WEAK SOLUTIONS FOR A STRONGLY-COUPLED NONLINEAR SYSTEM

OSMUNDO A. LIMA, ALDO T. LOURÉDO, ALEXANDRO O. MARINHO

Abstract. In this paper the authors study the existence of local weak solutions of the strongly nonlinear system

\begin{align*}
u'' + A_u + f(u, v)u &= h_1 \\
v'' + A_v + g(u, v)v &= h_2
\end{align*}

where \( A \) is the pseudo-Laplacian operator and \( f, g, h_1 \) and \( h_2 \) are given functions.

1. Introduction

Let \( \Omega \) be an open and bounded subset in \( \mathbb{R}^n \) with smooth boundary \( \Gamma \) and let \( T \) be a positive real number. In the cylinder \( Q = \Omega \times [0, T] \), with lateral boundary \( \Sigma = \Gamma \times [0, T] \), we consider the nonlinear system

\begin{align*}
&u'' + A_u + f(u, v)u = h_1 \\
&v'' + A_v + g(u, v)v = h_2 \\
&u(0) = u_0, \quad v(0) = v_0, \quad u'(0) = u_1, \quad v'(0) = v_1
\end{align*}

(1.1)

where

\[
A_u = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( | \frac{\partial u}{\partial x_i} |^{p-2} \frac{\partial u}{\partial x_i} \right), \quad p > 2,
\]

is the pseudo-Laplacian operator, \( f \) is a continuous function in the first variable and Lipschitz in the second variable and \( g \) is a Lipschitz’s function in the first variable and continuous in the second variable, with \( f(0, 0) = g(0, 0) = 0 \) and \( u_0, v_0, u_1, v_1, h_1 \) and \( h_2 \) are given functions.

When \( p \geq 2 \), many authors studied the system (1.1). For instance, we can mention: Segal [11], where the physical meaning of (1.1) is presented, Medeiros and Menzala [9], Medeiros and M. Miranda [10], Castro [3], Biazutti [1] and more recently, Clark and Lima [6] showed the existence, a local solution and its uniqueness for the system

\[
u'' - \Delta u + f(u, v)u = h_1 \quad \text{in} \quad Q = \Omega \times (0, T)
\]
\( v'' - \Delta u + g(u,v)v = h_2 \) in \( Q \)

\( u(0) = u_0, \quad u'(0) = u_1 \) in \( \Omega \)

\( v(0) = v_0, \quad v'(0) = v_1 \) in \( \Omega \)

\( u = 0, \quad v = 0 \) on \( \Sigma = \Gamma \times (0,T) \),

where the functions \( f \) and \( g \) satisfying the same conditions of the problem (1.1). Castro [3] showed the existence of solution for the system

\[
\begin{align*}
v'' + Au - \Delta u' + |v|^{p+2}|u|^\rho u &= f_1 \quad \text{in } Q \\
v'' + Av - \Delta v' + |u|^{p+2}|v|^\rho v &= f_2 \quad \text{in } Q \\
u(0) &= u_0, \quad u'(0) = u_1 \quad \text{in } \Omega \\
v(0) &= v_0, \quad v'(0) = v_1 \quad \text{in } \Omega \\
u &= 0, \quad v = 0 \quad \text{on } \Sigma,
\end{align*}
\]

where \( A \) is the pseudo-Laplacian operator. We can show that the functions \( f(u,v) = |u|^{p+2}|v|^\rho \) and \( g(u,v) = |v|^{p+2}|u|^\rho, \rho \geq -1 \), satisfy the conditions of the system (1.1). Consequently the above system, without the dissipations \( \Delta u' \) and \( \Delta v' \), is a particular case of (\*). Thus, we see that (1.1) generalizes the above mentioned problems.

To show the existence of a local solution for (1.1), we encounter following technical difficulties:

(i) The choices of the functional spaces;

(ii) In the a priori estimate for \( u_m'' \), we had that to use the projection operator, since, to derive the approximated equation we will have much technical difficulties because of the pseudo-Laplacian operator in the equation;

(iii) In the passage to the limit, we use strongly the fact that \( A \) is a monotonic and hemicontinuous operator.

We remark that these difficulties do not appear in [6].

**Notation.** We represent the Sobolev space of order \( m \) in \( \Omega \) by

\[
W^{m,p}(\Omega) = \{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \forall |\alpha| \leq m \},
\]

with the norm

\[
\| u \|_{m,p} = \left( \sum_{|\alpha| \leq m} |D^\alpha u|_{L^p(\Omega)}^p \right)^{1/p}, u \in W^{m,p}(\Omega), 1 \leq p < \infty.
\]

Let \( D(\Omega) \) be the space of test functions in \( \Omega \) and by \( W_0^{m,p}(\Omega) \) we represent the closure of \( D(\Omega) \) in \( W^{m,p}(\Omega) \). The dual space of \( W_0^{m,p}(\Omega) \) is denoted by \( W^{-m,p'}(\Omega) \) with \( p' \) is such that \( \frac{1}{p} + \frac{1}{p'} = 1 \). We use the symbols \( \langle \cdot, \cdot \rangle \) and \( | \cdot | \), to indicate the inner product and the norm in \( L^2(\Omega) \). We use \( \langle \cdot, \cdot \rangle_{W^{-1,p}(\Omega),W_0^{1,p}(\Omega)} \) to indicate the duality between \( W^{-1,p'}(\Omega) \) and \( W_0^{1,p}(\Omega) \) and \( \| \cdot \|_0 \) to indicate the norm \( W_0^{1,p}(\Omega) \).

The pseudo-Laplacian operator \( A \) is such that

\[
A : \quad W_0^{1,p}(\Omega) \quad \mapsto \quad W^{-1,p'}(\Omega)
\]

and it satisfies the following properties:

- \( A \) is monotonic, that is, \( \langle Au - A v, u - v \rangle \geq 0, \forall u, v \in W_0^{1,p}(\Omega) \);
Let \( P \) be a positive real function, \( \alpha, \beta \) and \( \gamma \), positive real constants, with \( \gamma > 1 \), such that
\[
\phi(t) \leq \alpha + \beta \int_0^t \{ \phi(s) + \phi^\gamma(s) \} ds.
\]
Then, there exists \( T_0 \in \mathbb{R} \), with \( 0 < T_0 < T \), such that \( \phi \) is bounded in \( [0, T_0] \).

**Definition.** A local weak solution of the problem (1.1) is a pair of functions \( u = u(x,t), v = v(x,t) \) defined for all \((x,t) \in Q_{T_0} = \Omega \times (0, T_0) \), and \( T_0 > 0 \) fixed, satisfying
\[
\begin{align*}
    u, v &\in L^\infty(0, T_0; W^{1,p}_0(\Omega)); \\
    u', v' &\in L^\infty(0, T_0; L^2(\Omega)); \\
    \frac{d}{dt}(u', w) + \langle Au, w \rangle + \langle f(u, v)u, w \rangle &= (h_1, w), \forall w \in W^{1,p}_0(\Omega) \text{ in } D'(0, T_0); \\
    \frac{d}{dt}(v', w) + \langle Av, w \rangle + \langle g(u, v)v, w \rangle &= (h_2, w), \forall w \in W^{1,p}_0(\Omega) \text{ in } D'(0, T_0); \\
    u(0) &= u_0, \quad u'(0) = u_1, \quad v(0) = v_0, \quad v'(0) = v_1.
\end{align*}
\]

2. Existence Results

**Theorem 2.1.** Let \( f \) and \( g \) be functions of two variables such that \( f \) is continuous in the first variable and Lipschitz in the second variable and \( g \) is Lipschitz in the first variable and continuous in the second variable, with \( f(0,0) = g(0,0) = 0 \).
\[
\begin{align*}
    h_1, h_2 &\in L^2(0, T; L^2(\Omega)); \\
    u_0, v_0 &\in W^{1,p}_0(\Omega); \\
    u_1, v_1 &\in L^2(\Omega). \quad \tag{2.3}
\end{align*}
\]
Then it exists \( T_0 > 0, T_0 \in \mathbb{R} \) and functions \( u : Q_{T_0} \to \mathbb{R} \) and \( v : Q_{T_0} \to \mathbb{R} \) satisfying
\[
\begin{align*}
    u, v &\in L^\infty(0, T_0; W^{1,p}_0(\Omega)); \\
    u', v' &\in L^\infty(0, T_0; L^2(\Omega)); \\
    \frac{d}{dt}(u', w) + \langle Au, w \rangle + \langle f(u, v)u, w \rangle &= (h_1, w), \forall w \in W^{1,p}_0(\Omega), \text{ in } D'(0, T_0); \\
    \frac{d}{dt}(v', w) + \langle Av, w \rangle + \langle g(u, v)v, w \rangle &= (h_2, w), \forall w \in W^{1,p}_0(\Omega), \text{ in } D'(0, T_0); \\
    u(0) &= u_0, \quad v(0) = v_0. \tag{2.8}
\end{align*}
\]
\[ u'(0) = u_1, \quad v'(0) = v_1. \] (2.9)

The main tools in the proof of this theorem are the Faedo-Galerkin method and compactness arguments. Let \( H_0^s(\Omega) \), with \( s > m = n\left(\frac{1}{2} - \frac{1}{p}\right) + 1 \), a separable Hilbert space such that \( H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \), is a continuous and dense immersion. In \( H_0^s(\Omega) \), there exists a complete orthonormal hilbertian base \( \{w_j\}_{j \in \mathbb{N}} \) in \( L^2(\Omega) \).

