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NULL CONTROLLABILITY OF SEMILINEAR DEGENERATE PARABOLIC EQUATIONS IN BOUNDED DOMAINS

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Abstract. In this paper we study controllability properties for semilinear degenerate parabolic equations with nonlinearities involving the first derivative in a bounded domain of $\mathbb{R}$. Due to degeneracy, classical null controllability results do not hold in general. Thus we investigate results of 'regional null controllability', showing that we can drive the solution to rest at time T on a subset of the space domain, contained in the set where the equation is nondegenerate.

1. Introduction

In this paper we study null controllability properties for the semilinear degenerate heat equation

$$\begin{cases} u_t - (a(x)u_x)_x + f(t,x,u,u_x) = h(t,x)\chi_{(\alpha,\beta)}(x), \\ u(t,1) = 0, \\ \begin{cases} u(t,0) = 0, & \text{for } (WDP), \quad \text{or} \\ (au_x)(t,0) = 0, & \text{for } (SDP), \end{cases} \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $(t,x) \in (0,T) \times (0,1)$, $h \in L^2((0,T)\times(0,1))$, $u_0 \in L^2(0,1)$, $(\alpha,\beta) \subset\subset [0,1]$ and a is degenerate. We shall admit two types of degeneracy for a, namely weak and strong degeneracy, each type being associated with its own boundary condition at $x = 0$. The Dirichlet boundary condition $u(t,0) = 0$ as in (1.1) will be imposed for *weakly degenerate* problems (WDP), that is, when

$$\begin{aligned} &\text{(i)} \quad a \in C([0,1]) \cap C^1((0,1]),\ a > 0 \text{ in } (0,1]\,,\ a(0) = 0, \\ &\text{(ii)} \quad \exists\, K \in [0,1) \text{ such that } xa_x(x) \le Ka(x)\ \forall\, x \in [0,1]. \end{aligned} \tag{1.2}$$

Notice that, in this case, $\frac{1}{a} \in L^1(0,1)$, as a consequence of (1.2)(ii) (see Remark 2.2 (3)). On the other hand, when the problem is *strongly degenerate* (SDP), that

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is

$$\begin{aligned} &\text{(i)} \quad a \in C^1([0,1]),\, a > 0 \text{ in } (0,1]\,,\, a(0) = 0\,, \\ &\text{(ii)} \quad \exists\, K \in [1,2) \text{ such that } xa_x(x) \le Ka(x)\ \forall\, x \in [0,1], \end{aligned} \tag{1.3}$$

the natural boundary condition to impose at $x = 0$ is of Neumann type:

$$(au_x)(t,0) = 0, \quad t \in (0,T)$$

(see [5] for the well-posedness of such problem in $C([0,1])$; see also the Appendix of [8]). We observe that, in this case, $\frac{1}{a} \notin L^1(0,1)$ because of (1.3)(ii) (see Remark 2.2 (3)), we now have $\frac{1}{\sqrt{a}} \in L^1(0,1)$.

In the nondegenerate case, i.e., when $a > 0$ on $[0,1]$, (global) null controllability is well-understood: for all $T > 0$ there exists a control $h \in L^2((0,T) \times (0,1))$ such that u, solution of (1.1), satisfies $u(T,x) = 0$ for all $x \in [0,1]$. The reader is referred to [10] for a seminal paper in this research direction, and to [16] and [25] for the approach based on Carleman estimates. Several results have also been obtained for semilinear nondegenerate equations, see, in particular, [10, 14, 15, 16, 18].

However, many problems that are relevant for applications are described by degenerate equations, with degeneracy occurring at the boundary of the space domain. For instance, degenerate parabolic equations can be obtained as suitable transformations of the Prandtl equations, see [19]. In a different context, degenerate operators have been extensively studied since Feller's investigations in [12, 13], whose main motivation was the probabilistic interest of (1.1) for transition probabilities. Indeed, in the linear case, e.g., $f(t,x,u,u_x) = b(t,x)u_x + c(t,x)u$, (1.1) is the backward equation coming from a one-dimensional diffusion process, where a and c model diffusion and absorption, respectively. The evolution equation in (1.1) has been studied under different boundary conditions that also have a genuine probabilistic meaning, see, for example, [11, 17, 21, 23, 24, 26]. In particular, [11, 21, 23, 24] develop a functional analytic approach to the construction of Feller semigroups generated by a degenerate elliptic operator with Wentzell boundary conditions. In [17], J.A. Goldstein and C.Y. Lin consider degenerate operators with boundary conditions of Dirichlet, Neumann, periodic, or nonlinear Robin type. Another example of degenerate elliptic operators arises in gene frequency models for population genetics, see, for instance, the Wright-Fischer model studied in [22].

For this kind of equations the classical null controllability property does not hold. In fact, simple examples (see, e.g., [8]) show that null controllability fails due to the degeneracy of a. Thus, it is important to introduce another notion of controllability, which is the *regional null controllability* (r.n.c.) (see [6, 8]). For the convenience of the reader, we recall here the definition of r.n.c.

Definition 1.1 (Regional null controllability)**.** Equation (1.1) is regional null controllable in time T if for all $u_0 \in L^2(0,1)$, and $\delta \in (0, \beta - \alpha)$, there exists $h \in L^2((0,T) \times (0,1))$ such that u, solution of (1.1), satisfies

$$u(T,x) = 0 \quad \text{for every} \quad x \in (\alpha + \delta, 1). \tag{1.4}$$

We note that global null controllability is a strong property in the sense that it is automatically preserved with time. More precisely, if $u(T) \equiv 0$ in $(0,1)$ and if we stop controlling the system at time T, then for all $t \ge T$, $u(t) \equiv 0$ in $(0,1)$. On the contrary, regional null controllability is a weaker property: due to the uncontrolled part on $(0, \alpha + \delta)$, (1.4) is no more preserved with time if we stop controlling at time T. Thus, it is important to improve the previous result, as shown in [6] and in [8], proving that the solution can be forced to vanish identically on $(\alpha+\delta,1)$ during a given time interval (T,T'), i.e. that the solution is persistent regional null controllable (p.r.n.c.).

Definition 1.2 (Persistent regional null controllability). Equation (1.1) is persistent regional null controllable in time $T'>T>0$ if for all $u_0\in L^2(0,1)$ and $\delta\in(0,\beta-\alpha)$, there exists $h\in L^2((0,T')\times(0,1))$ such that u, solution of (1.1), satisfies

$$u(t,x)=0 \quad \text{for every } (t,x)\in(T,T')\times(\alpha+\delta,1). \tag{1.5}$$

In [6, 7, 8], the regional and the persistent regional null controllability of (1.1) is analyzed in the special cases

$$f(t,x,u,u_x)=c(t,x)u(t,x), \tag{1.6}$$

$$f(t,x,u,u_x)=c(t,x)u(t,x)+b(t,x)u_x(t,x), \tag{1.7}$$

$$f(t,x,u,u_x)=b(t,x)u_x+g(t,x,u), \tag{1.8}$$

respectively. In these papers c and $b\in L^\infty((0,T)\times(0,1))$, $|b(t,x)|\le L\sqrt{a(x)}$, for some positive constant L, the function g satisfies some suitable assumptions to insure the well-posedness of the problem and the degenerate function a is such that

$$a:[0,1]\to[0,+\infty) \text{ is } C^1[0,1],\ a(0)=0, \text{ and } a>0 \text{ on } (0,1].$$

However, the previous results have been improved, recently, in [1] and in [20] (in the semilinear and in the linear case), where a global null controllability is proved in the weakly and in the strongly degenerate case. In particular in [20] P. Martinez and J. Vancostenoble consider the linear equation $u_t-(au_x)_x=h$, while in [1] the authors consider the semilinear case $u_t-(au_x)_x+f(t,x,u)=h$, where the function f satisfies conditions like those of Hypothesis 3.1. In both papers the main technique part is the proof of Carleman estimates for the adjoint problem of $u_t-(au_x)_x=h$.

On the other hand, in the present paper we consider, first of all, the linear equation

$$u_t-(a(x)u_x)_x+b(t,x)u_x+c(t,x)u=h(t,x)\chi_{(\alpha,\beta)}(x), \tag{1.9}$$

where a satisfies (1.2) or (1.3). For it we will prove regional and persistent regional null controllability results. Finally, with such linear null controllability results our disposal, we study the semilinear problem (1.1). Using the fixed point method developed in [14] for nondegenerate problems we obtain null controllability results for (1.1) when f satisfies generalized Lipschitz conditions. We note that, as in the nondegenerate case, our method relies on a compactness result for which, once again, the fact that $xa_x\le Ka$ $(K<2)$ is an essential assumption (see Theorem 4.1).

