xi} = 0 \\ -\xi\frac{d\sigma}{d\xi} + u\frac{d\sigma}{d\xi} - k^2\frac{du}{d\xi} = 0 \end{aligned} \tag{3.3}$$

for $\xi \in (-\infty,\infty)$ with boundary conditions

$$\begin{aligned} u(-\infty) = u_L, \quad u(\infty) = u_R, \\ \sigma(-\infty) = \sigma_L, \quad \sigma(\infty) = \sigma_R. \end{aligned} \tag{3.4}$$

The smooth solution $(u^\epsilon,\sigma^\epsilon)$ of (1.1) and (1.2) constructed in the previous section is an approximation to the problem (3.3) and (3.4). Because of the estimates (2.4), by compactnes, there exists a subsequence $(u^{\epsilon_n},\sigma^{\epsilon_n})$ converges to a BV function (u,σ) as $\epsilon_n \to 0$. This limit function is not in general continuous and hence the nonconservative product which appear in the equation (3.3) does not make sense in the theory of distribution. So we use the theory developed by LeFloch and Tzavaras [8] for nonconservative products. For completeness we briefly describe in short their results that we use.

Let $u_n : [a,b] \to R^n$ be a sequence of continuous functions uniformly bounded total variation:

$$\sup |u_n| + TV(u_n) \le C. \tag{3.5}$$

where $TV(u)$ denotes the total variation of u on $[a,b]$. Define the Radon measure

$$\langle \mu_n, g\rangle = \int_{[a,b]} g(u_n)du_n, g \in C(R^n)$$

We have

$$|\langle \mu_n, g\rangle| \le TV(u_n). \sup_{|\lambda|\le C} |g(\lambda)|.$$

So by weak* compactness of μ_n, there exists a subsequence n_k and a measure $\mu \in M(R^n)$ such that

$$\mu_{nk} \to \mu$$

in weak* $M(R^n)$. To characterize μ, we need the notion of generalized graph of u.

Definition 3.1. Generalized graph of u is defined as a Lipschitz continuous map

$$(X,U) : [0,1] \to [a,b] \times R^n$$

such that

(a) $(X(0),U(0)) = (a,u(a)),(X(1),U(1)) = (b,u(b))$
(b) X is increasing :$s_1 < s_2$ implies $X(s_1) \le X(s_2)$
(c) given $y \in [a,b]$, there exists $s \in [0,1]$ such that $X(s) = y$ and $U(s) = u(y)$.

Generalized graph for a continuous BV function u can be easily defined. Let

$$\sigma(x) = \frac{x - a - TV_{[a,x]}(u)}{b - a + TV_{[a,b]}(u)}$$

σ is strictly increasing , surjective and Lipschitz continuous. Let $X : [0,1] \to [a,b]$ is inverse of σ and $U : [0,1] \to R^n$ be defined as uoX, then $(X,U) : [0,1] \to [a,b] \times R^n$ is the generalized graph of u. LeFloch and Tzavaras [8] proved the following result.

Theorem 3.2 (LeFloch and Tzavaras [8])**.** *If u_n is a sequence of continuous functions satisfying a uniform bound* (3.5) *Then there exists a subsequence u_{nk} and associated generalized graph (X, U) such that for any continuous function g,*

$$\int_{[a,b]} \theta(x)g(u_{nk}(x)du_{nk}(x) \to \langle \mu_g, \theta \rangle$$

where $\mu : C_0(R^n) \to M[a, b]$ is defined by

$$\langle \mu_g, \theta \rangle = \int_0^1 (\theta(X(s))g(U(s))dU(s), \quad \theta \in C[a, b]$$

The product $g(u).u_x$ is defined as μ_g and is denoted by $[g(u).u_x]_{(X,U)}$.

LeFloch and Tzavaras [8] considered the Riemann problem for a general nonconservative hyperbolic system

$$-\xi \frac{du}{d\xi} + A(u)\frac{du}{d\xi} = 0, \quad u(-\infty) = u_L, \quad u(\infty) = u_R. \tag{3.6}$$

They introduced the following definition.

Definition 3.3. A vector function $u(\xi)$ defined on $(-\infty, \infty)$ and of bounded variation is a solution for the system (3.6) if there exists (X, U), a generalised graph of u such that as a Borel measure

$$[(-\xi + A(u))\frac{du}{d\xi}]_{(X,U)} = 0, \quad u(-\infty) = u_L, \quad u(\infty) = u_R$$

Under the condition that $|u_L - u_R|$ sufficiently small, the Riemann problem (3.6) was solved in by LeFloch and Tzavaras [8] using vanishing diffusion approximation. The Riemann problem for the general systems with diffusion - dispersion approximations were treated by LeFloch and Rohde [7] when $|u_l - u_R|$ is small. Joseph [2] treated (3.3) with only the diffusion terms but with large data. Present paper treats (3.3) with diffusion and dispersion terms and with arbitrary data (3.4). We shall prove the following result.

Theorem 3.4. *There exits a function of bounded variation $V = (u, \sigma)$ and an associated generalized graph (X, U) solving the the Riemann problem* (3.3) *and* (3.4) *in the sense of definition 3.3.*

Proof. As we have the estimate (2.4), we know $\gamma\epsilon^2 u''' + \epsilon u''$ and $\gamma\epsilon^2\sigma''' + \epsilon\sigma''$ goes to zero in distribution as ϵ goes to zero. So by the Theorem of LeFloch and Tzavaras [8] stated before, there exists a bounded measurable function $V = (u, \sigma)$ and associated generalized graph (X, U) such that

$$[-\xi + A(V)\frac{dV}{d\xi}]_{(X,U)} = 0.$$

where $A(V) = (A_{ij}(u, \sigma))$ is a 2×2 matrix with $A_{11}(u, \sigma) = u, A_{12}(u, \sigma) = -1, A_{21}(u, \sigma) = -k^2, A_{22}(u, \sigma) = u$. We also have $(u(\xi), \sigma(\xi) = (u_L, \sigma_L)$ on $\xi \in [a, \lambda_1^m]$ and so can be extended as (u_L, σ_L) on $(-\infty, a]$ and $(u(\xi), \sigma(\xi) = (u_R, \sigma_R)$ for $\xi \in [\lambda_2^M, b]$ and so can be extended as (u_R, σ_R) on $[b, \infty)$. The proof of the theorem is complete. □

Remarks. We note that the hyperbolic system (3.1) and (3.2) has finite speed of propagation with minimum speed λ_1^m and maximum speed λ_2^M defined by (2.2) which depends only on Riemann data $u_L, u_R, \sigma_L, \sigma_R$ and so the solution $(u(\xi), \sigma(\xi))$ is (u_L, σ_L) on $(-\infty, \lambda_1^m]$ and (u_R, σ_R) on $[\lambda_2^M, \infty)$. This is the meaning of the estimates (2.5) and (2.6).

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