POSITIVE SOLUTIONS FOR A CLASS OF SINGULAR BOUNDARY-VALUE PROBLEMS

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Abstract. This paper concerns the existence and multiplicity of positive solutions for Sturm-Liouville boundary-value problems. We use fixed point theorems and the sub-super solutions method to two solutions to the problem studied.

Introduction

Consider the boundary-value problem

\[ Lu = \lambda f(t, u), \quad 0 < t < 1 \]
\[ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \]

where \( Lu = -(ru')' + qu \), \( r, q \in C[0, 1] \) with \( r > 0, \ q \geq 0 \) on \([0, 1], \ \alpha, \beta, \gamma, \delta \geq 0 \) with \( \alpha \delta + \alpha \gamma + \beta \gamma > 0, \) \( f : (0, 1) \times [0, \infty) \to [0, \infty) \), and \( \lambda \) is a positive parameter.

The existence and nonexistence of positive solutions of problem (0.1) with \( f \) possibly singular have been established by Choi [1], Dalmasso [2], Wong [7], and recently by Erbe and Mathsen [4]. In this paper, we shall obtain positive solutions to (0.1) under assumptions less stringent than in [4]. In particular, we do not need the condition that \( f(t, u) \) be nondecreasing in \( u \), which is essential in [1, 2, 4, 7]. Our approach depends on fixed point theorems and sub-super solutions method.

1. Main results

Let \( G(t, s) \) be the Green’s function for (0.1). Then \( u \) is a solution of (0.1) if and only if

\[ u(t) = \lambda \int_0^1 G(t, s)f(s, u(s))ds. \]

Recall that

\[ G(t, s) = \begin{cases} c^{-1}\phi(t)\psi(s) & \text{if } t \leq s \\ c^{-1}\phi(s)\psi(t) & \text{if } t > s \end{cases} \]

where \( \phi \) and \( \psi \) satisfy

\[ L\phi = 0, \quad \phi(0) = \beta, \quad \phi'(0) = \alpha \]
\[ L\psi = 0, \quad \psi(1) = \delta, \quad \psi'(1) = -\gamma \]
and \( c = -r(t)(\phi(t) \psi'(t) - \phi'(t) \psi(t)) > 0 \). Note that \( \phi' > 0 \) on \((0, 1]\), \( \psi' < 0 \) on \([0, 1)\).

We shall make the following assumptions:

(H1) \( f : (0, 1) \times [0, \infty) \rightarrow [0, \infty) \) is continuous

(H2) For each \( M > 0 \), there exists a continuous function \( g_M \) on \((0, 1)\) such that

\[
|f(t, u)| \leq g_M(t) \quad \text{for } t \in (0, 1), \ 0 \leq u \leq M,
\]

\[
\int_0^1 G(s, s)g_M(s)ds < \infty.
\]

(H3) There exist an interval \( I \subset (0, 1) \) and a function \( m \in L^1(I) \) with \( m \geq 0 \), \( m \not\equiv 0 \) such that for every \( a > 0 \), there exists \( r_a > 0 \) such that

\[
f(t, u) \geq am(t)u \quad \text{for } t \in I, \ u \in (0, r_a).
\]

(H4) There exist an interval \( J \subset (0, 1) \) and a positive number \( d \) such that

\[
f(t, u) \geq du \quad \text{for } t \in J, \ u \geq 0.
\]

(H5) There exist an interval \( I_1 \subset (0, 1) \) and a function \( m_1 \in L^1(I_1) \) with \( m_1 \geq 0, m_1 \not\equiv 0 \) such that for every \( b > 0 \), there exists \( R_b > 0 \) such that

\[
f(t, u) \geq bm_1(t)u \quad \text{for } t \in I_1, \ u \geq R_b.
\]

Our main results are stated as follows.

**Theorem 1.1.** Let \((H1)-(H3)\) hold. Then there exists \( \lambda_0 > 0 \) such that \((0.1)\) has a positive solution for \( 0 < \lambda < \lambda_0 \). If, in addition, \((H5)\) holds, then \((0.1)\) has at least two positive solutions for \( 0 < \lambda < \lambda_0 \).

**Theorem 1.2.** Let \((H1)-(H4)\) hold. Then there exists \( \lambda^* > 0 \) such that \((0.1)\) has a positive solution for \( 0 < \lambda < \lambda^* \) and no positive solution for \( \lambda > \lambda^* \).