It is important to underline the fact that until now we are not able to prove Carleman estimates for the adjoint problem of (1.9) and, as a consequence, global null controllability for (1.9) and for (1.1).

The paper is organized as follows: in section 2 we will discuss the linear case. In particular, we introduce function spaces and operators that are needed for the well-posedness of the problem, we state the null controllability results and, as an application of them, we give the regional and the persistent regional observability properties. In section 3 we prove the regional and the persistent regional null controllability properties for the semilinear case. These results are based on some compactness theorems, whose proofs are given, for the reader's convenience, in the last section (see also [1]).

2. Linear degenerate parabolic equations

2.1. Well-posedness. In this subsection, we study the well-posedness of the linear degenerate parabolic equation

$$\begin{cases} u_t - (a(x)u_x)_x + b(t,x)u_x + c(t,x)u = h(t,x)\chi_{(\alpha,\beta)}(x), \\ u(t,1) = 0, \\ \begin{cases} u(t,0) = 0, & \text{for } (WDP), \quad \text{or} \\ (au_x)(t,0) = 0, & \text{for } (SDP), \end{cases} \\ u(0,x) = u_0(x), \end{cases} \tag{2.1}$$

where $(t,x) \in (0,T') \times (0,1)$, $u_0 \in L^2(0,1)$ and $h \in L^2((0,T') \times (0,1))$.

Here we make the following assumptions.

Hypothesis 2.1. Let $0 < \alpha < \beta < 1$ and $T' > T > 0$ be fixed. Assume that $b, c \in L^\infty((0,T') \times (0,1))$, there exists $L > 0$ such that $|b(t,x)| \leq L\sqrt{a(x)}$ for $(t,x) \in (0,T') \times (0,1)$ and that $a : [0,1] \to \mathbb{R}_+$ is $C[0,1] \cap C^1(0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and

Case (WDP). there exists $K \in [0,1)$ such that $xa_x \leq Ka$ for all $x \in [0,1]$ (e.g. $a(x) = x^\alpha,\ 0 < \alpha < 1$).

Case (SDP). there exists $K \in [1,2)$ such that $xa_x \leq Ka$ for all $x \in [0,1]$ (e.g. $a(x) = x^\alpha,\ \alpha \geq 1$).

Remark 2.2. Observe that as an immediate consequence of Hypothesis 2.1 one has that

(1) The Markov process described by the operator $Cu := -(au_x)_x + bu_x$ in $[0,1]$ doesn't reach the point $x = 0$, while the point $x = 1$ is an absorbing barrier since $u(t,1) = 0$. This implies that, if we set the problem in $C[0,1]$ instead of $L^2(0,1)$, then we don't need a boundary condition at $x = 0$ (see, e.g., [9]);

(2) in both cases the function
$$x \to x^\theta/a(x) \text{ is nondecreasing on } (0,1]$$
for all $\theta \geq K$;

(3) the assumption $xa_x \leq Ka$ implies that $\frac{1}{\sqrt{a}} \in L^1(0,1)$. In particular if $K < 1$, then $\frac{1}{a} \in L^1(0,1)$.

Proof. Since (1)and (2) are very easy to prove, we will put our attention only on the last point: using (2) we have $\frac{x^K}{a(x)} \leq \frac{1}{a(1)}$. Thus
$$\frac{1}{\sqrt{a(x)}} \leq \frac{1}{\sqrt{a(1)x^K}}.$$
Since $K < 2$, the above right-hand side is integrable. In the same way, one can prove that, if $K < 1$, then $\frac{1}{a} \in L^1(0,1)$. $\square$

Now, let us introduce the following weighted spaces:

Case (WDP).

$$\begin{aligned} H^1_a := \big\{&u \in L^2(0,1) : u \text{ absolutely continuous in } [0,1], \\ &\sqrt{a}u_x \in L^2(0,1) \text{ and } u(1) = u(0) = 0\big\} \end{aligned}$$

and

$$H^2_a := \{u \in H^1_a(0,1) |\, au_x \in H^1(0,1)\}. \tag{2.2}$$

Case (SDP).

$$\begin{aligned} H^1_a := \big\{&u \in L^2(0,1) : u \text{ locally absolutely continuous in } (0,1], \\ &\sqrt{a}u_x \in L^2(0,1) \text{ and } u(1) = 0\big\} \end{aligned}$$

and

$$\begin{aligned} H^2_a &:= \big\{u \in H^1_a(0,1) : au_x \in H^1(0,1)\big\} \\ &= \{u \in L^2(0,1) : u \text{ is locally absolutely continuous on } (0,1], \\ &\quad au \in H^1_0(0,1),\ au_x \in H^1(0,1) \text{ and } (au_x)(0) = 0\big\}, \end{aligned}$$

with the norms

$$\begin{aligned} \|u\|^2_{H^1_a} &:= \|u\|^2_{L^2(0,1)} + \|\sqrt{a}u_x\|^2_{L^2(0,1)}, \\ \|u\|^2_{H^2_a} &:= \|u\|^2_{H^1_a} + \|(au_x)_x\|^2_{L^2(0,1)}. \end{aligned}$$

To prove the well-posedness of (2.1), we define the operator $(A, D(A))$ by

$$D(A) = H^2_a \quad \text{and} \quad Au := (au_x)_x. \tag{2.3}$$

Observe that if $u \in D(A)$ (or even $u \in H^1_a(0,1)$), then u satisfies the boundary conditions $u(0) = u(1) = 0$, in case (WDP), and $u(1) = 0$, $(au_x)(0) = 0$, in case (SDP).

For the operator $(A, D(A))$ the following proposition holds (see [8] for the proof in our case and also [5] for a proof in the case $a(0) = a(1) = 0$):

Proposition 2.3. *The operator $A : D(A) \to L^2(0,1)$ is closed, self-adjoint and negative with dense domain.*

Hence A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on $L^2(0,1)$. Since A is a generator, and setting $B(t)u := -b(t,x)u_x - c(t,x)u$, working in the spaces considered above, we can prove that (2.1) is well-posed in the sense of semigroup theory using some well-known perturbation technique.

Theorem 2.4. *Assume that Hypothesis 2.1 holds. Then, for all $u_0 \in L^2(0,1)$ and $h \in L^2((0,T') \times (0,1))$, there exists a unique solution $u \in C^0([0,T']; L^2(0,1)) \cap L^2(0,T'; H^1_a)$ of (2.1) and*

$$\begin{aligned} &\sup_{t\in[0,T']} \|u(t)\|^2_{L^2(0,1)} + \int_0^{T'} \|\sqrt{a}u_x(t)\|^2_{L^2(0,1)} \\ &\le C_{T'}(\|u_0\|^2_{L^2(0,1)} + \|h\|^2_{L^2((0,T')\times(0,1))}). \end{aligned} \tag{2.4}$$

Moreover, if $u_0 \in H^1_a(0,1)$, then

$$u \in \mathcal{U} := H^1(0,T'; L^2(0,1)) \cap L^2(0,T'; H^2_a) \cap C^0([0,T']; H^1_a),$$

and there exists a positive constant C such that

$$\sup_{t\in[0,T']}\left(\|u(t)\|^2_{H^1_a}\right)+\int_0^{T'}\left(\|u_t\|^2_{L^2(0,1)}+\|(au_x)_x\|^2_{L^2(0,1)}\right)dt \\ \le C_{T'}\left(\|u_0\|^2_{H^1_a}+\|h\|^2_{L^2((0,T')\times(0,1))}\right).$$

2.2. **Controllability results.** Assume that Hypothesis 2.1 is satisfied. Using the fact that there is no degeneracy on $(\alpha,1)$ and using the classical result known for linear nondegenerate parabolic equations in bounded domain (see for example [16, 18]), we give a direct proof of the regional null controllability for the linear degenerate problem (2.1):