We consider \( V_m = [w_1, \ldots, w_m] \) the subspace of \( H_0^s(\Omega) \) generated by the \( m \) first vectors of the base \( \{w_j\}_{j \in \mathbb{N}} \). Also, we have the following chain of continuous and dense immersions.

\[
H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,p'}(\Omega) \hookrightarrow H^{-s}(\Omega). \tag{2.10}
\]

We will divide the proof in three parts: (i) Approximated Problem, (ii) A Priori Estimates I and (iii) A Priori Estimates II.

**Approximated Problem.** We want to find \( u_m(t), v_m(t) \) in \( V_m \) satisfying the approximated problem.

\[
\begin{align*}
(u''_m(t), w) + \langle Au_m(t), w \rangle + (f(u_m(t), v_m(t))u_m(t), w) &= (h_1(t), w), \tag{2.11}
(v''_m(t), w) + \langle Av_m(t), w \rangle + (g(u_m(t), v_m(t))v_m(t), w) &= (h_2(t), w), \tag{2.12}
\end{align*}
\]

for all \( w \in V_m \); and

\[
\begin{align*}
&u_m(0) = u_{0m}, \quad u'_m(0) = u_{1m}, \\
v_m(0) = v_{0m}, \quad v'_m(0) = v_{1m}; \tag{2.13}
\end{align*}
\]

So that

\[
\begin{align*}
&u_{0m} \to u_0, \quad v_{0m} \to v_0, \quad \text{in} \ W_0^{1,p}(\Omega); \\
&u_{1m} \to u_1, \quad v_{1m} \to v_1, \quad \text{in} \ L^2(\Omega).
\end{align*}
\]

It can be shown that the above system satisfies the Carathéodory's conditions; therefore there exists solutions \( u_m(t), v_m(t) \) in \([0, t_m), t_m < T \) satisfying (2.11) - (2.13).

**A priori estimates I.** Let us consider \( w = 2u'_m(t) \) in (2.11). It follows that

\[
2(u''_m(t), u'_m(t)) + 2\langle Au_m(t), u'_m(t) \rangle + 2(f(u_m(t), v_m(t))u_m(t), u'_m(t))
= (h_1(t), u'_m(t)).
\]

Thus

\[
\frac{d}{dt}|u'_m(t)|^2 + 2 \frac{d}{dt} \|u_m(t)\|^p_0 = 2(h_1(t), u'_m(t)) - 2(f(u_m(t), v_m(t))u_m(t), u'_m(t)).
\]

Similarly, setting \( w = 2v'_m(t) \) in (2.12) it follows that

\[
\frac{d}{dt}|v'_m(t)|^2 + 2 \frac{d}{dt} \|v_m(t)\|^p_0 = 2(h_2(t), v'_m(t)) - 2(g(u_m(t), v_m(t))v_m(t), v'_m(t)).
\]

Summing the two equalities above, then integrating from 0 to \( t, t < t_m \), and using the Cauchy-Schwarz's inequality and \( ab \leq \frac{a^2 + b^2}{2} \), we obtain

\[
\begin{align*}
&\|u'_m(t)|^2 + |v'_m(t)|^2 + \frac{2}{p} \|u_m(t)\|^p_0 + \frac{2}{p} \|v_m(t)\|^p_0 \\
&\leq |u'_m(0)|^2 + |v'_m(0)|^2 + \frac{2}{p} \|u_m(0)\|^p_0 + \frac{2}{p} \|v_m(0)\|^p_0 \\
&+ 2 \int_0^t \int_{\Omega} |f(u_m(s), v_m(s))||u_m(s)||u'_m(s)|ds
\end{align*}
\]
\[ + 2 \int_0^t \int_\Omega |g(u_m(s), v_m(s))||v_m(s)||v'_m(s)|ds \]
\[ + \int_0^t (|u'_m(s)|^2 + |v'_m(s)|^2)ds + \int_0^T (|h_1(t)|^2 + |h_2(t)|^2)dt. \]

From (2.1) and (3), it follows that
\[
|u'_m(t)|^2 + |v'_m(t)|^2 + \frac{2}{p} |u_m(t)|_p^p + \frac{2}{p} |v_m(t)|_p^p \leq C + \int_0^t (|u'_m(s)|^2 + |v'_m(s)|^2)ds \]
\[ + 2 \int_0^t |f(u_m(s), v_m(s))||u_m(s)||u'_m(s)|ds \]
\[ + 2 \int_0^t |g(u_m(s), v_m(s))||v_m(s)||v'_m(s)|ds. \]

From the Sobolev immersions it is well known that

\[ W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \forall 1 \leq q \leq \frac{np}{n-p}. \]

Let \( \alpha, \beta > 0 \), such that \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \), with \( 1 \leq \alpha, \beta \leq \frac{np}{n-p} \).

Now, using Holder and Young inequalities, the inequality \( ab \leq \frac{a^2 + b^2}{2} \) and the hypothesis over \( f \), we have

\[
2 \int_0^t \int_\Omega |f(u_m(s), v_m(s))||u_m(s)||u'_m(s)|ds \leq C \int_0^t \int_\Omega |v_m(s)||u_m(s)||u'_m(s)|ds \]
\[ \leq C \int_0^t \left( \int_\Omega |v_m(s)|^\alpha \right)^{\frac{1}{\alpha}} \left( \int_\Omega |u_m(s)|^\beta \right)^{\frac{1}{\beta}} \left( \int_\Omega |u'_m(s)|^2 \right)^2 \]
\[ = C \int_0^t \left| v_m(s) \right|_{L^\alpha(\Omega)} || u_m(s) ||_{L^\beta(\Omega)} || u'_m(s) ||_{L^2(\Omega)} ds \]
\[ \leq C \int_0^t \left\{ \frac{1}{p} |v_m(s)|_{L^\alpha(\Omega)}^p + \frac{1}{p} |u_m(s)|_{L^\beta(\Omega)}^{p-1} \left( \frac{p-2}{p-1} \right) \right\} |u'_m(s)|_{L^2(\Omega)} ds \]
\[ \leq C \int_0^t \left\{ \frac{1}{p} |v_m(s)|_{L^\alpha(\Omega)}^p + \frac{1}{p} |u_m(s)|_{L^\beta(\Omega)}^{p-1} + \frac{p-2}{p-1} \right\} |u'_m(s)|_{L^2(\Omega)} ds \]
\[ = C \int_0^t \left\{ \frac{1}{p} |v_m(s)|_{L^\alpha(\Omega)}^p + \frac{1}{p} |u_m(s)|_{L^\beta(\Omega)}^{p-1} + \frac{p-2}{p-1} \right\} |u'_m(s)|_{L^2(\Omega)} ds \]
\[ \leq C \int_0^t \left\{ \frac{1}{p} |v_m(s)|_{L^\alpha(\Omega)}^{2p} + \frac{1}{p} |u_m(s)|_{L^\beta(\Omega)}^{2p-1} + \left( \frac{p-2}{p-1} \right)^2 + |u'_m(s)|_{L^2(\Omega)}^2 \right\} ds \]
\[ \leq C \int_0^t \left\{ \frac{1}{p^2} |v_m(s)|_{L^\alpha(\Omega)}^{2p} + \frac{1}{p^2} |u_m(s)|_{L^\beta(\Omega)}^{2p-1} + \left( \frac{p-2}{p-1} \right)^2 + |u'_m(s)|_{L^2(\Omega)}^2 \right\} ds. \]
Since $W^1,p_0(\Omega) \hookrightarrow L^\alpha(\Omega)$ and $W^1,p_0(\Omega) \hookrightarrow L^3(\Omega)$, it follows that

\[ 2 \int_0^t \int_0^s \left| f(u_m(s), v_m(s)) \right| u_m(s) \, |u'_m(s)| \, ds \leq C \int_0^t \left\{ \frac{1}{p^2} |v_m(s)|^{2p} + \frac{1}{p^2} |u_m(s)|^{2p} + 1 + |u'_m(s)|^2_{L^2(\Omega)} \right\} ds. \]

