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A REMARK ON C^2 INFINITY-HARMONIC FUNCTIONS

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Abstract. In this paper, we prove that any nonconstant, C^2 solution of the infinity Laplacian equation $u_{x_i}u_{x_j}u_{x_ix_j} = 0$ can not have interior critical points. This result was first proved by Aronsson [2] in two dimensions. When the solution is C^4, Evans [6] established a Harnack inequality for $|Du|$, which implies that non-constant C^4 solutions have no interior critical points for any dimension. Our method is strongly motivated by the work in [6].

1. Introduction

In the 1960's, Aronsson introduced the notion of the absolutely minimizing Lipschitz extension. Namely, $u \in W^{1,\infty}(\Omega)$ is said to be an *absolutely minimizing Lipschitz extension* in some bounded open subset Ω if for any open set $V \subset \Omega$, we have that

$$\sup_{x\neq y\in\partial V} \frac{|u(x)-u(y)|}{|x-y|} = \sup_{x\neq y\in\bar{V}} \frac{|u(x)-u(y)|}{|x-y|}.$$

The results in Crandall-Evans-Gariepy [5] imply that the above definition is in fact equivalent to say that for any open set $V \subset \Omega$ and $v \in W^{1,\infty}(V)$,

$$u|_{\partial V} = v|_{\partial V} \;\Rightarrow\; ||Du||_{L^\infty(V)} \le ||Dv||_{L^\infty(V)}.$$

The second characterization is what Jensen used in his influential paper [9] where he proved that $u \in W^{1,\infty}(\Omega)$ is an absolutely minimizing Lipschitz extension with given Lipschitz continuous boundary date g if and only if it is a viscosity solution of the following infinity Laplacian equation.

$$\begin{gathered} u_{x_i}u_{x_j}u_{x_ix_j} = 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} = g. \end{gathered}$$

He also showed that the above infinity Laplacian equation has a unique viscosity solution with given continuous boundary data. A direct consequence is that absolute minimizing Lipschitz extension is unique with given boundary data. We also name a viscosity solution of the infinity Laplacian equation as an infinity harmonic function. Recently, people have tremendous interest in this degenerate elliptic equation. The interested readers can find most of relevant works in the note Crandall [4].

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The focus of this work is on classical solutions (i.e, C^2) of the infinity Laplacian equation. As observed by Aronsson [2], smooth solutions of the infinity Laplacian equation have some special properties which are in general not possessed by viscosity solutions. In our paper, we study one of them, i.e, the non-vanishing gradient. From now on, we assume that Ω is a connected bounded open set. By carefully studying the gradient flows of C^2 solutions (note that $|Du|$ is constant along the gradient flow of a C^2 solution u), Aronsson proved in [2] that $|Du|$ will nowhere be zero unless u is constant when $n = 2$. Recently Jensen mentioned a simple proof of Aronsson's result in a seminar talk. Using some elementary maximum principle argument, Evans [6] extended Aronsson's result to $n \geq 3$ for C^4 infinity harmonic functions. In fact, Evans established a harnack inequality for $|Du|$. We found that part of Evans's argument can be interpreted in viscosity sense. From that, we are able to establish a weak Hopf-type lemma for $|Du|$ instead of the Harnack inequality, which is sufficient to prove the following new result.

Theorem 1.1. *Let Ω be a connected open subset of $\mathbb{R}^n$. Assume that u is a C^2 solution of*

$$\Delta_\infty u = 0 \quad \textit{in } \Omega.$$

If $Du(z) = 0$ for some $z \in \Omega$, then $u \equiv u(z)$.

Remark 1.2. In general, infinity harmonic functions might not be C^2. For example, $u(x, y) = x^{\frac{4}{3}} - y^{\frac{4}{3}}$ is a $C^{1,\frac{1}{3}}$ infinity harmonic function in $\mathbb{R}^2$. See Aronsson [3]. It is clear that Theorem 1.1 does not hold for this non-classical solution since $(0, 0)$ is its critical point. A main open problem of the infinity laplacian equation is whether any viscosity solution is C^1. Savin [8] proved the C^1 regularity when $n = 2$. We just learned that in a forthcoming paper, Evans and Savin [7] proved the $C^{1,\alpha}$ regularity when $n = 2$. For higher dimensions, the regularity issue remains a very challenging problem.

2. Proof of the main theorem

In this section, we prove Theorem 1.1. Following the notations in Evans [6], we denote $v(x) = |Du(x)|$. If $v(x) \neq 0$, set

$$\nu^i = \frac{u_{x_i}}{|Du|} = \frac{u_{x_i}}{v} \quad (1 \leq i \leq n),$$

and also write

$$h_{ij} = \nu^i \nu^j.$$

Then we have the following lemma.

Lemma 2.1. *If $v \neq 0$ in Ω, then v is a viscosity solution of*

$$h_{ij} v_{x_i x_j} = -\frac{|Dv|^2}{v} \quad \textit{in } \Omega. \tag{2.1}$$

Proof. First we want to remark that (2.1) was derived in [6] for $u \in C^3$. Since $u \in C^2$, we have that

$$\nu^i v_{x_i} = 0 \quad \text{in } \Omega.$$

Hence

$$(h_{ij} v_{x_i})_{x_j} = 0.$$

Assume that for $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$,

$$\phi(x) - v(x) > \phi(x_0) - v(x_0) = 0,$$

for $x \in \Omega \backslash \{x_0\}$. Then a standard argument shows that

$$(h_{ij}\phi_{x_i})_{x_j}(x_0) \geq 0.$$

Therefore, following the calculations in [6],

$$\begin{aligned}
h_{ij}\phi_{x_ix_j}(x_0) &\geq -\nu^i_{x_j}\nu^j\phi_{x_i}(x_0) - \nu^j_{x_j}\nu^i\phi_{x_i}(x_0) \\
&= -\nu^i_{x_j}\nu^j v_{x_i}(x_0) - \nu^j_{x_j}\nu^i v_{x_i}(x_0) \\
&= -\nu^i_{x_j}\nu^j v_{x_i}(x_0) \\
&= -(\frac{u_{x_ix_j}}{v} - \frac{u_{x_i}v_{x_j}}{v^2})\nu^j v_{x_i}(x_0) \\
&= -\frac{\nu^j v_{x_i}u_{x_ix_j}}{v}(x_0) = -\frac{|Dv(x_0)|^2}{v(x_0)}.
\end{aligned}$$