Theorem 2.5. *Assume that Hypothesis 2.1 holds. Then the following holds.*
(i) Regional null controllability. *Given $T>0$, $u_0\in L^2(0,1)$, and $\delta\in(0,\beta-\alpha)$, there exists $h\in L^2((0,T)\times(0,1))$ such that the solution u of (2.1) satisfies*

$$u(T,x)=0 \quad \text{for every} \quad x\in(\alpha+\delta,1).$$

Moreover, there exists a constant $C_T>0$ such that

$$\int_0^T\int_0^1 h^2(t,x)dxdt\le C_T\int_0^1 u_0^2(x)dx.$$

(ii) Persistent regional null controllability. *Given $T'>T>0$, $u_0\in L^2(0,1)$, and $\delta\in(0,\beta-\alpha)$, there exists $h\in L^2((0,T')\times(0,1))$ such that the solution u of (2.1) satisfies*

$$u(t,x)=0 \qquad \text{for every } (t,x)\in[T,T']\times(\alpha+\delta,1).$$

Moreover, there exists a constant $C_{T,T'}>0$ such that

$$\int_0^{T'}\int_0^1 h^2(t,x)dxdt\le C_{T,T'}\int_0^1 u_0^2(x)dx.$$

As an application of Theorem 2.5 (i), we will deduce directly the *regional* observability inequality found in [8] (for the proof see [6]). Consider the adjoint problem associated with

$$\begin{cases} u_t-(a(x)u_x)_x+b(t,x)u_x+c(t,x)u=h(t,x)\chi_{(\alpha,\beta)}(x),\\ u(t,1)=0,\\ \begin{cases} u(t,0)=0, & \text{for } (WDP), \quad \text{or}\\ (au_x)(t,0)=0, & \text{for } (SDP),\end{cases}\\ u(0,x)=u_0(x),\end{cases} \tag{2.5}$$

where $(t,x)\in(0,T)\times(0,1)$, i.e.

$$\begin{cases} \varphi_t+(a\varphi_x)_x+(b\varphi)_x-c\varphi=0, \quad (t,x)\in(0,T)\times(0,1),\\ \varphi(t,1)=0,\\ \begin{cases} \varphi(t,0)=0, & \text{for } (WDP), \quad \text{or}\\ (a\varphi_x)(t,0)=0, & \text{for } (SDP),\end{cases} \quad t\in(0,T).\end{cases} \tag{2.6}$$

Then the following corollary holds.

Corollary 2.6. *Let φ a solution in $\mathcal{U}$ of* (2.6)*. Then for all $\delta \in (0, \beta - \alpha)$ there exists a positive constant K_T such that*

$$\int_0^1 \varphi^2(0,x)dx \le K_T\Big(\int_0^T \int_\alpha^\beta \varphi^2(t,x)dxdt + \int_0^{\alpha+\delta} \varphi^2(T,x)dx\Big). \tag{2.7}$$

Moreover, as a consequence of the persistent regional null controllability result one has the second observability inequality given in [8] for the *non homogeneous* adjoint problem. In fact given

$$\begin{cases} \varphi_t + (a\varphi_x)_x - c\varphi + (b\varphi)_x = G(T,x)\chi_{(T,T')}(t), \quad (t,x) \in (0,T') \times (0,1), \\ \varphi(t,1) = 0, \quad t \in (0,T'), \\ \begin{cases} \varphi(t,0) = 0, & \text{for } (WDP), \quad \text{or} \\ (a\varphi_x)(t,0) = 0, & \text{for } (SDP), \end{cases} \quad t \in (0,T'), \end{cases} \tag{2.8}$$

where $G \in L^2((T,T') \times (0,1))$, and using the same technique of the previous corollary, one can prove the next result.

Corollary 2.7. *Let φ a solution in $\mathcal{U}$ of* (2.8)*. Then for all $\delta \in (0, \beta - \alpha)$ there exists a positive constant $K_{T'}$ such that*

$$\begin{aligned} &\int_0^1 \varphi^2(0,x)dx \\ &\le K_{T'}\Big(\int_0^{T'} \int_\alpha^\beta \varphi^2(t,x)dxdt + \int_0^{\alpha+\delta} \varphi^2(T',x)dx + \int_T^{T'} \int_0^{\alpha+\delta} G^2(t,x)dxdt\Big). \end{aligned} \tag{2.9}$$

3. Semilinear degenerate parabolic equations

In this section we extend the result of Theorem 2.5 to the semilinear degenerate parabolic equation (1.1)

$$\begin{cases} u_t - (a(x)u_x)_x + f(t,x,u,u_x) = h(t,x)\chi_{(\alpha,\beta)}(x), \\ u(t,1) = 0, \\ \begin{cases} u(t,0) = 0, & \text{for } (WDP), \quad \text{or} \\ (au_x)(t,0) = 0, & \text{for } (SDP), \end{cases} \quad t \in (0,T'), \\ u(0,x) = u_0(x), \end{cases} \tag{3.1}$$

where $(t,x) \in (0,T') \times (0,1)$ and a satisfies Hypothesis 2.1. Moreover, we assume the following:

Hypothesis 3.1. Let $0 < \alpha < \beta < 1$ and $T' > T > 0$ be fixed. Let $f : [0,T'] \times [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be such that

$$\forall\,(u,p) \in \mathbb{R}^2, \quad (t,x) \mapsto f(t,x,u,p) \text{ is measurable}, \tag{3.2}$$

$$\forall\,(t,x) \in (0,T') \times (0,1), \quad f(t,x,0,0) = 0; \tag{3.3}$$

for all $(t,x,u) \in (0,T') \times (0,1) \times \mathbb{R}$,

$$f(t,x,u,p) \text{ is locally Lipschitz continuous in the fourth variable} \tag{3.4}$$

and there exists $L > 0$ such that $\forall\,(t,x,u,p) \in (0,T') \times (0,1) \times \mathbb{R} \times \mathbb{R}$,

$$|f_p(t,x,u,p)| \le L\sqrt{a(x)}. \tag{3.5}$$

Suppose that there exist a nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and a positive number ρ with

$$\rho > \begin{cases} 0 & K < 1, \\ \frac{1}{4} & K \geq 1, \end{cases} \tag{3.6}$$

such that

$$|f(t,x,\lambda,p) - f(t,x,\mu,p)| \leq \varphi\Big(a^\rho(x)(|\lambda| + |\mu|)\Big)|\lambda - \mu|, \tag{3.7}$$

$$\forall s \in \mathbb{R}_+, \quad \varphi(s) \leq M(1+|s|), \tag{3.8}$$

for some positive constant M.

Moreover, assume that there exists a positive constant C such that

$$\forall\, \lambda,\, \mu \in \mathbb{R} \quad \Big(f(t,x,\lambda+\mu,p) - f(t,x,\mu,p)\Big)\lambda \geq -C\lambda^2. \tag{3.9}$$

The previous assumptions on f guarantee that for (3.1), Theorem 2.4 still holds (see [7]). However, for the well-posedness of (3.1) it is sufficient to require (3.9) with $\mu = 0$, which is equivalent, thanks to (3.3)-(3.7), to the following apparently more general condition

$$\exists\, C \geq 0 \text{ such that } -f(t,x,\lambda,p)\lambda \leq C(1+|\lambda|^2)$$

(see, e.g., [7]).

As a first step, we study (3.1) with $u_0 \in H^1_a(0,1)$ and $h \in L^2((0,T') \times (0,1))$. To prove the controllability results we will use, as in [4], a fixed point method. To this aim, we rewrite, first of all, the function f in the following way $f(t,x,u,u_x) = b(t,x,u)u_x + c(t,x,u)u$, where

$$b(t,x,u) := \int_0^1 f_p(t,x,\lambda u,\lambda u_x)d\lambda,$$
$$c(t,x,u) := \int_0^1 f_u(t,x,\lambda u,\lambda u_x)d\lambda$$

(f_u exists a.e. since by condition (3.7) the function f is locally Lipschitz continuous in the third variable). In fact

$$\begin{aligned} f(t,x,u,u_x) &= \int_0^1 \frac{d}{d\lambda} f(t,x,\lambda u,\lambda u_x)d\lambda \\ &= \int_0^1 f_u(t,x,\lambda u,\lambda u_x)u d\lambda + \int_0^1 f_p(t,x,\lambda u,\lambda u_x)u_x d\lambda. \end{aligned}$$

Proposition 3.2. *For the functions b and c one has the following properties:*

- *$b(t,x,u(t,x))$ and $c(t,x,u(t,x))$ belong to $L^\infty((0,T') \times (0,1))$;*
- *$|b(t,x,u)| \leq L\sqrt{a(x)}$;*
- *if $\lim_{k\to+\infty} v_k = v$ in $X := C(0,T;L^2(0,1)) \cap L^2(0,T;H^1_a(0,1))$, then*

$$\lim_{k\to+\infty} \frac{b(t,x;v_k)}{\sqrt{a(x)}} = \frac{b(t,x;v)}{\sqrt{a(x)}}, \quad a.e.,$$
$$\lim_{k\to+\infty} c(t,x;v_k) = b(t,x;v), \quad a.e..$$

Here L is the same constant of (3.5).