(2.15)

Similarly, we have

\[ 2 \int_0^t \int_0^s \left| g(u_m(s), v_m(s)) \right| v_m(s) \, |v'_m(s)| \, ds \leq C \int_0^t \left\{ \frac{1}{p^2} |u_m(s)|^{2p} + \frac{1}{p^2} |v_m(s)|^{2p} + 1 + |v'_m(s)|^2_{L^2(\Omega)} \right\} ds. \]

(2.16)

Substituting (2.15) and (2.16) in (2.14),

\[ |u'_m(t)|^2 + |v'_m(t)|^2 + \frac{2}{p} |u_m(t)|^p + \frac{2}{p} |v_m(t)|^p \leq C + C \int_0^t \left( |u'_m(s)|^2 + |v'_m(s)|^2 \right) ds + C \int_0^t \left\{ |u_m(s)|^{2p} + |v_m(s)|^{2p} \right\} \]

\[ + C \int_0^t 2 \, ds \]

\[ \leq C + C \int_0^t \left( |u'_m(s)|^2 + |v'_m(s)|^2 \right) ds + C \int_0^t \left\{ |u_m(s)|^{2p} + |v_m(s)|^{2p} \right\} \]

\[ + C \int_0^t 2 \, ds \]

\[ \leq C + C \int_0^t \left( |u'_m(s)|^2 + |v'_m(s)|^2 \right) ds + C \int_0^t \left\{ |u_m(s)|^{2p} + |v_m(s)|^{2p} \right\}. \]

(2.17)

Note that

\[ \frac{2}{p} |u'_m(t)|^2 + \frac{2}{p} |v'_m(t)|^2 + \frac{2}{p} |u_m(t)|^p + \frac{2}{p} |v_m(t)|^p \leq |u'_m(t)|^2 + |v'_m(t)|^2 + \frac{2}{p} |u_m(t)|^p + \frac{2}{p} |v_m(t)|^p, \]

with $p > 2$, it follows that

\[ |u'_m(t)|^2 + |v'_m(t)|^2 + |u_m(t)|^p + |v_m(t)|^p \leq C + C \int_0^t \left( |u'_m(s)|^2 + |v'_m(s)|^2 \right) ds + C \int_0^t \left\{ |u_m(s)|^{2p} + |v_m(s)|^{2p} \right\} \]

\[ \leq C + C \int_0^t \left\{ (|u'_m(s)|^2 + |v'_m(s)|^2)^2 + \left( |u_m(s)|^p + |v_m(s)|^p \right)^2 + 2 (|u'_m(s)|^2 + |v'_m(s)|^2) (||u_m(s)||^p + ||v_m(s)||^p) \right\} ds \]

\[ + C \int_0^t \left\{ |u'_m(s)|^2 + |v'_m(s)|^2 + |u_m(s)|^p + |v_m(s)|^p \right\} ds \]

\[ = C + C \int_0^t \left\{ |u'_m(s)|^2 + |v'_m(s)|^2 + |u_m(s)|^p + |v_m(s)|^p \right\}^2 ds. \]
+ C \int_0^t \left\{ |u_m'(s)|^2 + |v_m'(s)|^2 + \|u_m(s)\|_0^p + \|v_m(s)\|_0^p \right\} ds.

By setting

\[ \phi(t) = |u_m'(t)|^2 + |v_m'(t)|^2 + \|u_m(t)\|_0^p + \|v_m(t)\|_0^p, \]

the above inequality can be rewritten as

\[ \phi(t) \leq C + C \int_0^t \{ \phi(s) + \phi^2(s) \} ds. \] (2.18)

Then, by Lemma 1.1 there exists \( T_0 \in \mathbb{R} \), with \( 0 < T_0 < T \), such that \( \phi \) is bounded in \([0, T_0)\). From this, we have

\[ |u_m'(t)|^2 + |v_m'(t)|^2 + \|u_m(t)\|_0^p + \|v_m(t)\|_0^p \leq C \quad \forall t \in [0, T_0), \quad \forall m \in \mathbb{N}. \] (2.19)

Therefore, by prolongation results, we can extend the solutions \( u_m(t), v_m(t) \), to the interval \([0, T_0]\).

We will estimate, now, the second derivatives \( u_m''(t), v_m''(t) \). Since, the procedure, to estimates \( u_m''(t) \) and \( v_m''(t) \) are similar, we will fix our attention only on bounding \( u_m''(t) \).

### 2.1. A priori Estimates II

Let \( P_m : L^2(\Omega) \rightarrow V_m \subset L^2(\Omega) \) be

\[ P_m(h) = \sum_{j=1}^m (h, w_j) w_j, \]

the projection operator on \( L^2(\Omega) \). Observe that \( P_m = P_m^* \) and \( P_m \in \mathcal{L}(H_0^1(\Omega)) \).

Now, by the approximate equation [2.12],

\[ (u_m''(t), w) + (Au_m(t), w) + \langle f(u_m(t), v_m(t))u_m(t), w \rangle = (h_1(t), w) \] (2.20)

for all \( w \in V_m \). By the chain of immersions [2.10] we have

\[ \langle u_m''(t) + Au_m(t) + f(u_m(t), v_m(t))u_m(t) - h_1(t), w \rangle \in H^{-s}(\Omega), H_0^s(\Omega) = 0, \]

for all \( w \in V_m \). From this equality and the fact that \( P_m w = w, \forall w \in V_m \), we have

\[ P_m^*(u_m''(t) + Au_m(t) + f(u_m(t), v_m(t))u_m(t) - h_1(t)) = 0 \]

in \( V_m \). From this, by the linearity of \( P_m^* \), the fact that \( u_m'' \in V_m \), and by the continuous and dense immersions, we have

\[ u_m''(t) = -P_m^*(Au_m(t)) - P_m^*(f(u_m(t), v_m(t))u_m(t)) + P_m^*(h_1(t)) \]

in \( H^{-s}(\Omega) \). Thus

\[ \|u_m''(t)\|_{H^{-s}(\Omega)} \leq \|P_m^*(f(u_m(t), v_m(t))u_m(t))\|_{H^{-s}(\Omega)} + \|P_m^*(Au_m(t))\|_{H^{-s}(\Omega)} + \|P_m^*(h_1(t))\|_{H^{-s}(\Omega)} \]

With \( P_m \in \mathcal{L}(H_0^1(\Omega)) \) which implies \( P_m^* \in \mathcal{L}(H^{-s}(\Omega)) \). Since \( W^{-1,s'}(\Omega) \hookrightarrow H^{-s}(\Omega) \), it follows that \( P_m^* \in \mathcal{L}(W^{-1,s'}(\Omega), H^{-s}(\Omega)) \). Then

\[ \|P_m^*(Au_m(t))\|_{H^{-s}(\Omega)} \leq C\|Au_m(t)\|_{W^{-1,s'}(\Omega)} \leq C\|u_m(t)\|_{0}^{p-1}. \] (2.21)

Since, \( L^2(\Omega) \hookrightarrow H^{-s}(\Omega) \), we have \( P_m^* \in \mathcal{L}(L^2(\Omega), H^{-s}(\Omega)) \). Furthermore,

\[ \|P_m^*(h_1(t))\|_{H^{-s}(\Omega)} \leq C|h_1(t)|_{L^2(\Omega)}. \] (2.22)