**Remark 1.3.** Let \( f(t, u) = m(t)g(u) \), where \( g : [0, \infty) \rightarrow [0, \infty) \) be continuous with \( \lim_{u \to 0^+} \frac{g(u)}{u} = \infty \), \( \lim_{u \to \infty} \frac{g(u)}{u} = \infty \), and \( m \in L^1(0, 1) \) with \( m \geq 0, m \not\equiv 0 \). Then \( f \) satisfies \((H1)-(H3)\) and \((H5)\) and therefore Theorem 1.1 applies. If we take \( m(t) = 1/\sqrt{t} \), \( g(u) = u^p + u^q + h(u) \), where \( p < 1 \leq q \) and \( h \) is a nonnegative continuous function, then it is easily seen that \( f(t, u) \) satisfies \((H1)-(H5)\) and Theorem 1.2 applies. However, the results in \([1, 2, 4, 7]\) may not apply since \( g \) may not be nondecreasing.

To prove our main results, we first establish the following results.

**Lemma 1.4.** Let \( h \in L^1(0, 1) \) be such that \( h \geq 0 \) and let \( u \) satisfy

\[
Lu = h \quad \text{in } (0, 1)
\]

\[
\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0.
\]

Then

\[
u(t) \geq |u|_0 p(t),
\]

where \( p(t) = \min \left( \frac{\phi(t)}{\|\phi\|_0}, \frac{\psi(t)}{\|\psi\|_0} \right) \), and \( \| \cdot \|_0 \) denotes the supremum norm.

**Proof.** We proceed as in \([3]\). It is easy to see that

\[
u(t) = \int_0^1 G(t, s)h(s)ds.
\]
Let \(|u|_0 = u(t_0)\) for some \(t_0 \in (0, 1)\). We verify that
\[
\frac{G(t, s)}{G(t_0, s)} \geq p(t).
\]
If \(t, t_0 \leq s\) then
\[
\frac{G(t, s)}{G(t_0, s)} = \frac{\phi(t)}{\phi(t_0)} \geq \frac{\phi(t)}{|\phi|_0},
\]
and if \(t_0 \leq s \leq t\) then
\[
\frac{G(t, s)}{G(t_0, s)} = \frac{\phi(s)\psi(t)}{\phi(t_0)\psi(s)} \geq \frac{\psi(t)}{|\psi|_0}
\]
since \(\phi(s) \geq \phi(t_0)\). The other two cases are treated in a similar manner. Hence
\[
u(t) \geq p(t)u(t_0) = |u|_0 p(t).
\]

Lemma 1.5. Let \((H1)-(H3)\) hold. Then for each \(\lambda > 0\), there exists \(c_\lambda > 0\) such that if \(u\) is a nonzero solution of (0.1) then \(|u|_0 \geq c_\lambda\). Furthermore, \((c_\lambda)\) is nondecreasing in \(\lambda\).

Proof. Let \(p_0 = \min_{t \in I} p(t)\), where \(p\) is defined in Lemma 1.4, and
\[
K = \int_I G(\frac{1}{2}, s)m(s)ds.
\]
By \((H3)\), there exists \(r_\lambda \in (0, 1)\) such that
\[
\frac{f(t, u)}{u} \geq \frac{2m(t)}{\lambda p_0 K} \quad \text{for } t \in I, \; 0 < u < r_\lambda.
\]
Define
\[
c_\lambda = \sup \{ r \in (0, 1) : \frac{f(t, u)}{u} \geq \frac{2m(t)}{\lambda p_0 K} \text{ for } t \in I, \; 0 < u < r \}.
\]
Then \(0 < c_\lambda \leq 1\) and
\[
\frac{f(t, u)}{u} \geq \frac{2m(t)}{\lambda p_0 K} \quad \text{for } t \in I, \; 0 < u \leq c_\lambda. \tag{1.2}
\]
Clearly \((c_\lambda)\) is nondecreasing in \(\lambda\). Let \(u\) be a nonzero solution of (0.1) and suppose that \(|u|_0 < c_\lambda\). Using Lemma 1.4 and (1.2), we obtain
\[
u(t) = \lambda \int_0^1 G(t, s)f(s, u(s))ds
\geq \lambda \int_I \frac{2m(s)}{\lambda p_0 K} G(t, s)u(s)ds
\geq 2K^{-1}|u|_0 \int_I G(t, s)m(s)ds,
\]
which implies
\[
|u|_0 \geq u(\frac{1}{2}) \geq 2K^{-1}\left( \int_I G(\frac{1}{2}, s)m(s)ds \right)|u|_0 = 2|u|_0,
\]
a contradiction. This completes the proof.

