BOUNDARY LAYERS FOR TRANSMISSION PROBLEMS WITH SINGULARITIES

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Abstract. We study two-dimensional transmission problems for the Laplace operator for two diffusion coefficients. We describe the boundary layers of this problem and show that the layers appear only in the part where the coefficient is large. The relationship with the singularities of the limit problem is also described.

1. Introduction

We study two-dimensional transmission problems (also called interface problems) for the Laplace operator on polygonal domains consisting of different materials connected via an interface line. Dirichlet boundary conditions on the exterior boundary and standard transmission conditions are imposed. Such problems appear in diffusion problems where the conductivity of the materials are different on some parts of the domain. It is well known that the solutions of such problems have corner singularities due the jump of the coefficients \([6, 7, 9, 10, 12, 13, 14]\). On the other hand, for a homogeneous medium having a large diffusion coefficient, the solution exhibits boundary layers added to corner singularities. Their relationship and description are well understood nowadays \([1, 4, 5, 8, 11]\). But to our knowledge, the description of such a phenomenon is not known for transmission problems where only one of the diffusion coefficients is large. Therefore in this paper we study a relatively simple example of a transmission problem that has corner singularities and boundary layers.

For a standard problem

\[-\varepsilon \Delta u_\varepsilon + u_\varepsilon = f \quad \text{in } \Omega,\]

when \(\Omega\) is a polygonal domain of the plane, \(f\) is smooth and \(\varepsilon > 0\) is a fixed (but small) parameter. An asymptotic expansion of \(u_\varepsilon\) is well known \([1, 4, 5, 8, 11]\) and may be written as

\[u_\varepsilon = w_\varepsilon + w^{BL} + w^{CL} + r_\varepsilon,\]

where \(w_\varepsilon\) is the outer expansion, \(w^{BL}\) describes the boundary layer, \(w^{CL}\) describes the corner layer, and \(r_\varepsilon\) is a remainder that is estimated as a function of \(\varepsilon\) in some appropriate norms. Usually the terms \(w_\varepsilon, w^{BL}\) and \(w^{CL}\) are explicit, which means
that, for numerical purposes for instance, the behaviour of \( u_\varepsilon \) is fully understood by the behaviour of the terms \( w_\varepsilon, w^{BL}, \) and \( w^{CL} \).

The goal of the present paper is to reproduce a similar but simpler expansion for a transmission problem where on a part \( \Omega_+ \) of the domain we consider the problem

\[
-\varepsilon \Delta u_\varepsilon + u_\varepsilon = f \quad \text{in} \quad \Omega_+,
\]

and on the other part \( \Omega_- \), the problem

\[
-\Delta u_\varepsilon + u_\varepsilon = f \quad \text{in} \quad \Omega_-,
\]

with, of course, transmission conditions on the interface. By a simpler expansion, we mean that \( w_\varepsilon, w^{BL}, w^{CL} \) will be reduced to one term. As we shall see the situation is more complicated than in the standard case of problem (1.1). The main reason is that the solution of the limit problem has singularities in the domain \( \Omega_- \). Let us further notice that surprisingly the solution of our problem has only layers in the domain \( \Omega_+ \).

In this paper, the spaces \( H^s(\Omega) \), with \( s \geq 0 \), are the standard Sobolev spaces in \( \Omega \) with norm \( \| \cdot \|_s,\Omega \) and semi-norm \( | \cdot |_s,\Omega \). The space \( H^1_0(\Omega) \) is defined, as usual, by \( H^1_0(\Omega) := \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma \} \). \( L^p(\Omega), \ p > 1, \) are the usual Lebesgue spaces with norm \( \| \cdot \|_{0,p,\Omega} \) (as usual we drop the index \( p \) for \( p = 2 \)). Finally, the notation \( a \lesssim b \) means the existence of a positive constant \( C \), which is independent of the quantities \( a \) and \( b \) under consideration and of the parameter \( \varepsilon \), such that \( a \leq C b \).

This paper is organized as follows: In section 2 we start with a one-dimensional problem in order to describe and understand the typical phenomena. Section 3 is devoted to the introduction of the two-dimensional problem and to the (weak) convergence of the solution to the solution of the limit problem. We go on with the description of the boundary and corner layers in section 4, paying a particular attention to the interface layers due to the singularities. Finally in section 5 we give the expansion of the solution of our problem.