Now, to bound the term \( \|P_m^*(f(u_m(t), v_m(t))u_m(t))\|_{H^{-s}(\Omega)} \), it is necessary to place \( f(u_m(t), v_m(t))u_m(t) \) in some space contained in \( H^{-s}(\Omega) \). Let \( \gamma, \theta \in [1, \frac{np}{n-p}] \), such
that $\frac{1}{q} + \frac{1}{q'} = 1$. Since $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q \leq \frac{np}{n-p}$, we have, in particular $W_0^{1,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$. Therefore,

$$(L^\gamma(\Omega))' \hookrightarrow W^{-1,p'}(\Omega).$$

From the chain of immersions (2.10), we have $W^{-1,p'}(\Omega) \hookrightarrow H^{-s}(\Omega)$, from where

$$(L^\theta(\Omega))' = (L^\gamma(\Omega))' \hookrightarrow H^{-s}(\Omega)$$

(2.23)

Now, it is sufficient to show that $f(u_m(t), v_m(t))u_m(t) \in L^\theta(\Omega)$. From the Hölder inequality and the hypothesis on $f$ we have

$$\int_\Omega |f(u_m(s), v_m(s))u_m(s)|^\theta dx = \int_\Omega |f(u_m(s), v_m(s))|^{\theta} |u_m(s)|^\theta dx$$

$$\leq C_f^\theta \int_\Omega |v_m(s)|^\theta |u_m(s)|^\theta dx$$

$$\leq C_f^\theta \left( \int_\Omega |v_m(s)|^{\alpha' \theta} \right)^{1/\alpha'} \left( \int_\Omega |u_m(s)|^{\beta' \theta} \right)^{1/\beta'},$$

(2.24)

where $C_f$ is the Lipschitz constant, associated $f$ and $\frac{1}{\alpha'} + \frac{1}{\beta'} = 1$.

If $\theta \alpha' \leq \frac{np}{n-p}$ and $\theta \beta' \leq \frac{np}{n-p}$, then

$$\theta \leq \frac{1}{\alpha'} \frac{np}{n-p}, \quad \text{and} \quad \theta \leq \frac{1}{\beta'} \frac{np}{n-p},$$

from which,

$$2\theta \leq \left( \frac{1}{\alpha'} + \frac{1}{\beta'} \right) \frac{np}{n-p}.$$

Then, we have

$$1 \leq \theta \leq \frac{np}{2(n-p)} < \frac{np}{n-p}.$$

Noticing that $W_0^{1,p}(\Omega) \hookrightarrow L^{\alpha'}(\Omega)$ and $W_0^{1,p}(\Omega) \hookrightarrow L^{\beta'}(\Omega)$, we have

$$\int_\Omega |f(u_m(s), v_m(s))u_m(s)|^\theta dx \leq C_f^\theta \|v_m(t)\|^{\theta}_{L^{\alpha' \theta}} \|u_m(t)\|^{\theta}_{L^{\beta' \theta}} \leq C\|u_m(t)\|^{\theta}_{\theta} \|v_m(t)\|^{\theta}_{\theta}.$$

From this estimate and (2.19), it follows

$$\int_\Omega |f(u_m(s), v_m(s))u_m(s)|^\theta dx < \infty;$$

(2.25)

that is,

$$f(u_m(t), v_m(t))u_m(t) \in L^\theta(\Omega) = (L^\gamma(\Omega))', \quad \text{for} \ 1 \leq \theta \leq \frac{np}{2(n-p)};$$

(2.26)

and

$$\|f(u_m(t), v_m(t))u_m(t)\|_{L^\theta(\Omega)} \leq C, \quad \forall m, \ t \in [0, T_0]$$

(2.27)

Similarly, we have

$$\|g(u_m(t), v_m(t))v_m(t)\|_{L^\theta(\Omega)} \leq C, \quad \forall m, \ t \in [0, T_0]$$

(2.28)

We will also need that $f(u_m(t), v_m(t))u_m^2(t) \in L^\theta(\Omega)$. In fact, by Hölder inequality,

$$\int_\Omega |f(u_m(s), v_m(s))u_m^2(s)|^\theta dx$$
\[
\begin{align*}
&\int_{\Omega} |f(u_m(s), v_m(s))|^{\theta} |u_m^2(s)|^{\theta} \, dx \\
&\leq C_f^\theta \int_{\Omega} |v_m(s)|^{\theta} |u_m(s)|^{\theta} u_m^2(s) \, dx \\
&\leq C_f^\theta \left( \int_{\Omega} |v_m(s)|^{\theta} \right)^{\frac{1}{\theta}} \left( \int_{\Omega} |u_m(s)|^{\delta \theta} \right)^{\frac{1}{\delta \theta}} \left( \int_{\Omega} |u_m(s)|^{-\omega} \right)^{1/\omega},
\end{align*}
\]
where \(C_f\) is the Lipschitz constant, associated to \(f\) and \(\frac{1}{\delta} + \frac{1}{\omega} + \frac{1}{\xi} = 1\). If \(\theta \xi \leq \frac{np}{n-p}, \theta \delta \leq \frac{np}{n-p}\) and \(\theta \omega \leq \frac{np}{n-p}\), then

\[
\theta \leq \frac{1}{\xi} \frac{np}{n-p}, \quad \theta \leq \frac{1}{\delta} \frac{np}{n-p}, \quad \theta \leq \frac{1}{\omega} \frac{np}{n-p}
\]

which implies

\[
3\theta \leq \left( \frac{1}{\xi} + \frac{1}{\delta} + \frac{1}{\omega} \right) \frac{np}{n-p}.
\]

Then

\[
1 \leq \theta \leq \frac{np}{3(n-p)} < \frac{np}{n-p}.
\]

Observing that \(W^{1,p}_0(\Omega) \hookrightarrow L^{\theta \xi}(\Omega), W^{1,p}_0(\Omega) \hookrightarrow L^{\theta \delta}(\Omega)\) and \(W^{1,p}_0(\Omega) \hookrightarrow L^{\theta \omega}(\Omega)\), it follows that

\[
\int_{\Omega} |f(u_m(s), v_m(s))u_m^2(s)|^{\theta} \, dx \leq C_f^\theta \|v_m(t)\|_{L^{\theta \xi}}^{\theta} \|u_m(t)\|_{L^{\theta \omega}}^{\theta} \|u_m(t)\|_{L^{\theta \omega}}^{\theta} \\
\leq C \|u_m(t)\|_0^{2\theta} \|v_m(t)\|_0^{\theta}.
\]

This estimate and (2.19) lead us to

\[
\int_{\Omega} |f(u_m(s), v_m(s))u_m^2(s)|^{\theta} \, dx < \infty;
\]

that is,

\[
f(u_m(t), v_m(t))u_m^2(t) \in L^{\theta}(\Omega) = (L^{\gamma}(\Omega))^\prime, \quad \text{for } 1 \leq \theta \leq \frac{np}{3(n-p)}, \tag{2.30}
\]

\[
\|f(u_m(t), v_m(t))u_m^2(t)\|_{L^{\theta}(\Omega)} \leq C, \quad \forall m, t \in [0, T_0] \tag{2.31}
\]

Similarly, we have

\[
\|g(u_m(t), v_m(t))v_m^2(t)\|_{L^{\theta}(\Omega)} \leq C, \quad \forall m, t \in [0, T_0] \tag{2.32}
\]

Note that if \(\theta \leq \frac{np}{3(n-p)}\), we still have (2.26) and (2.30), because \(\frac{np}{3(n-p)} < \frac{np}{n-p}\).