Observe that the proof of the last point is an easy consequence of the Lebesgue Theorem. The null controllability result for (3.1) may be obtained as a consequence of the approximate null controllability property for it (see, e.g., [14]).

Definition 3.3. **(i):** The system (3.1) is regional approximate null controllable if for all $\epsilon > 0$ there exists $h^\epsilon \in L^2((0,T)\times(0,1))$ such that

$$\|u^{h^\epsilon}(T)\|_{L^2(\alpha+\delta,1)} \le \epsilon \tag{3.10}$$

and

$$\int_0^T \int_\alpha^\beta |h^\epsilon(t,x)|^2 dxdt \le C_T \int_0^1 |u_0(x)|^2 dx, \tag{3.11}$$

for some positive constant C_T.
(ii): The system (3.1) is persistent regional approximate null controllable if for all $\epsilon > 0$ there exists $h^\epsilon \in L^2((0,T)\times(0,1))$ such that

$$\|u^{h^\epsilon}(t)\|_{L^2(\alpha+\delta,1)} \le \epsilon, \quad \forall\, t \in (T,T'), \tag{3.12}$$

and

$$\int_0^{T'} \int_\alpha^\beta |h^\epsilon(t,x)|^2 dxdt \le C_{T,T'} \int_0^1 |u_0(x)|^2 dx, \tag{3.13}$$

for some positive constant $C_{T,T'}$. Here u^{h^ϵ} is the solution of (3.1) associated to h^ϵ.

To prove that the system (3.1) satisfies (3.10)-(3.13) we need a priori estimates on the solution and on the control of a suitable linear system. Fix $\epsilon > 0$, $v \in X := C(0,T';L^2(0,1)) \cap L^2(0,T';H^1_a)$ and, for any $(t,x) \in (0,T')\times(0,1)$, set $b^v(t,x) := b(t,x,v(t,x))$ and $c^v(t,x) := c(t,x,v(t,x))$. Now, let us consider the following problem:

$$\begin{cases} u^\epsilon_t - (a(x)u^\epsilon_x)_x + b^v(t,x)u^\epsilon_x + c^v(t,x)u^\epsilon = h^{v,\epsilon}(t,x)\chi_{(\alpha,\beta)}(x),\\ u^\epsilon(t,1) = 0,\\ \begin{cases} u^\epsilon(t,0) = 0, & \text{for } (WDP), \quad \text{or}\\ (au^\epsilon_x)(t,0) = 0, & \text{for } (SDP), \end{cases}\\ u^\epsilon(0,x) = u_0(x). \end{cases} \tag{3.14}$$

Then the next proposition holds.

Proposition 3.4. *Let $u^{\epsilon,v}$ be the solution of (3.14) associated to the control $h^{v,\epsilon}$ given by Theorem 2.5. Then, for all $\sigma(\epsilon) > 0$, there exists a positive constant K_T such that*

$$\frac{1}{\sigma(\epsilon)} \int_{\alpha+\delta}^1 |u^{\sigma(\epsilon),v}(T,x)|^2 dx + \frac12 \int_0^T \int_\alpha^\beta |h^{\sigma(\epsilon),v}|^2 dxdt \le \frac{K_T}{2} \int_0^1 u_0^2(x) dx. \tag{3.15}$$

Proof. By Theorem 2.5 one has that there exists a control $h^{v,\epsilon} \in L^2((0,T)\times(0,1))$ such that the solution $u^{\epsilon,v} := u^{\epsilon,v,h^{v,\epsilon}}$ of (3.14) satisfies

$$u^{\epsilon,v}(T,x) = 0, \quad \forall\, x \in (\alpha+\delta,1),$$

and there exists a constant $C_T > 0$ such that

$$\int_0^T \int_0^1 |h^{\epsilon,v}(t,x)|^2 dxdt \le C_T \int_0^1 u_0^2(x) dx. \tag{3.16}$$

Moreover, there exists $h^{\epsilon,v} \in L^2((0,T') \times (0,1))$ such that

$$u^{\epsilon,v}(t,x) = 0, \quad \forall\,(t,x) \in (T,T') \times (\alpha+\delta,1)$$

and

$$\int_0^{T'}\int_\alpha^\beta |h^{\epsilon,v}(t,x)|^2 dxdt \le C_{T,T'}\int_0^1 |u_0(x)|^2 dx,$$

for some positive constant $C_{T,T'}$. Observe that, since $u_0 \in H^1_a$, by Theorem 2.4, the solution $u^{\epsilon,v}$ of (3.14) belongs to $Y := H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2_a)$.

For all $\sigma(\epsilon) > 0$, consider the penalized problem

$$\min\{J_{\sigma(\epsilon)}(h^v) : h^v \in L^2((0,T)\times(0,1))\}, \tag{3.17}$$

where

$$J_{\sigma(\epsilon)}(h^v) := \frac{1}{2}\int_0^T\int_\alpha^\beta (h^v)^2 dxdt + \frac{1}{2\sigma(\epsilon)}\int_{\alpha+\delta}^1 |u^{h^v}(T,x)|^2 dx,$$

with u^{h^v} the solution of (3.14) associated to h^v. As in [8], one can prove that problem (3.17) has a unique solution $h^{\sigma(\epsilon),v}$ and we can verify that it is characterized by

$$h^{\sigma(\epsilon),v} = -\varphi^{\sigma(\epsilon),v}\chi_{(\alpha,\beta)}. \tag{3.18}$$

Here $\varphi^{\sigma(\epsilon),v}$ is the solution of the associated adjoint problem

$$\begin{cases} \varphi_t^{\sigma(\epsilon),v} + (a\varphi_x^{\sigma(\epsilon),v})_x - c\varphi^{\sigma(\epsilon),v} + (b\varphi^{\sigma(\epsilon),v})_x = 0, \quad (t,x) \in (0,T)\times(0,1),\\ \varphi^{\sigma(\epsilon),v}(t,1) = 0, \quad t \in (0,T),\\ \begin{cases} \varphi^{\sigma(\epsilon),v}(t,0) = 0, & \text{for } (WDP), \quad \text{or}\\ (a\varphi_x^{\sigma(\epsilon),v})(t,0) = 0, & \text{for } (SDP), \end{cases}\\ \varphi^{\sigma(\epsilon),v}(T,x) = \frac{1}{\sigma(\epsilon)}u^{\sigma(\epsilon),v}(T,x)\chi_{(\alpha+\delta,1)}, \quad x \in (0,1). \end{cases}$$

Therefore, by Corollary 2.6, there exists a positive constant K_T such that

$$\int_0^1 (\varphi^{\sigma(\epsilon),v})^2(0,x)dx \le K_T\Big(\int_0^T\int_\alpha^\beta (\varphi^{\sigma(\epsilon),v})^2(t,x)dxdt + \int_0^{\alpha+\delta}(\varphi^{\sigma(\epsilon),v})^2(T,x)dx\Big). \tag{3.19}$$

Multiplying

$$\varphi_t^{\sigma(\epsilon),v} + (a\varphi_x^{\sigma(\epsilon),v})_x - c\varphi^{\sigma(\epsilon),v} + (b\varphi^{\sigma(\epsilon),v})_x = 0$$

by $u^{\sigma(\epsilon)}$ and

$$u_t^{\sigma(\epsilon),v} - (au_x^{\sigma(\epsilon),v})_x + cu^{\sigma(\epsilon),v} + bu_x^{\sigma(\epsilon),v} = h^{\sigma(\epsilon),v}$$

by $\varphi^{\sigma(\epsilon),v}$, summing up and integrating over $(0,1)$ and over $(0,T)$ one has

$$\int_0^1 \frac{d}{dt}(u^{\sigma(\epsilon),v}\varphi^{\sigma(\epsilon),v})dx = \int_0^1 h^{\sigma(\epsilon),v}\chi_{(\alpha,\beta)}\varphi^{\sigma(\epsilon),v}.$$