Hence v is a viscosity subsolution of (2.1). Similarly, we can show that v is a viscosity supersolution of (2.1). □

Next we prove a weak Hopf type Lemma.

Lemma 2.2. *Suppose that $v \neq 0$ in Ω and $\bar{B}_r(x_0) \subset \Omega$ for some $r > 0$. Assume that $\min_{\bar{B}_{\frac{r}{2}}(x_0)} v \geq \delta$. Then there exists $\epsilon_0 > 0$ which only depends on r and δ such that if $0 < \epsilon = \min_{\partial B_r(x_0)} v < \epsilon_0$, then*

$$\epsilon = \min_{\partial B_r(x_0)} v > \min_{\bar{\Omega}} v.$$

Proof. Choose $x_\epsilon \in \partial B_r(x_0)$ such that

$$v(x_\epsilon) = \epsilon = \min_{\partial B_r(x_0)} v.$$

Let

$$w_\epsilon = \log v - \log \epsilon.$$

Since $\nu^i v_{x_i} = 0$, owing to Lemma 2.1, we discover that w_ϵ is a viscosity solution of

$$h_{ij}w_{\epsilon,x_ix_j} = -|Dw_\epsilon|^2.$$

For $k > 0$, denote

$$f_k(x) = k(r^2 - |x - x_0|^2).$$

A simple calculation shows that if we choose $k = 4/r^2$,

$$h_{ij}f_{k,x_ix_j} > -|Df_k|^2 \quad \text{in } \{\frac{r}{2} \leq |x - x_0| \leq r\}.$$

Since $\min_{B_{\frac{r}{2}}(x_0)} \log v \geq \log \delta$, there exists a ϵ_0 depending only on r and δ such that if $\epsilon < \epsilon_0$, we have that

$$w_\epsilon \geq f_{4/r^2} \quad \text{on } \partial B_{\frac{r}{2}}(x_0).$$

Also,

$$w_\epsilon \geq 0 = f_{4/r^2} \quad \text{on } \partial B_r(x_0).$$

Since f_{4/r^2} is smooth, by comparison, we derive that

$$w_\epsilon \geq f_{4/r^2} \quad \text{in } \{\frac{r}{2} \leq |x - x_0| \leq r\}.$$

In particular,

$$\frac{\partial w_\epsilon}{\partial n}(x_\epsilon) \geq \frac{\partial f_{4/r^2}}{\partial n}(x_\epsilon) = \frac{8}{r} > 0,$$

where n is the inward normal vector of $\partial B_r(x_0)$ at x_ϵ. Hence Lemma 2.2 holds. □

Proof of Theorem 1.1. We argue by contradiction. If u is not constant, then there exists $x_0 \in \Omega$ and $r > 0$ such that $v > 0$ in $B_r(x_0)$ and

$$\partial B_r(x_0) \cap \{x \in \Omega|\ v(x) = 0\} \neq \Phi.$$

For $\epsilon > 0$, denote

$u_\epsilon(x, x_{n+1}) = u(x) + \epsilon x_{n+1}, B_r(x_0, 0) = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}|\ |x - x_0|^2 + x_{n+1}^2 \leq r^2\}.$

Then we have that for any $\epsilon > 0$,

$$\min_{\partial B_r(x_0,0)} |Du_\epsilon| = \epsilon = \min_{\Omega\times\mathbb{R}} |Du_\epsilon|,$$
$$\min_{\bar{B}_{\frac{r}{2}}(x_0,0)} |Du_\epsilon| > \min_{B_{\frac{r}{2}}(x_0)} |Du| > 0.$$

Since u_ϵ is a C^2 infinity harmonic function in $\Omega \times \mathbb{R}$ and $|Du_\epsilon| > 0$, applying Lemma 2.2 to u_ϵ, we get contradiction for small ϵ. □

Remark 2.3. Evans [6] showed that if $u \in C^4$, then $z = \frac{|D|Du||}{|Du|}$ is a subsolution of the following equation

$$-h_{ij} z_{x_i x_j} \leq -z^2 + w_{x_i} z_{x_i}, \tag{2.2}$$

where $w = \log |Du|$. Owing to the quadratic term z^2, he is able to derive that z is locally bounded, which implies that $|Du|$ satisfies a Harnack inequality. Evans's proof also implies that the only entirely C^4 solutions (i,e, $u \in C^4(\mathbb{R}^n)$) in $\mathbb{R}^n$ are linear functions. Aronsson [2] proved this Liouville type theorem for C^2 solutions when $n = 2$. It is not clear to us whether z is a viscosity subsolution of (2.2) if we only assume that $u \in C^2$. If it is true, we can show that the only entirely classical solutions (i.e, $u \in C^2(\mathbb{R}^n)$) in $\mathbb{R}^n$ are linear functions.

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