Thus, as \(L^{\theta}(\Omega) \hookrightarrow H^{-s}(\Omega)\), we have that \(P_m^{u} \in \mathcal{L}(L^{\theta}(\Omega), H^{-s}(\Omega))\). Therefore

\[
\|P_m^{u} f(u_m(t), v_m(t))u_m(t)\|_{H^{-s}(\Omega)} \leq C \|f(u_m(t), v_m(t))u_m(t)\|_{L^{\theta}(\Omega)} \tag{2.33}
\]

Hence, from the estimates (2.21), (2.22) and (2.33), we have

\[
\|u_m''(t)\|_{H^{-s}(\Omega)} \leq C \left\{ \|u_m(t)\|_0^{p-1} + \|f(u_m(t), v_m(t))u_m(t)\|_{L^{\theta}(\Omega)} + |h_1(t)| \right\}.
\]

From this inequality, it results

\[
\int_{0}^{T_0} \|u_m''(t)\|_{H^{-s}(\Omega)}^2 \, dt \leq C \left\{ \int_{0}^{T_0} \|u_m(t)\|_0^{2(p-1)} \, dt + \int_{0}^{T_0} |h_1(t)|^2 \, dt \right. \\
+ \left. \int_{0}^{T_0} \|f(u_m(t), v_m(t))u_m(t)\|_{L^{\theta}(\Omega)}^2 \, dt \right\}.
\]
Therefore, from \((2.17), (2.25)\) and \((2.1)\), we conclude that

\[
\|u_m''(t)\|_{L^2(0,T_0;H^{-s}(\Omega))} \leq C, \quad \forall m \in \mathbb{N}. \tag{2.34}
\]

Arguing in a similar way, one can deduce that

\[
\|v_m''(t)\|_{L^2(0,T_0;H^{-s}(\Omega))} \leq C, \forall m \in \mathbb{N}. \tag{2.35}
\]

From \((2.19)\), we have

\[
\|u_m(t)\|_0 \leq C \quad \text{and} \quad \|v_m(t)\|_0 \leq C, \quad \forall m, \ t \in [0,T_0].
\]

\[
|u_m'(t)| \leq C \quad \text{and} \quad |v_m'(t)| \leq C, \quad \forall m, \ t \in [0,T_0].
\]

From where, it follows that \(\text{ess sup}_{t \in [0,T_0]} \|u_m(t)\|_0 \leq C\); that is

\[
\|u_m\|_{L^\infty(0,T_0;W^{1,p}_0(\Omega))} \leq C, \quad \forall m \in \mathbb{N}. \tag{2.36}
\]

Similarly, we have

\[
\|v_m\|_{L^\infty(0,T_0;W^{1,p}_0(\Omega))} \leq C, \quad \forall m \in \mathbb{N}; \tag{2.37}
\]

\[
\|u_m'\|_{L^\infty(0,T_0;L^2(\Omega))} \leq C, \quad \forall m \in \mathbb{N}; \tag{2.38}
\]

\[
\|v_m'\|_{L^\infty(0,T_0;L^2(\Omega))} \leq C, \quad \forall m \in \mathbb{N}. \tag{2.39}
\]

Therefore, from \((2.27), (2.28), (2.31), (2.32), (2.34), (2.35), (2.36), (2.37), (2.38), (2.39)\), we have

\[
(u_m)_m, (v_m)_m \text{ are bounded in } L^\infty(0,T_0;W^{1,p}_0(\Omega)); \tag{2.40}
\]

\[
(u_m')_m, (v_m')_m \text{ are bounded in } L^\infty(0,T_0;L^2(\Omega)); \tag{2.41}
\]

\[
(u_m'')_m, (v_m'')_m \text{ are bounded in } L^2(0,T_0;H^{-s}(\Omega)); \tag{2.42}
\]

\[
(f(u_m,v_m)u_m)_m, (g(u_m,v_m)v_m)_m \text{ are bounded in } L^\infty(0,T_0;L^\theta(\Omega)); \tag{2.43}
\]

\[
(f(u_m,v_m)u_m^2)_m, (g(u_m,v_m)v_m^2)_m \text{ are bounded in } L^\infty(0,T_0;L^\theta(\Omega)); \tag{2.44}
\]

Furthermore, since \(A\) is bounded, we have

\[
(Au_m)_m, (Av_m)_m \text{ are bounded in } L^\infty(0,T_0;W^{-1,p'}(\Omega)).
\]

**Taking Limits.** From the estimates and Banach-Alaoglu-Bouharki theorem guarantee the existence of subsequences \((u_\nu)_\nu, (v_\nu)_\nu\) of \((u_m)_m, (v_m)_m\), respectively, such that

\[
u \overset{\ast} \rightarrow u, \quad v_\nu \overset{\ast} \rightarrow v \quad \text{in } L^\infty(0,T_0;W^{1,p}_0(\Omega)). \tag{2.45}
\]

\[
u' \overset{\ast} \rightarrow u', \quad v_\nu' \overset{\ast} \rightarrow v' \quad \text{in } L^\infty(0,T_0;L^2(\Omega)). \tag{2.46}
\]

\[
u'' \overset{\ast} \rightarrow u'', \quad v_\nu'' \overset{\ast} \rightarrow v'' \quad \text{in } L^\infty(0,T_0;H^{-s}(\Omega)). \tag{2.47}
\]

\[
u \overset{\ast} \rightarrow \chi, \quad Av_\nu \overset{\ast} \rightarrow \eta \quad \text{in } L^\infty(0,T_0;W^{-1,p'}(\Omega)). \tag{2.48}
\]

As \(L^2(0,T_0;H^{-s}(\Omega))\) is reflexive, the convergence \((2.47)\) becomes

\[
u'' \rightarrow u'', v_\nu'' \rightarrow v'' \quad \text{in } L^2(0,T_0;H^{-s}(\Omega)). \tag{2.49}
\]

Let us consider the approximate equation \((2.11)\) in the form

\[
(u_\nu'', t)(w) + (Au_\nu(t), w) + (f(u_\nu(t), v_\nu(t))u_\nu(t), w) = (h_1(t), w) \quad \forall w \in V_m, \ \nu \geq m
\]
Now, multiplying the above equality by \( \varphi \in D(0, T_0) \) and integrating from 0 for \( T_0 \) we obtain

\[
\int_0^{T_0} (u'_\nu(t), w) \varphi dt + \int_0^{T_0} \langle Au_\nu(t), w \rangle \varphi dt + \int_0^{T_0} \langle f(u_\nu(t), v_\nu(t))u_\nu(t), w \rangle \varphi dt = \int_0^{T_0} (h_1(t), w) \varphi dt \quad \forall w \in V_m, \, \nu \geq m.
\]

Integrating by parts, we obtain

\[
- \int_0^{T_0} (u''_\nu(t), w) \varphi' dt + \int_0^{T_0} \langle Au_\nu(t), w \rangle \varphi dt + \int_0^{T_0} \langle f(u_\nu(t), v_\nu(t))u_\nu(t), w \rangle \varphi dt = \int_0^{T_0} (h_1(t), w) \varphi dt \quad \forall w \in V_m, \, \nu \geq m.
\]

With \( u'_\nu \rightharpoonup u' \) in \( L^\infty(0, T_0; L^2(\Omega)) = (L^1(0, T_0; L^2(\Omega)))' \) then

\[
\langle u'_\nu, \phi \rangle \to \langle u', \phi \rangle, \quad \forall \phi \in L^1(0, T_0; L^2(\Omega)).
\]

Convergence \((2.51)\) with \( \langle u'_\nu, \phi \rangle = \int_0^{T_0} (u'_\nu(t), \phi(t)) dt \), and assuming \( \phi(x, t) = w(x) \psi(t) \) imply that

\[
\int_0^{T_0} (u'_\nu(t), \psi(t)) dt = \int_0^{T_0} (u'_\nu(t), w(x) \psi(t)) dt, \quad \forall w \in L^2(\Omega), \quad \forall \psi \in L^1(0, T_0).
\]

Consequently, for all \( w \in L^2(\Omega) \) and all \( \psi \in L^1(0, T_0) \),

\[
\int_0^{T_0} (u'_\nu(t), w(x) \psi(t)) dt \to \int_0^{T_0} (u'(t), w(x) \psi(t)) dt.
\]

In fact,

\[
\int_0^{T_0} (u'_\nu(t), (w(x) \psi(t)) \varphi'(t) dt \to \int_0^{T_0} (u'(t), w(x) \psi(t)) \varphi'(t) dt,
\]

for all \( w \in V_m \subset W^{1,p}_0(\Omega) \subset L^2(\Omega) \) and all \( \psi = \varphi', \, \varphi \in D(0, T_0) \subset L^1(0, T_0) \). In a similar way,

\[
\int_0^{T_0} (Au_\nu(t), w(x) \psi(t)) dt \to \int_0^{T_0} (\chi(t), w(x) \psi(t)) dt,
\]

for all \( w \in W^{1,p}_0(\Omega) \) and all \( \psi \in L^1(0, T_0) \). In fact,

\[
\int_0^{T_0} (Au_\nu(t), w(x)) \varphi(t) dt \to \int_0^{T_0} (\chi(t), w(x)) \varphi(t) dt,
\]

for all \( w \in V_m \subset W^{1,p}_0(\Omega) \) and all \( \varphi \in D(0, T_0) \subset L^1(0, T_0) \).