Here we have used the fact that $|b(t,0)| \le L\sqrt{a(0)} = 0$. Integrating over $(0,T)$, using (3.18) and the fact that $\varphi^{\sigma(\epsilon),v}(T,x) = \frac{1}{\sigma(\epsilon)}u^{\sigma(\epsilon),v}(T,x)\chi_{(\alpha+\delta,1)}$ we have:

$$\int_0^1 u^{\sigma(\epsilon),v}(T,x)\varphi^{\sigma(\epsilon),v}(T,x)dx - \int_0^1 u_0(x)\varphi^{\sigma(\epsilon),v}(0,x) = \int_0^T\int_0^1 h^{\sigma(\epsilon),v}\chi_{(\alpha,\beta)}\varphi^{\sigma(\epsilon),v}$$

if and only if

$$\begin{aligned}
&\frac{1}{\sigma(\epsilon)}\int_{\alpha+\delta}^{1}|u^{\sigma(\epsilon),v}(T,x)|^2dx+\int_0^T\int_\alpha^\beta|\varphi^{\sigma(\epsilon),v}(t,x)|^2dxdt\\
&=\int_0^1 u_0(x)\varphi^{\sigma(\epsilon),v}(0,x)dx\\
&\le\frac{1}{2K_T}\int_0^1\varphi^2(0,x)dx+\frac{K_T}{2}\int_0^1 u_0^2(x)dx.
\end{aligned}$$

From (3.19), it results

$$\begin{aligned}
&\frac{1}{\sigma(\epsilon)}\int_{\alpha+\delta}^{1}|u^{\sigma(\epsilon),v}(T,x)|^2dx+\int_0^T\int_\alpha^\beta|\varphi^{\sigma(\epsilon),v}(t,x)|^2dxdt\\
&\le\frac12\Big(\int_0^T\int_\alpha^\beta|\varphi^{\sigma(\epsilon),v}(t,x)|^2dxdt+\int_0^{\alpha+\delta}(\varphi^{\sigma(\epsilon),v})^2(T,x)dx\Big)+\frac{K_T}{2}\int_0^1 u_0^2(x)dx.
\end{aligned}$$

But $\int_0^{\alpha+\delta}(\varphi^{\sigma(\epsilon),v})^2(T,x)dx=0$ since $\varphi^{\sigma(\epsilon),v}(T,x)=\frac{1}{\sigma(\epsilon)}u^{\sigma(\epsilon),v}(T,x)\chi_{(\alpha+\delta,1)}$. Thus

$$\frac{1}{\sigma(\epsilon)}\int_{\alpha+\delta}^{1}|u^{\sigma(\epsilon),v}(T,x)|^2dx+\frac12\int_0^T\int_\alpha^\beta|\varphi^{\sigma(\epsilon),v}(t,x)|^2dxdt\le\frac{K_T}{2}\int_0^1 u_0^2(x)dx.$$

The thesis follows from (3.18). □

Observe that (3.15) gives a priori estimates that allows us to pass to the limit in (3.14) as $\epsilon\to0$.

Theorem 3.5. *Let $T'>T>0$ and $u_0\in H^1_a$. Assume that Hypotheses 2.1 and 3.1 hold. Then system* (1.1) *satisfies* (3.10)-(3.13).

Proof. Let $\epsilon>0$ and consider the function

$$\mathcal{T}_\epsilon : v\in X\mapsto u^{\epsilon,v}\in X. \tag{3.20}$$

Here $X:=C(0,T;L^2(0,1))\cap L^2(0,T;H^1_a(0,1))$ and $u^{\epsilon,v}$ is the unique solution of (3.14), where $c^v(t,x)=\int_0^1 f_v(t,x,\lambda v,\lambda v_x)d\lambda$. By Theorem 2.5, problem (3.14) is regional and persistent regional null controllable. Hence, if we prove that $\mathcal{T}_\epsilon$ has a fixed point $u^{\epsilon,v}$, i.e. $\mathcal{T}_\epsilon(u^{\epsilon,v})=u^{\epsilon,v}$, then $u^{\epsilon,v}$ is solution of (3.1) and satisfies (3.10)- (3.13).

To prove that $\mathcal{T}_\epsilon$ has a fixed point, by the Schauder's Theorem, it is sufficient to prove that

(1) $\mathcal{T}_\epsilon : B_X\to B_X$,
(2) $\mathcal{T}_\epsilon$ is a compact function,
(3) $\mathcal{T}_\epsilon$ is a continuous function.

Here $B_X:=\{v\in X:\|v\|_X\le R\}$, $\|v\|_X:=\sup_{t\in[0,T']}\big(\|u(t)\|^2_{L^2}\big)+\int_0^T\|\sqrt{a}u_x\|^2_{L^2}dt$ and $R:=C_T(\|u_0\|^2_{L^2}+\|h\|^2_{L^2((0,T)\times(0,1))})$ (C_T is the same constant of Theorem 2.4).

The first point is a consequence of Theorem 2.4. Indeed, one has that $\mathcal{T}_\epsilon : X\to B_X$ and in particular $B_X\to B_X$. Moreover, it is easy to see that point (2) is a simple consequence of the compactness Theorem 4.4 below. This theorem is also useful for the proof of point (3). Indeed, let $v_k\in X$ be such that $v_k\to v$ in X, as $k\to+\infty$. We want to prove that $u^{\epsilon,v_k}\to u^{\epsilon,v}$ in X, as $k\to+\infty$. Here u^{ϵ,v_k} and $u^{\epsilon,v}$ are the solutions of (3.14) associated to v_k, h^{ϵ,v_k} and v, $h^{\epsilon,v}$ respectively. Moreover,

$h^{\epsilon,v_k} = \min J_{\sigma(\epsilon),v_k} = -\varphi^{\sigma(\epsilon),v_k}\chi_{(\alpha,\beta)}$ and $h^{v,\epsilon} = \min J_{\sigma(\epsilon),v} = -\varphi^{\sigma(\epsilon),v}\chi_{(\alpha,\beta)}$. For simplicity, set $u_k := u^{\epsilon,v_k}$ and $u := u^{\epsilon,v}$. By (3.15), it follows that h^{ϵ,v_k} is bounded, thus, up to subsequence, $h_k := h^{\epsilon,v_k}$ converges weakly to $\bar h$ in $L^2((0,T)\times(0,1))$. Moreover, proceeding as in the proof of Theorem 4.2 (see below), one has that, up to subsequence, u_k converges weakly to $\bar u$ in Y and, thanks to Theorem 4.4 (see below), strongly in X. Moreover, it holds that $\bar u$ is solution of

$$\begin{cases} \bar u_t - (a(x)\bar u_x)_x + b^v(t,x)\bar u_x + c^v(t,x)\bar u = \bar h(t,x)\chi_{(\alpha,\beta)}(x), \\ \bar u(t,1) = 0, \\ \begin{cases} \bar u(t,0) = 0, & \text{for } (WDP), \quad \text{or} \\ (a\bar u_x)(t,0) = 0, & \text{for } (SDP), \end{cases} \\ \bar u(0,x) = u_0(x). \end{cases}$$

Indeed, one has

$$u_k(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}[B^k(s)u_k(s) + \chi_{(\alpha,\beta)}h_k(s)]ds, \tag{3.21}$$

where $B^k(s)u := b(s,\cdot;v^k)u_x + c(s,\cdot;v^k)u$. Then

$$\begin{aligned} &\Big\|\int_0^t e^{(t-s)A}[B^k(s)u_k(s) - \bar B(s)\bar u(s)]ds\Big\| \\ &\le \Big\|\int_0^t e^{(t-s)A}B^k(s)(u_k(s)-\bar u(s))ds\Big\| + \Big\|\int_0^t e^{(t-s)A}(B^k(s)-\bar B(s))\bar u(s)ds\Big\|, \end{aligned}$$

where $\bar B(s)u := b(s,\cdot;v)\bar u_x + c(s,\cdot;v)\bar u$. Moreover,

$$\begin{aligned} &\Big\|\int_0^t e^{(t-s)A}B^k(s)(u_k(s)-\bar u(s))ds\Big\| \\ &\le \int_0^t \|B^k(s)(u_k(s)-\bar u(s))\|_{L^2}ds \\ &\le \int_0^t \Big(\int_0^1 |c(s,x;v^k)(u_k(s,x)-\bar u(s,x))|^2dx\Big)^{1/2}ds \\ &\quad + \int_0^t \Big(\int_0^1 \Big|\frac{b^2(s,x;v^k)}{a(x)}\Big||((u_k)_x(s,x)-\bar u_x(s,x))\sqrt{a(x)}|^2dx\Big)^{1/2}ds. \end{aligned}$$