Lemma 1.6. Let \((H1), (H2), (H4)\) hold. Then (0.1) has no positive solution for \(\lambda\) large.
Proof. Let $u$ be a positive solution of (0.1). Using (H4) and Lemma 1.4, we obtain
\[ u\left(\frac{1}{2}\right) = \lambda \int_0^{1} G\left(\frac{1}{2}, s\right)f(s, u(s))ds \geq \lambda d \int_0^{1} G\left(\frac{1}{2}, s\right)u(s)ds \geq \lambda d C |u|_0, \]
where $C = \left( \min_{s \in J} p(s) \right) \left( \int_J G\left(\frac{1}{2}, s\right)ds \right)$, which implies $\lambda \leq (dC)^{-1}$. \hfill $\square$

The next Lemma establishes the existence of a solution once a pair of ordered sub- and supersolution are known, without assuming monotonicity of $f(t, u)$ in $u$.

**Lemma 1.7.** Let (H1), (H2) hold. Suppose that $u$ and $\bar{u}$ in $C[0, 1] \cap C^1(0, 1)$ are sub- and supersolutions of (0.1) respectively with $0 \leq u \leq \bar{u}$, i.e.,
\[ Lu(t) \leq \lambda f(t, u) \quad \text{in } (0, 1) \]
\[ \alpha u(0) - \beta u'(0) \leq 0, \quad \gamma u(1) + \delta u'(1) \leq 0 \]
and
\[ L\bar{u}(t) \geq \lambda f(t, \bar{u}(t)) \quad \text{in } (0, 1) \]
\[ \alpha \bar{u}(0) - \beta \bar{u}'(0) \geq 0, \quad \gamma \bar{u}(1) + \delta \bar{u}'(1) \geq 0. \]
Then (0.1) has a solution $u$ with $u \leq u \leq \bar{u}$.

**Proof.** The proof is essentially given in [6], where nonsingular problems were considered. For convenience, we give a proof. Without loss of generality, we assume that $\lambda = 1$. Define
\[
\bar{f}(t, v) = \begin{cases} 
 f(t, \bar{u}(t)) + \frac{\bar{u}(0) - v}{1 + \bar{u}(0)} & \text{if } v > \bar{u}(t) \\
 f(t, v) & \text{if } \bar{u}(t) \leq v \leq \bar{u}(t) \\
 f(t, \bar{u}(t)) + \frac{\bar{u}(0) - v}{1 + \bar{u}(0)} & \text{if } v < \bar{u}(t).
\end{cases}
\]
For each $v \in C[0, 1]$, let $u = T v$ be the solution of
\[ Lu = \bar{f}(t, v), \quad 0 < t < 1 \]
\[ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0. \]
Then $T : C[0, 1] \rightarrow C[0, 1]$ is completely continuous. Since $T$ is bounded, $T$ has a fixed point $u$ by the Schauder fixed point Theorem. We verify that $u \leq u \leq \bar{u}$. Suppose to the contrary that there exists $t_0 \in (0, 1)$ such that $u(t_0) > \bar{u}(t_0)$. Let $w = u - \bar{u}$ and $t_1 \in [0, 1]$ be such that $w(t_1) = \max_{0 \leq t \leq 1} w(t) > 0$. If $t_1 \in (0, 1)$ then $w'(t_1) = 0$ and $(rw'(t_1)) \leq 0$, which implies that $Lw(t_1) \geq 0$. On the other hand,
\[ Lw(t_1) = Lu(t_1) - L\bar{u}(t_1) \leq - \frac{w(t_1)}{1 + w^2(t_1)} < 0, \]
a contradiction. Suppose that $t_1 = 0$. Then $w'(0) \leq 0$, and since $\alpha w(0) - \beta w'(0) \leq 0$, we have a contradiction if $\alpha > 0$. If $\alpha = 0$ then $\beta > 0$ and therefore $w'(0) = 0$. Since $-(rw'(t_1)) + qw(t_1) \equiv Lw(t) < 0$ for small $t > 0$, it follows by integrating that $(rw'(t)) > 0$ and so $w'(t) > 0$ for small $t > 0$, a contradiction. Similarly, we reach a contradiction if $t_1 = 1$. Hence $u \leq \bar{u}$ on $(0, 1)$. The lower inequality can be derived in a similar manner. \hfill $\square$

In view of Lemmas 1.4 and 1.5, we see that $u$ is a positive solution of (0.1) if and only if $u$ is a solution of
\[ Lu = \lambda \bar{f}(t, u), \quad 0 < t < 1 \]
\[ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \]
(1.3)
where \( \tilde{f}(t, u(t)) = f(t, \max(u(t), c_\lambda p(t))) \), or equivalently, \( u \) is a fixed point of \( A_\lambda \), where

\[
A_\lambda u(t) = \lambda \int_0^1 G(t, s) \tilde{f}(s, u(s))ds
\]

Note that \( A_\lambda : C[0, 1] \to C[0, 1] \) is completely continuous (see [3]).