From \((2.24)\), we have the existence of a subsequence \((f(u_\nu, v_\nu)u_\nu)_{\nu} \) such that

\[
f(u_\nu, v_\nu)u_\nu \rightharpoonup \lambda, \quad \text{in} \quad L^\infty(0, T_0; L^\theta(\Omega)).
\]

Since \( L^\infty(0, T_0; L^\theta(\Omega)) \hookrightarrow L^\theta(0, T_0; L^\theta(\Omega)) \), we have from \((2.29)\) that

\[
(f(u_m(t), v_m(t))_m(t), (g(u_m(t), v_m(t))_m(t) \text{ are bounded in } L^\theta(0, T_0; L^\theta(\Omega)); \text{ thus we guarantee the existence of a subsequence, denoted as above, such that}
\]

\[
f(u_\nu, v_\nu)u_\nu \rightharpoonup \lambda, \quad \text{in} \quad L^\theta(0, T_0; L^\theta(\Omega)).
\]
Since

\[ (u'_m)_m, \text{ is bounded in } L^\infty(0,T_0; L^2(\Omega)), \]

\[ (u_m)_m, \text{ is bounded in } L^\infty(0,T_0; W^{1,p}_0(\Omega)) W^{1,p}_0(\Omega) \xrightarrow{w} L^2(\Omega), \]

we have by Aubin-Lions theorem, the existence of a subsequence \((u_\nu)_\nu\) such that

\[ u_\nu \rightharpoonup u, \text{ in } L^2(0,T_0; L^2(\Omega)) \equiv L^2(Q_{T_0}) \quad (2.54) \]

\[ u_\nu \to u, \text{ a.e. in } Q_{T_0} \quad (2.55) \]

Since, the sequences \((v_m)_m, (v'_m)_m\) satisfy the same conditions, it follows that, there exists a subsequence \((v_\nu)_\nu\) such that

\[ v_\nu \rightharpoonup v, \text{ in } L^2(0,T_0; L^2(\Omega)) \equiv L^2(Q_{T_0}) \quad (2.56) \]

\[ v_\nu \to v, \text{ a.e. in } Q_{T_0} \quad (2.57) \]

From \((2.55), (2.57)\), and of the hypothesis on \(f, g\), we have

\[ f(u_\nu, v_\nu)u_\nu \to f(u,v)u, \text{ a.e. in } Q_{T_0}, \quad (2.58) \]

\[ g(u_\nu, v_\nu)v_\nu \to g(u,v)v, \text{ a.e. in } Q_{T_0}. \quad (2.59) \]

From \((2.27)\), we have

\[ \|f(u_m, v_m)u_m\|_{L^\theta(Q_{T_0})} \leq C, \quad \forall m, \]

where \(L^\theta(Q_{T_0}) \equiv L^\theta(0,T_0; L^\theta(\Omega))\). From this and \((2.58)\), by means of Lion’s Lemma, it follows that

\[ f(u_\nu, v_\nu)u_\nu \to f(u,v)u, \text{ in } L^\theta(Q_{T_0}), \]

for \(1 \leq \theta \leq \frac{np}{3n - p}\). Therefore, from \((2.53)\), we have \(\lambda = f(u,v)u\) and from \((2.52)\). This implies

\[ f(u_\nu, v_\nu)u_\nu \rightharpoonup f(u,v)u, \text{ in } L^\infty(0,T_0; L^\theta(\Omega)). \quad (2.60) \]

Similarly,

\[ g(u_\nu, v_\nu)v_\nu \rightharpoonup g(u,v)v, \text{ in } L^\infty(0,T_0; L^\theta(\Omega)). \]

The convergence in \((2.60)\) implies

\[ \int_0^{T_0} \langle f(u_\nu(t), v_\nu(t))u_\nu(t), w(x)\rangle \psi(t) dt \to \int_0^{T_0} \langle f(u(t), v(t))u(t), w(x)\rangle \psi(t) dt, \]

for all \(w \in W^{1,p}_0(\Omega) \subset L^\gamma(\Omega)\), for all \(\psi \in L^1(0,T_0)\). In fact,

\[ \int_0^{T_0} \langle f(u_\nu(t), v_\nu(t))u_\nu(t), w(x)\rangle \varphi(t) dt \to \int_0^{T_0} \langle f(u(t), v(t))u(t), w(x)\rangle \varphi(t) dt, \]

for all \(w \in V_m \subset W^{1,p}_0(\Omega) \subset L^\gamma(\Omega)\), for all \(\varphi \in D(0,T_0) \subset L^1(0,T_0)\). Taking the limit, as \(\nu \to \infty\), in \((2.50)\) and using the convergences obtained above, we have

\[ - \int_0^{T_0} (u'(t), w)\varphi' dt + \int_0^{T_0} (\chi(t), w)\varphi dt + \int_0^{T_0} \langle f(u(t), v(t))u(t), w\rangle \varphi dt \]

\[ = \int_0^{T_0} \langle h_1(t), w\rangle \varphi dt, \quad \forall w \in V_m, \varphi \in D(0,T_0). \quad (2.61) \]
Note that, with a similar reasoning for the approximate equation (2.12) we obtain
\[ -\int_0^{T_0} (v'(t), w)\varphi' dt + \int_0^{T_0} \langle \eta(t), w \rangle \varphi dt + \int_0^{T_0} \langle g(u(t), v(t))v(t), w \rangle \varphi dt \]
\[ = \int_0^{T_0} (h_2(t), w) \varphi dt, \quad \forall w \in V_m, \varphi \in D(0, T_0). \tag{2.62} \]

Now, using the basis definition and the fact that \( V_m \) is dense in \( W^{1,p}_0(\Omega) \), expressions (2.61) and (2.62) take the form
\[ -\int_0^{T_0} (u'(t), w)\varphi' dt + \int_0^{T_0} \langle \eta(t), w \rangle \varphi dt + \int_0^{T_0} \langle f(u(t), v(t))u(t), w \rangle \varphi dt \]
\[ = \int_0^{T_0} (h_1(t), w) \varphi dt, \quad \forall w \in W^{1,p}_0(\Omega), \varphi \in D(0, T_0), \tag{2.63} \]

and
\[ -\int_0^{T_0} (v'(t), w)\varphi' dt + \int_0^{T_0} \langle \eta(t), w \rangle \varphi dt + \int_0^{T_0} \langle g(u(t), v(t))v(t), w \rangle \varphi dt \]
\[ = \int_0^{T_0} (h_2(t), w) \varphi dt, \quad \forall w \in W^{1,p}_0(\Omega), \varphi \in D(0, T_0). \tag{2.64} \]

Note that, the mappings \( t \mapsto (u'(t), w), t \mapsto (v'(t), w) \) being functions in \( L^\infty(0, T_0) \), they define distributions on \( (0, T_0) \). Therefore, the first integrals of (2.63), (2.64) are the derivative of these distributions. Thus, from (2.63) we have
\[ \int_0^{T_0} \left\{ \frac{d}{dt} (u'(t), w) + \langle \chi(t), w \rangle + \langle f(u(t), v(t))u(t), w \rangle - (h_1(t), w) \right\} \varphi dt = 0 \]
for all \( w \in W^{1,p}_0(\Omega) \) and all \( \varphi \in D(0, T_0) \). Thus,
\[ \frac{d}{dt} (u'(t), w) + \langle \chi(t), w \rangle + \langle f(u(t), v(t))u(t), w \rangle = (h_1(t), w), \]
for all \( w \in W^{1,p}_0(\Omega) \), in \( D'(0, T_0) \). Similarly,
\[ \frac{d}{dt} (v'(t), w) + \langle \eta(t), w \rangle + \langle g(u(t), v(t))v(t), w \rangle = (h_2(t), w), \]
for all \( w \in W^{1,p}_0(\Omega) \), in \( D'(0, T_0) \).

If one shows that \( A_u(t) = \chi(t) \) and \( A_v(t) = \eta(t) \), the proof of the theorem will be complete; since the verification of the initial conditions can be done in a standard way.