Using the assumptions on c and b, one has

$$\begin{aligned} &\Big\|\int_0^t e^{(t-s)A}B^k(s)(u_k(s)-\bar u(s))ds\Big\| \\ &\le C\int_0^t \Big(\int_0^1 |(u_k(s,x)-\bar u(s,x))|^2dx\Big)^{1/2}ds \\ &\quad + L\int_0^t \Big(\int_0^1 |((u_k)_x(s,x)-\bar u_x(s,x))\sqrt{a(x)}|^2dx\Big)^{1/2}ds \to 0, \end{aligned}$$

as $k \to +\infty$. Therefore,

$$
\begin{aligned}
&\Big\| \int_0^t e^{(t-s)A}(B^k(s) - \bar{B}(s))\bar{u}(s)ds \Big\| \\
&\le \int_0^t \|(B^k(s) - \bar{B}(s))\bar{u}(s)\|_{L^2} ds \\
&\le \int_0^t \Big(\int_0^1 |(c(s,x;v^k) - c(s,x;v))\bar{u}(s,x)|^2 dx \Big)^{1/2} ds \\
&\quad + \int_0^t \Big(\int_0^1 \Big| \frac{b(s,x;v^k) - b(s,x;v)}{\sqrt{a(x)}} \Big|^2 a(x)\bar{u}_x^2(s,x)dx \Big)^{1/2} ds \to 0,
\end{aligned}
$$

as $k \to +\infty$.

By (3.21) and using the weakly convergence of h_k, one has

$$
\bar{u}(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}[\bar{B}(s)\bar{u}(s) + \chi_{(\alpha,\beta)}\bar{h}(s)]ds.
$$

The thesis will follow if we prove that $\bar{h} = h^v$. Since h_k is the minimum of $J_{\sigma(\epsilon),v_k}$, then, for all $h \in L^2((0,T)\times(0,1))$,

$$
\begin{aligned}
&\frac{1}{2}\int_0^T \int_\alpha^\beta |h_k|^2 dxdt + \frac{1}{2\sigma(\epsilon)} \int_{\alpha+\delta}^1 |u_k(T,x)|^2 dx \\
&\le \frac{1}{2}\int_0^T \int_\alpha^\beta |h|^2 dxdt + \frac{1}{2\sigma(\epsilon)} \int_{\alpha+\delta}^1 |u^{\epsilon,v_k,h}(T,x)|^2 dx.
\end{aligned} \tag{3.22}
$$

Passing to the limit in (3.22), one has, for all $h \in L^2((0,T)\times(0,1))$,

$$
\begin{aligned}
&\frac{1}{2}\int_0^T \int_\alpha^\beta |\bar{h}|^2 dxdt + \frac{1}{2\sigma(\epsilon)} \int_{\alpha+\delta}^1 |\bar{u}(T,x)|^2 dx \\
&\le \frac{1}{2}\int_0^T \int_\alpha^\beta |h|^2 dxdt + \frac{1}{2\sigma(\epsilon)} \int_{\alpha+\delta}^1 |u^{v,h}(T,x)|^2 dx.
\end{aligned}
$$

Thus $\bar{h} = \min J_{\sigma(\epsilon),v}(h)$, i.e. $\bar{h} = h^{\bar{v}}$. ☐

The previous theorem yields regional and persistent regional null controllability properties for (1.1) for initial data $u_0 \in H_a^1$.

Theorem 3.6. *Consider $T' > T > 0$ and $u_0 \in H_a^1(0,1)$. Assume that Hypotheses 2.1 and 3.1 hold.*

(i) Regional null controllability. *Given $\delta \in (0, \beta - \alpha)$, there exists $h \in L^2((0,T)\times(0,1))$ such that the solution u of (1.1) satisfies*

$$
u(T,x) = 0 \quad \text{for every} \quad x \in (\alpha+\delta, 1). \tag{3.23}
$$

Moreover, there exists a positive constant C_T such that

$$
\int_0^T \int_0^1 h^2(t,x)dxdt \le C_T \int_0^1 u_0^2(x)dx. \tag{3.24}
$$

(ii) Persistent regional null controllability. *Given $\delta \in (0, \beta - \alpha)$, there exists $h \in L^2((0,T')\times(0,1))$ such that the solution u of (1.1) satisfies*

$$
u(t,x) = 0 \text{ for every } (t,x) \in [T,T'] \times (\alpha+\delta, 1). \tag{3.25}
$$

Moreover, there exists a positive constant $C_{T,T'}$ such that

$$\int_0^{T'}\int_0^1 h^2(t,x)dxdt \le C_{T,T'}\int_0^1 u_0^2(x)dx. \tag{3.26}$$

Proof. By Theorem 3.5, problem (3.1) is approximate null controllable. Thus, for all $\epsilon > 0$, there exists $h^\epsilon \in L^2((0,T)\times(0,1))$ such that (3.10)-(3.13) hold. By (3.11) or (3.13) one has that h^ϵ converges weakly to h_0 in $L^2((0,T)\times(0,1))$ as $\epsilon \to 0$ and, by the semicontinuity of the norm, it results

$$\int_0^T\int_\omega |h_0(t,x)|^2dxdt \le \liminf_{\epsilon\to 0}\int_0^T\int_\omega |h^\epsilon(t,x)|^2dxdt \le C_T\int_0^1 |u_0(x)|^2dx.$$

Moreover, proceeding as in Theorem 3.5, one can prove that, for all $t \in [0,T]$,

$$u^{h^\epsilon}(t,\cdot) \to u^{h_0}(t,\cdot) \tag{3.27}$$

strongly in $X := L^2(0,T;H^1_a)\cap C(0,T;L^2(0,1))$, as $\epsilon \to 0$. Using (3.7) and (3.8), we can prove that u^{h^0} solves (1.1) with $h \equiv h^0$ and, by (3.10), (3.12) and (3.27),

$$u^{h_0}(T,x) = 0 \quad \forall x \in (\alpha+\delta,1)$$

and

$$u^{h_0}(t,x) = 0 \quad \forall (t,x) \in (T,T')\times(\alpha+\delta,1),$$

□

To prove that the null controllability result of Theorem 3.6 holds also if the initial data u_0 is in $L^2(0,1)$, we observe that (3.5) implies

$$\forall\ (t,x,u) \in (0,T')\times(0,1)\times\mathbb{R}, \quad |f(t,x,u,p)-f(t,x,u,q)| \le L\sqrt{a(x)}|p-q|. \tag{3.28}$$

Moreover, by (3.9) and (3.28), it follows that $\forall\,(t,x) \in (0,T')\times(0,1)$

$$|(f(t,x,u,p)-f(t,x,v,q))(u-v)| \le M[|u-v|^2+\sqrt{a(x)}|p-q||u-v|], \tag{3.29}$$

for some positive constant M.

Theorem 3.7. *The problem*

$$\begin{cases} u_t-(au_x)_x+f(t,x,u,u_x)=0, \quad (t,x)\in(0,T)\times(0,1),\\ u(t,0)=0, \quad t\in(0,T),\\ \begin{cases} u(t,0)=0, & \textit{for } (WDP), \quad \textit{or}\\ (au_x)(t,0)=0, & \textit{for } (SDP), \end{cases} \quad t\in(0,T),\\ u(0,x)=u_0(x)\in L^2(0,1), \quad x\in(0,1), \end{cases} \tag{3.30}$$

has a solution $u \in X$.