We are now in a position to prove our main result.

**Proof of Theorem 1.1.** Let

\[
\lambda_0 = \left( \int_0^1 G(s, s)g_1(s)ds \right)^{-1}
\]

and suppose that \( 0 < \lambda < \lambda_0 \), where \( g_1 \) is defined in (H2). Let \( u \) be a solution of

\[
u = \theta A_\lambda u \quad \text{for some} \quad \theta \in [0, 1].
\]

We claim that \( |u|_0 \neq 1 \). Indeed, if \( |u|_0 = 1 \) then since \( c_\lambda |p|_0 \leq c_\lambda \leq 1 \), it follows from (H2) that \( \tilde{f}(s, u(s)) \leq g_1(s) \), which implies

\[1 = |u|_0 \leq \lambda \int_0^1 G(s, s)g_1(s)ds < 1\]

for \( \lambda < \lambda_0 \), a contradiction, and the claim is proved. Hence the Leray-Schauder fixed point Theorem gives the existence of a fixed point \( u \) of \( A_\lambda \) with \( |u|_0 < 1 \).

Next, suppose that (H5) holds. We shall employ fixed point theorems in a cone to show the existence of a second solution. Let \( \mathbb{K} \) be the cone of nonnegative functions in \( C[0, 1] \). By the above arguments, we have

\[u \in \mathbb{K} \quad \text{and} \quad u \leq A_\lambda u \Rightarrow |u|_0 \neq 1.\]

Let

\[b = 2 \left( \lambda p_1 \int_{I_1} G\left(\frac{1}{2}, s\right)m_1(s)ds \right)^{-1},\]

where \( p_1 = \min_{s \in I_1} p(s) \). By (H5), there exists \( R_b > p_1 \) such that

\[\tilde{f}(s, u) \geq bm_1(s)u \quad \text{for} \quad s \in I_1, \quad u \geq R_b.\]

We claim that

\[u \in \mathbb{K} \quad \text{and} \quad u \geq A_\lambda u \Rightarrow |u|_0 \neq R_b p_1^{-1}.\]

Suppose that \( u \in \mathbb{K} \) and \( u \geq A_\lambda u \). If \( |u|_0 = R_b p_1^{-1} \) then it follows from Lemma 1.4 that

\[u(s) \geq R_b p_1^{-1} p(s) \geq R_b \quad \text{for} \quad s \in I_1.\]

Hence

\[R_b p_1^{-1} = |u|_0 \geq u\left(\frac{1}{2}\right)\]

\[\geq \lambda \int_0^1 G\left(\frac{1}{2}, s\right)\tilde{f}(s, u(s))ds\]

\[\geq bR_b \lambda \left( \int_{I_1} G\left(\frac{1}{2}, s\right)m_1(s)ds \right) = 2R_b p_1^{-1},\]

a contradiction, and the claim is proved. By Krasnoselskii’s fixed point Theorem, [5], \( A_\lambda \) has a fixed point \( \tilde{u} \) in \( \mathbb{K} \) with \( 1 < |\tilde{u}|_0 < R_b p_1^{-1} \). This completes the proof.
Proof of Theorem 1.2. Let \( \Lambda \) be the set of all \( \lambda > 0 \) such that (0.1) has a positive solution and let \( \lambda^* = \sup \Lambda \). By Theorem 1.1 and Lemma 1.6, \( 0 < \lambda^* < \infty \). Let \( 0 < \lambda < \lambda^* \). Then there exists \( \lambda_0 > 0 \) such that \( \lambda < \lambda_0 \) and (0.1)\(_{\lambda_0}\) has a positive solution \( u_{\lambda_0} \). Then \( u_{\lambda_0} \) satisfies
\[
u_{\lambda_0}(t) \geq c_{\lambda_0} p(t) \geq c_\lambda p(t),
\]
and therefore
\[
u_{\lambda_0}(t) = \lambda_0 f(t, u_{\lambda_0}(t))
\[
\geq \lambda f(t, \max(u_{\lambda_0}(t), c_\lambda p(t))
\]
\[
= \lambda f(t, u_{\lambda_0}(t)),
\]
i.e., \( u_{\lambda_0} \) is a supersolution of (1.3). Since \( 0 \) is a subsolution of (1.3), it follows from Lemma 1.7 that (1.3) has a solution \( u_\lambda \) with \( 0 \leq u_\lambda \leq u_{\lambda_0} \). Thus \( u_\lambda \) is a positive solution of (0.1), completing the proof of Theorem 1.2. \( \square \)

References


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