The monotonicity of \( A \) implies that
\[ \int_0^{T_0} \langle A u_\nu(t) - A w, u_\nu - w \rangle dt \geq 0, \quad \forall w \in W^{1,p}_0(\Omega); \]
that is,
\[ 0 \leq \int_0^{T_0} \langle A u_\nu(t), u_\nu \rangle dt - \int_0^{T_0} \langle A u_\nu(t), w \rangle dt - \int_0^{T_0} \langle A w, u_\nu(t) - w \rangle dt \]
for all \( w \in W^{1,p}_0(\Omega) \).
\[ 0 \leq \lim \sup \int_0^{T_0} \langle A u_\nu(t), u_\nu \rangle dt - \int_0^{T_0} \langle \chi(t), w \rangle dt - \int_0^{T_0} \langle A w, u(t) - w \rangle dt, \]
for all \( w \in W_{0}^{1,p}(\Omega) \). Considering the approximate equation (2.11) with \( m = \nu \) and \( w = u_{\nu}(t) \) we have

\[
(u_{\nu}''(t), u_{\nu}(t)) + \langle Au_{\nu}(t), u_{\nu}(t) \rangle + \langle f(u_{\nu}, v_{\nu})u_{\nu}, u_{\nu} \rangle = (h_{1}(t), u_{\nu}(t)).
\]

Therefore,

\[
\frac{d}{dt}(u_{\nu}'(t), u_{\nu}(t)) - |u_{\nu}'(t)|^{2} + \langle Au_{\nu}(t), u_{\nu}(t) \rangle + \langle f(u_{\nu}, v_{\nu})u_{\nu}, u_{\nu} \rangle = (h_{1}(t), u_{\nu})
\]

Integrating from 0 the \( T_{0} \) we have

\[
\int_{0}^{T_{0}} \langle Au_{\nu}(t), u_{\nu}(t) \rangle dt = (u_{\nu}'(0), u_{\nu}(0)) - (u_{\nu}'(T_{0}), u_{\nu}(T_{0})) + \int_{0}^{T_{0}} |u_{\nu}'(t)|^{2} dt
\]

Recall that \( W_{0}^{1,p}(\Omega) \hookrightarrow L^{2}(\Omega) \). Since \( u_{\nu}(0) \rightarrow u(0) \) in \( W_{0}^{1,p}(\Omega) \) it implies \( u_{\nu}(0) \rightarrow u(0) \) in \( L^{2}(\Omega) \). Since \( u_{\nu}'(0) \rightarrow u'(0) \) in \( L^{2}(\Omega) \), it implies

\[
(u_{\nu}'(0), u_{\nu}(0)) \rightarrow (u'(0), u(0)) \quad \text{ in } \mathbb{R}
\]

Recall that \( (u_{m}(T_{0}))_{m} \) is bounded in \( W_{0}^{1,p}(\Omega) \) and \( (u_{m}'(T_{0}))_{m} \) is bounded in \( L^{2}(\Omega) \). Thus, there exists subsequences \( (u_{\nu}(T_{0}))_{\nu} \) and \( (u_{\nu}'(T_{0}))_{\nu} \) such that

\[
u_{\nu}(T_{0}) \rightarrow u(T_{0}) \quad \text{ in } W_{0}^{1,p}(\Omega) \hookrightarrow L^{2}(\Omega),
\]

which implies

\[
u_{\nu}(T_{0}) \rightarrow u(T_{0}), \text{ in } L^{2}(\Omega),
\]

\[
u_{\nu}'(T_{0}) \rightarrow u'(T_{0}), \text{ in } L^{2}(\Omega)
\]

Consequently,

\[
(u_{\nu}'(0), u_{\nu}(T_{0})) \rightarrow (u'(T_{0}), u(T_{0})) \quad \text{ in } \mathbb{R}.
\]

We have that \( (u_{m}') \) bounded in \( L^{\infty}(0, T_{0}; L^{2}(\Omega)) \). Since

\[
L^{\infty}(0, T_{0}; L^{2}(\Omega)) \hookrightarrow L^{2}(0, T_{0}; L^{2}(\Omega)),
\]

it follows that \( (u_{m}') \) is bounded in \( L^{2}(0, T_{0}; L^{2}(\Omega)) \). We also have that \( (u_{m}') \) is bounded in \( L^{2}(0, T_{0}; H^{-\gamma}(\Omega)) \). Therefore, by the Aubin-Lions Theorem, there exists a subsequence \( (u_{\nu}') \) such that

\[
u_{\nu}' \rightarrow u' \quad \text{ in } L^{2}(0, T_{0}; L^{2}(\Omega)) \equiv L^{2}(Q_{T_{0}}).
\]

Hence

\[
\int_{0}^{T_{0}} |u_{\nu}'(t)|^{2} dt \rightarrow \int_{0}^{T_{0}} |u'(t)|^{2} dt
\]

Note that

\[
\langle f(u_{m}(t), v_{m}(t))u_{m}(t), u_{m}(t) \rangle_{L^{p}, L^{\gamma}} = \langle f(u_{m}(t), v_{m}(t))u_{m}^{2}(t), 1 \rangle_{L^{p}, L^{\gamma}}.
\]

From (2.68) we have \( u_{\nu}^{2} \rightarrow u^{2} \) a.e. in \( Q_{T_{0}} \). Similarly

\[
\int_{0}^{T_{0}} |v_{\nu}'(t)|^{2} dt \rightarrow \int_{0}^{T_{0}} |v'(t)|^{2} dt
\]

hence, we have \( v_{\nu}^{2} \rightarrow v^{2} \) a.e. in \( Q_{T_{0}} \). From (2.31), we have

\[
\|f(u_{\nu}, v_{\nu})u_{\nu}^{2}\|_{L^{p}(0, T_{0}; L^{p}(\Omega))} \equiv L^{p}(Q_{T_{0}}) \leq C, \quad \forall m.
\]
From this inequality and (2.44), we guarantee the existence of a subsequence such that
\[ f(u_\nu, v_\nu) u_\nu^2 \xrightarrow{\ast} \sigma \quad \text{in} \ L^\infty(0, T_0; L^\theta(\Omega)) \]  
(2.70)
\[ f(u_\nu, v_\nu) u_\nu^2 \rightarrow \sigma \quad \text{in} \ L^\theta(0, T_0; L^\theta(\Omega)) \]  
(2.71)
Thus, from (2.55), (2.57) and the hypotheses on \( f, g \), we have that
\[ f(u_\nu, v_\nu) u_\nu^2 \rightarrow f(u, v)u^2 \quad \text{a.e. in} \ Q_{T_0}, \]  
(2.72)
\[ g(u_\nu, v_\nu) u_\nu^2 \rightarrow g(u, v)u^2 \quad \text{a.e in} \ Q_{T_0}. \]  
(2.73)

From (2.69), (2.72) and the Lions’ Lemma it follows that
\[ f(u_\nu, v_\nu) u_\nu^2 \rightarrow f(u, v)u^2 \text{in} L^\theta(Q_{T_0}) \equiv L^\theta(0, T_0; L^\theta(\Omega)), \quad \text{for } 1 \leq \theta \leq \frac{np}{3(n-p)}. \]

From this convergence and (2.71), we have \( \sigma = f(u, v)u^2 \) and from (2.70),
\[ f(u_\nu, v_\nu) u_\nu^2 \xrightarrow{\ast} f(u, v)u^2 \text{ in } L^\infty(0, T_0; L^\theta(\Omega)). \]  
(2.74)

Similarly,
\[ g(u_\nu, v_\nu) u_\nu^2 \xrightarrow{\ast} g(u, v)u^2 \text{in } L^\infty(0, T_0; L^\theta(\Omega)). \]