Proof. Let $(u_0^j)_j \in H^1_a$ be such that $\lim_{j\to+\infty}\|u_0^j-u_0\|_{L^2} = 0$. Denote with u^j and u the solutions of (3.30) with respect to u_0^j and u_0. Then $(u^j)_j$ is a Cauchy sequence in X. In fact u^j-u^i solves the system

$$\begin{cases} (u^j-u^i)_t-(a(u^j-u^i)_x)_x+f(t,x,u^j,u^j_x)-f(t,x,u^i,u^i_x)=0,\\ (u^j-u^i)(t,1)=0,\\ \begin{cases} (u^j-u^i)(t,0)=0, & \text{for } (WDP), \quad \text{or}\\ (a(u^j-u^i)_x)(t,0)=0, & \text{for } (SDP), \end{cases}\\ (u^j-u^i)(0,x)=(u_0^j-u_0^i)(x), \end{cases}$$

where $(t,x) \in (0,T) \times (0,1)$. Multiplying
$$(u^j - u^i)_t - (a(u^j - u^i)_x)_x + f(t,x,u^j,u^j_x) - f(t,x,u^i,u^i_x) = 0$$
by $u^j - u^i$ and integrating over $(0,1)$, one has, using (3.29),
$$\frac{1}{2}\frac{d}{dt}\int_0^1 |u^j - u^i|^2 dx + \int_0^1 a|u^j_x - u^i_x|^2 dx \le \int_0^1 M[|u^j - u^i|^2 + \sqrt{a}|u^j_x - u^i_x||u^j - u^i|]dx.$$
Integrating over $(0,t)$:
$$\begin{aligned}
&\frac{1}{2}\|(u^j - u^i)(t)\|^2_{L^2} + \int_0^t\int_0^1 a|(u^j - u^i)_x|^2 dxds \\
&\le \frac{1}{2}\|u^j_0 - u^i_0\|^2_{L^2} + M\int_0^t\int_0^1 |u^j - u^i|^2 dxds + \frac{\epsilon M}{2}\int_0^t\int_0^1 a|u^j_x - u^i_x|^2 dxds \\
&\quad + \frac{M}{2\epsilon}\int_0^t\int_0^1 |u^j - u^i|^2 dxds.
\end{aligned}$$
Thus
$$\begin{aligned}
&\frac{1}{2}\|(u^j - u^i)(t)\|^2_{L^2} + \big(1 - \frac{\epsilon M}{2}\big)\int_0^t\int_0^1 a|u^j_x - u^i_x|^2 dxds \\
&\le \frac{1}{2}\|u^j_0 - u^i_0\|^2_{L^2} + M_\epsilon \int_0^t\int_0^1 |u^j - u^i|^2 dxds.
\end{aligned} \tag{3.31}$$
By Gronwall's Lemma
$$\|(u^j - u^i)(t)\|^2_{L^2} \le e^{M_\epsilon t}\|u^j_0 - u^i_0\|^2_{L^2}, \tag{3.32}$$
and
$$\sup_{t\in[0,T]}\|(u^j - u^i)(t)\|^2_{L^2} \le e^{M_\epsilon T}\|u^j_0 - u^i_0\|^2_{L^2}.$$
This implies that $(u^j)_j$ is a Cauchy sequence in $C(0,T;L^2(0,1))$. Moreover, by (3.31), one has
$$\big(1 - \frac{\epsilon M}{2}\big)\int_0^t\int_0^1 a|u^j_x - u^i_x|^2 dxds \le \frac{1}{2}\|u^j_0 - u^i_0\|^2_{L^2} + M_\epsilon\int_0^t\int_0^1 |u^j - u^i|^2 dxds.$$
Using (3.32), it follows
$$\begin{aligned}
\int_0^t \|\sqrt{a}(u^j_x - u^i_x)\|^2_{L^2} ds &\le M_{\epsilon,T}(\|u^j_0 - u^i_0\|^2_{L^2} + \sup_{t\in[0,T]}\|u^j - u^i\|^2_{L^2}) \\
&\le M_{\epsilon,T}\|u^j_0 - u^i_0\|^2_{L^2}.
\end{aligned}$$
Thus $(u^j)_j$ is a Cauchy sequence also in $L^2(0,T;H^1_a)$. Then there exists $\bar{u} \in X$ such that
$$\lim_{j\to+\infty}\|u^j - \bar{u}\|_X = 0.$$
Proceeding as in the proof of Theorem 3.5 and using assumptions (3.7), (3.8), (3.9) and (3.28), one can prove that $\bar{u}$ is a solution of (3.30). □

Theorem 3.8. *Let $T' > T > 0$ and $u_0 \in L^2(0,1)$. Assume that Hypotheses 2.1 and 3.1 hold. Then the following properties hold.*
(i) Regional null controllability. *Given $\delta \in (0, \beta - \alpha)$, there exists $h \in L^2((0,T)\times(0,1))$ such that the solution u of (1.1) satisfies*
$$u(T,x) = 0 \quad \textit{for every } x \in (\alpha + \delta, 1). \tag{3.33}$$

Moreover, there exists a positive constant C_T such that

$$\int_0^T \int_0^1 h^2(t,x)dxdt \le C_T \int_0^1 u_0^2(x)dx. \tag{3.34}$$

(ii) Persistent regional null controllability. *Given $\delta \in (0, \beta - \alpha)$, there exists $h \in L^2((0,T') \times (0,1))$ such that the solution u of (1.1) satisfies*

$$u(t,x) = 0 \text{ for every } (t,x) \in [T,T'] \times (\alpha + \delta, 1). \tag{3.35}$$

Moreover, there exists a positive constant $C_{T,T'}$ such that

$$\int_0^{T'} \int_0^1 h^2(t,x)dxdt \le C_{T,T'} \int_0^1 u_0^2(x)dx. \tag{3.36}$$

Proof. (i). **Step 1:** Consider the problem

$$\begin{cases} v_t - (av_x)_x + f(t,x,v,v_x) = 0, & (t,x) \in \left(0, \frac{T}{2}\right) \times (0,1), \\ v(t,1) = 0, \quad t \in \left(0, \frac{T}{2}\right), \\ \begin{cases} v(t,0) = 0, & \text{for } (WDP), \quad \text{or} \\ (av_x)(t,0) = 0, & \text{for } (SDP), \end{cases} & t \in (0,T), \\ v(0,x) = u_0(x), \quad x \in (0,1). \end{cases}$$

Then, by Theorem 3.7, $v(t,\cdot) \in H_a^1$ a.e.. Thus $\exists\, t_0 \in (0, \frac{T}{2})$, such that $v(t_0,x) =: u_1(x) \in H_a^1$.
Step 2: Consider the problem

$$\begin{cases} w_t - (aw_x)_x + f(t,x,w,w_x) = h_1 \chi_{(\alpha,\beta)}, & (t,x) \in (t_0,T) \times (0,1), \\ w(t,1) = 0, \quad t \in (t_0,T), \\ \begin{cases} w(t,0) = 0, & \text{for } (WDP), \quad \text{or} \\ (aw_x)(t,0) = 0, & \text{for } (SDP), \end{cases} & t \in (0,T), \\ w(t_0,x) = u_1(x), \quad x \in (0,1). \end{cases}$$

By Theorem 3.6, we have that there exists a control $h_1 \in L^2((0,T) \times (0,1))$ such that

$$w(T,x) = 0, \quad \forall\, x \in (\alpha + \delta, 1)$$

and

$$\int_{t_0}^T \int_0^1 h_1^2(t,x)dxdt \le C_T \int_0^1 u_1^2(x)dx,$$

for some positive constant C_T.
Step 3: Finally, we define u and h by

$$u := \begin{cases} v, & [0,t_0], \\ w, & [t_0,T], \end{cases} \qquad h := \begin{cases} 0, & [0,t_0], \\ h_1, & [t_0,T]. \end{cases}$$

Then u is a solution of (1.1) and satisfies (3.33).
(ii). The proof of this part is the same of the previous part. $\square$

4. Appendix: Compactness Theorems

In this section we will give some compactness theorems that we have used in the previous section.