The convergence (2.74) implies
\[ \langle f(u_\nu, v_\nu) u_\nu^2, \psi \rangle \rightarrow \langle f(u, v)u^2, \psi \rangle, \quad \forall \psi \in L^1(0, T_0; L^\gamma(\Omega)) \]
or better
\[ \int_0^{T_0} \langle f(u_\nu, v_\nu) u_\nu^2, w(x) \rangle \varphi(t) dt \rightarrow \int_0^{T_0} \langle f(u, v)u^2, w(x) \rangle \varphi(t) dt, \]
for all \( w \in L^\gamma(\Omega) \) and all \( \varphi \in L^1(0, T_0) \). When fixing \( w \equiv 1 \) and \( \varphi \equiv 1 \), we have
\[ \int_0^{T_0} \langle f(u_\nu(t), v_\nu(t)) u_\nu(t), u_\nu(t) \rangle dt = \int_0^{T_0} \langle f(u_\nu(t), v_\nu(t)) u_\nu^2(t), 1 \rangle dt \]
which approaches
\[ \int_0^{T_0} \langle f(u(t), v(t)) u^2(t), 1 \rangle dt = \int_0^{T_0} \langle f(u(t), v(t)) u(t), u(t) \rangle dt. \]
hence
\[ \int_0^{T_0} \langle f(u_\nu(t), v_\nu(t)) u_\nu(t), u_\nu(t) \rangle dt \rightarrow \int_0^{T_0} \langle f(u(t), v(t)) u(t), u(t) \rangle dt, \]  
(2.75)
as \( \nu \rightarrow \infty \). Therefore, taking the limit in (2.65), using the convergence (2.66), (2.67), (2.68) and (2.75), as \( \nu \rightarrow +\infty \), we have
\[ \limsup \int_0^{T_0} \langle Au_\nu(t), u_\nu(t) \rangle dt = (u'(0), u(0)) - (u'(T_0), u(T_0)) + \int_0^{T_0} |u'(t)|^2 dt \]
\[ - \int_0^{T_0} \langle f(u(t), v(t)) u(t), u(t) \rangle dt + \int_0^{T_0} \langle h_1(t), u(t) \rangle dt \]
From this equality and (2.75), we have
\[ 0 \leq (u'(0), u(0)) - (u'(T_0), u(T_0)) + \int_0^{T_0} |u'(t)|^2 dt - \int_0^{T_0} \langle f(u, v) u, u \rangle dt \]
\[ - \int_0^{T_0} \langle \chi(t), w \rangle dt - \int_0^{T_0} \langle Aw, u(t) - w \rangle dt + \int_0^{T_0} \langle h_1(t), u(t) \rangle dt, \]  
(2.76)
for all \( w \in W^{0,p}_0(\Omega) \). From the approximate equation \( (2.11) \), we have
\[
(u''_w(t), w) + (Au_w(t), w) + \langle f(u_v(t), v_v(t))u_v(t), w \rangle = (h_1(t), w), \quad \forall w \in V_m, \nu \geq m.
\]

Now, let \( \varphi \in C^1([0,T_0]) \). Then
\[
\int_0^{T_0} (u''_w(t), w)\varphi + \int_0^{T_0} (Au_w(t), w)\varphi + \int_0^{T_0} \langle f(u_v(t), v_v(t))u_v(t), w \rangle \varphi
\]
\[
= \int_0^{T_0} (h_1(t), w),
\]
for all \( w \in V_m \) and all \( \nu \geq m \). Setting
\[
(u'_w(t), w)\varphi(T_0) - (u'_w(0), w)\varphi(0) - \int_0^{T_0} (u'_w(t), w)\varphi' \, dt
\]
\[
+ \int_0^{T_0} (Au_w(t), w)\varphi \, dt + \int_0^{T_0} \langle f(u_v(t), v_v(t))u_v(t), w \rangle \varphi(t) \, dt
\]
\[
= \int_0^{T_0} (h_1(t), w)\varphi(t) \, dt, \quad \forall w \in V_m, \varphi \in C^1([0,T_0]), \nu \geq m.
\]

Taking into account the previous convergence statements, it follows that
\[
(u'_w(t), w)\varphi(T_0) - (u'_w(0), w)\varphi(0) - \int_0^{T_0} (u'_w(t), w)\varphi' \, dt
\]
\[
+ \int_0^{T_0} (Au_w(t), w)\varphi \, dt + \int_0^{T_0} \langle f(u_v(t), v_v(t))u_v(t), w \rangle \varphi(t) \, dt
\]
\[
= \int_0^{T_0} (h_1(t), w)\varphi(t) \, dt, \quad \forall w \in V_m, \varphi \in C^1([0,T_0])
\]

Using a basis argument and the fact that \( V_m \) is dense in \( W^{1,p}_0(\Omega) \), it follows that
\[
(u'_w(t), w)\varphi(T_0) - (u'_w(0), w)\varphi(0) - \int_0^{T_0} (u'_w(t), w)\varphi' \, dt
\]
\[
+ \int_0^{T_0} (Au_w(t), w)\varphi \, dt + \int_0^{T_0} \langle f(u_v(t), v_v(t))u_v(t), w \rangle \varphi(t) \, dt
\]
\[
= \int_0^{T_0} (h_1(t), w)\varphi(t) \, dt, \quad \forall w \in W^{1,p}_0(\Omega), \varphi \in C^1([0,T_0]).
\]

Observing that the set of the linear combinations of the type \( w\varphi \), with \( w \in W^{1,p}_0(\Omega) \) and \( \varphi \in C^1([0,T_0]) \), is dense in the space
\[
V = \{ v \in L^2(0,T_0;W^{1,p}_0(\Omega)), v' \in L^2(0,T_0;L^2(\Omega)) \}.
\]

It follows that \( (2.77) \) is true in the space \( V \).

Using the fact that,
\[
\begin{align*}
u \in L^\infty(0,T_0;W^{1,p}_0(\Omega)) & \hookrightarrow L^2(0,T_0;W^{1,p}_0(\Omega)), \\
u' \in L^\infty(0,T_0;L^2(\Omega)) & \hookrightarrow L^2(0,T_0;L^2(\Omega)),
\end{align*}
\]
we obtain that \( u \in V \). So \( (2.77) \) takes the form
\[
\begin{align*}
(u'(T_0), w)\varphi(T_0) - (u'(0), w)\varphi(0) \\
- \int_0^{T_0} (u'(t), u'(t)) \, dt + \int_0^{T_0} (\chi(t), u(t)) \, dt + \int_0^{T_0} \langle f(u, v)u, w \rangle \, dt
\end{align*}
\]
\[ = \int_0^{T_0} (h_1(t), u(t))dt \]

Substituting this expression in (2.76), it follows that
\[ 0 \leq \int_0^{T_0} \langle \chi(t), u(t) - w \rangle dt - \int_0^{T_0} \langle Aw, u(t) - w \rangle dt, \quad \forall w \in W_0^{1,p}(\Omega). \]

Let us take \( w = u(t) + \lambda v(t), \lambda > 0. \) Thus
\[ 0 \leq - \int_0^{T_0} \langle \chi(t), \lambda v(t) \rangle dt + \int_0^{T_0} \langle A(u(t) + \lambda v(t)), \lambda v(t) \rangle dt, \quad \forall v \in W_0^{1,p}(\Omega) \]

which implies
\[ 0 \leq - \int_0^{T_0} \langle \chi(t), \lambda v(t) \rangle dt + \int_0^{T_0} \langle A(u(t) + \lambda v(t)), \lambda v(t) \rangle dt. \]

Dividing the previous inequality by \( \lambda \) and letting \( \lambda \to 0^+ \), by the hemicontinuity of \( A \), we have
\[ 0 \leq - \int_0^{T_0} \langle \chi(t), v(t) \rangle dt + \int_0^{T_0} \langle A(u(t)), v(t) \rangle dt, \quad \forall v \in W_0^{1,p}(\Omega). \]

Hence
\[ 0 \leq \int_0^{T_0} \langle Au(t) - \chi(t), v(t) \rangle dt, \quad \forall v \in W_0^{1,p}(\Omega). \]

Now, for \( \lambda < 0 \) it follows that
\[ \int_0^{T_0} \langle Au(t) - \chi(t), v(t) \rangle dt \leq 0, \quad \forall v \in W_0^{1,p}(\Omega). \]

Therefore,
\[ 0 \leq \int_0^{T_0} \langle Au(t) - \chi(t), v(t) \rangle dt \leq 0, \quad \forall v \in W_0^{1,p}(\Omega). \]

Thus \( Au(t) = \chi(t) \). Similarly, \( Av(t) = \eta(t) \). This completes the proof of the theorem.

**References**


Osmundo A. Lima  
*Universidade Estadual da Paraíba*, DME, Campina Grande - PB, Brazil  
E-mail address: osmundo@hs24.com.br

Aldo T. Lourêdo  
*Universidade Estadual da Paraíba*, DME, Campina Grande - PB, Brazil  
E-mail address: aldotl@bol.com.br

Alexandro O. Marinho  
*Universidade Federal da Paraíba*, DM, João Pessoa - PB, Brazil  
E-mail address: nagasak@ig.com.br