Theorem 4.1. *The space H^1_a is compactly imbedded in $L^2(0,1)$.*

Proof. First of all we have to observe that H^1_a is continuously imbedded in $L^2(0,1)$. Indeed, let $u \in H^1_a$, then

$$|u(x)|^2 \le \Big| \int_x^1 \frac{1}{\sqrt{a(y)}} \sqrt{a(y)} u_x(y) dy \Big|^2 \le \|u\|^2_{1,a} \int_x^1 \frac{1}{a(y)} dy.$$

Integrating over $(0,1)$, we have

$$\int_0^1 |u(x)|^2 dx \le \|u\|^2_{1,a} \int_0^1 \frac{1}{a(y)} dy \int_0^y dx = \|u\|^2_{1,a} \int_0^1 \frac{y^K}{a(y) y^{K-1}} dy.$$

Using the fact that the function $y \mapsto y^K/a(y)$ in nondecreasing, it follows

$$\int_0^1 |u(x)|^2 dx \le \frac{\|u\|^2_{1,a}}{a(1)} \int_0^1 y^{1-K} dy = \frac{\|u\|^2_{1,a}}{a(1)(2-K)}.$$

Now, let $\epsilon > 0$. We want to prove that there exists $\delta > 0$ such that for all $u \in H^1_a$ and for all $|h| < \delta$ it results

$$\int_\delta^{1-\delta} |u(x+h) - u(x)|^2 dx < \epsilon, \tag{4.1}$$

$$\int_{1-\delta}^1 |u(x)|^2 dx + \int_0^\delta |u(x)|^2 dx < \epsilon. \tag{4.2}$$

Hence, let $u \in H^1_a$. Proceeding as before, it results that, taking $\delta < g(\epsilon)$, where $g(\epsilon)$ depends also on $a(1)$, K and $\|u\|^2_{1,a}$ and goes to zero as ϵ goes to zero, one has

$$\begin{aligned}
\int_0^\delta |u(x)|^2 dx &\le \frac{\|u\|^2_{1,a}}{a(1)} \int_0^\delta y^{1-K} dy + \|u\|^2_{1,a} \int_0^\delta dx \int_\delta^1 \frac{dy}{a(y)} \\
&\le \frac{\|u\|^2_{1,a}}{a(1)(2-K)} \Big(\delta^{2-K} + C(\delta + \delta \ln \delta + \delta^{2-K}) \Big) < \frac{\epsilon}{2}.
\end{aligned}$$

Analogously,

$$\int_{1-\delta}^1 |u(x)|^2 dx \le \frac{\|u\|^2_{1,a}}{a(1)} \int_{1-\delta}^1 y^{1-K} dy = \frac{\|u\|^2_{1,a}}{a(1)(2-K)} (1 - (1-\delta)^{2-K}) < \frac{\epsilon}{2}.$$

Now, let h be such that $|h| < \delta$ and, for simplicity, assume $h > 0$ (the case $h < 0$ can be treated in the same way). Then

$$|u(x+h) - u(x)|^2 \le \|u\|^2_{1,a} \int_x^{x+h} \frac{dy}{a(y)}.$$

Integrating over $(\delta, 1-\delta)$, it results

$$\begin{aligned}
\int_\delta^{1-\delta} |u(x+h)-u(x)|^2 dx &\le \|u\|_{1,a}^2 \int_\delta^{1-\delta} dx \int_x^{x+\delta} \frac{dy}{a(y)} \\
&\le \|u\|_{1,a}^2 \int_\delta^1 \frac{dy}{a(y)} \int_{y-\delta}^{y} dx \\
&= \|u\|_{1,a}^2 \delta \int_\delta^1 \frac{y^K}{a(y)y^K} dy \\
&\le \begin{cases} \frac{\|u\|_{1,a}^2}{a(1)} \delta \log \frac{1}{\delta}, & K=1, \\ \frac{\|u\|_{1,a}^2}{a(1)(1-K)} (\delta - \delta^{2-K}) < \epsilon, & \text{otherwise.} \end{cases}
\end{aligned}$$

Moreover, since $\lim_{\delta\to 0} \delta \log \delta = 0$, there exists $\eta(\epsilon) > 0$ such that if $\delta < \eta(\epsilon)$, then $|\delta \log \delta| < \epsilon$. Thus, taking $\delta < \min\{g(\epsilon), \eta(\epsilon)\}$, (4.1) and (4.2) are verified and the thesis follows (see, e.g., [3, Chapter IV]). $\square$

We have to observe that the assumption $xa_x \le Ka,\ K \in [0,2)$ is very important to prove the previous theorem. In fact if we consider $a(x) = x^\alpha$ with $\alpha > 2$, then $\frac{1}{\sqrt{a}} \notin L^1(0,1)$. Hence the estimate $xa_x \le Ka$ is not satisfied and the compact immersion fails (for the proof one can take $u_n(x) = \frac{1}{x^{1/2-1/n}}$).

Using Theorem 4.1 one can prove the next theorem.

Theorem 4.2. *The space H^2_a is compactly imbedded in H^1_a.*

Proof. Take $(u_n)_n \in \overline{B}_{H^2_a}$. Here $B_{H^2_a}$ denotes the unit ball of H^2_a. Since H^2_a is reflexive, then, up to subsequence, there exists $u \in H^2_a$ such that u_n converges weakly to u in H^2_a. In particular, u_n converges weakly to u in H^1_a and in L^2. But, since by the previous theorem H^1_a is compactly imbedded in $L^2(0,1)$, then, up to subsequence, there exists $v \in L^2$ such that u_n converges strongly to v in L^2. Thus u_n converges weakly to v in L^2. By uniqueness $v \equiv u$. Then we can conclude that the sequence u_n converges strongly to u in L^2.

Now it remains to prove that

$$\|\sqrt{a}u_{n,x} - \sqrt{a}u_x\|_{L^2} \to 0, \quad \text{as } n \to +\infty.$$

To this aim we will use the following facts:

(1) $a(1)(u_n(t,1)-u(t,1))_x(u_n(t,1)-u(t,1)) = a(0)(u_n(t,0)-u(t,0))_x(u_n(t,0)-u(t,0)) = 0$, for all $t \in (0,T)$.
(2) $(a(u_n-u)_x)_x \in L^2$.

Indeed:

(1) : it is an immediate consequence of the fact that $(u_n)_n$ and u belong to H^1_a.
(2) : $\int_0^1 [(a(u_n-u)_x)_x]^2 dx < +\infty$, since u_n converges weakly to u in H^2_a.

Thus, using the Hölder inequality and the previous properties, one has

$$\begin{aligned}
\|\sqrt{a}(u_n-u)_x\|_{L^2}^2 &= \int_0^1 a(u_n-u)_x(u_n-u)_x dx \\
&= -\int_0^1 (a(u_n-u)_x)_x(u_n-u) dx \\
&\le \|(a(u_n-u)_x)_x\|_{L^2} \|u_n-u\|_{L^2} \to 0,
\end{aligned}$$

as $n \to +\infty$. $\square$

For the proof of Theorem 4.4 below we will use the Aubin's Theorem, that we give here for the reader's convenience.

Theorem 4.3 ([2, Chapter 5])**.** *Let X_0, X_1 and X_2 be three Banach spaces such that $X_0 \subset X_1 \subset X_2$, X_0, X_2 are reflexives and the injection of X_0 into X_1 is compact. Let r_0, $r_1 \in (1, +\infty)$ and $a, b \in \mathbb{R}$, $a < b$. Then the space*

$$L^{r_0}(a, b; X_0) \cap W^{1,r_1}(a, b; X_2)$$

is compactly imbedded in $L^{r_0}(a, b; X_1)$.

Now we are ready to prove the last compactness theorem.

Theorem 4.4. *The space $H^1(0, T; L^2(0, 1)) \cap L^2(0, T; D(A))$ is compactly imbedded in $L^2(0, T; H^1_a) \cap C(0, T; L^2(0, 1))$.*

Proof. Using the Aubin's Theorem with $r_0 = r_1 = 2$, $X_0 = D(A)$, $X_1 = H^1_a$, $X_2 = L^2(0, 1)$, $a = 0$ and $b = T$, one has

$$H^1(0, T; L^2(0, 1)) \cap L^2(0, T; D(A)) \text{ is compactly imbedded in } L^2(0, T; H^1_a).$$

Moreover, since $H^1(0, T; L^2(0, 1))$ is compactly imbedded in $C(0, T; L^2(0, 1))$ and $H^1(0, T; L^2(0, 1)) \cap L^2(0, T; D(A))$ is continuously imbedded in $H^1(0, T; L^2(0, 1))$, then

$$H^1(0, T; L^2(0, 1)) \cap L^2(0, T; D(A)) \text{ is compactly imbedded in } C(0, T; L^2(0, 1)).$$

Thus $H^1(0, T; L^2(0, 1)) \cap L^2(0, T; D(A))$ is compactly imbedded in $L^2(0, T; H^1_a) \cap C(0, T; L^2(0, 1))$. □

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