### 2. The one-dimensional case

Let \( \varepsilon \in]0,1[ \) be a fixed parameter. Consider the following transmission problem in \( ]-1,1[ \):

\[
\begin{align*}
-\varepsilon^2 u''_\varepsilon + u_\varepsilon &= 1 \quad \text{in } ]-1,0[, \\
- w''_\varepsilon + w_\varepsilon &= 0 \quad \text{in } ]0,1[, \\
 u_\varepsilon(-1) &= w_\varepsilon(1) = 0, \\
u_\varepsilon(0) - w_\varepsilon(0) &= 0, \\
\varepsilon^2 u'_\varepsilon(0) - w'_\varepsilon(0) &= 0. \\
\end{align*}
\]

(2.1)

We remark that in this problem the small parameter \( \varepsilon \) appears only on \( ]-1,0[ \). Consequently the formal limit problem is the non standard transmission problem

\[
\begin{align*}
 u_0 &= 1 \quad \text{in } ]-1,0[, \\
- w''_0 + w_0 &= 0 \quad \text{in } ]0,1[, \\
 u_0(-1) &= w_0(1) = 0, \\
u_0(0) - w_0(0) &= 0, \\
w'_0(0) &= 0.
\end{align*}
\]

(2.2)
This limit problem has a solution \( w_0 \equiv 0 \), but has no solution \( u_0 \) since \( u_0 = 1 \) does not satisfy the boundary condition \( u_0(-1) = 0 \) and the transmission condition \( u_0(0) - w_0(0) = 0 \). Therefore, we may expect that \( u_\varepsilon \) will develop boundary layers at 0 (transmission layer) and at \(-1\) (standard boundary layer). We now justify this formal argument.

The exact solution of this problem (2.1) is

\[
\begin{align*}
\tilde{u}_\varepsilon(x) &= \alpha \cosh \frac{x}{\varepsilon} + \beta \sinh \frac{x}{\varepsilon} + 1, \\
\tilde{w}_\varepsilon(x) &= \gamma \cosh x + \delta \sinh x,
\end{align*}
\]

where \( \alpha, \beta, \gamma \) and \( \delta \) are constants (depending on \( \varepsilon \)) determined in order to check the boundary and transmission conditions. This yields a \( 4 \times 4 \) linear system that gives after resolution:

\[
\begin{align*}
\alpha + 1 &= \gamma, \\
\beta &= \delta/\varepsilon, \\
\gamma &= -\delta \tanh 1, \\
\delta &= -\varepsilon \psi(\varepsilon),
\end{align*}
\]

where the function \( \psi \) is

\[
\psi(\varepsilon) = \frac{\cosh \frac{1}{\varepsilon} - 1}{\varepsilon \tanh 1 \cosh \frac{1}{\varepsilon} + \sinh \frac{1}{\varepsilon}}.
\]

Since one easily sees that \( \psi(\varepsilon) \) approaches 1 as \( \varepsilon \) approaches 0, we deduce that \( \delta = -\varepsilon \psi(\varepsilon) \sim -\varepsilon \) as \( \varepsilon \to 0 \). Due to the identities (2.3) and (2.4), we can show that, as \( \varepsilon \) approaches 0, \( u_\varepsilon \to 1 \) and \( w_\varepsilon \to 0 \), as well as

\[
\begin{align*}
u_\varepsilon'(-1) &= -\frac{\alpha}{\varepsilon} \sinh \frac{1}{\varepsilon} + \frac{\beta}{\varepsilon} \cosh \frac{1}{\varepsilon} \sim \frac{1}{\varepsilon}, \\
u_\varepsilon'(0) &= \frac{\beta}{\varepsilon} \sim -\frac{1}{\varepsilon}, \\
\tilde{w}_\varepsilon'(1) &= \gamma \sinh 1 + \delta \cosh 1 \sim -\frac{\varepsilon}{\cosh 1}, \\
\tilde{w}_\varepsilon'(0) &= \delta \sim -\varepsilon.
\end{align*}
\]

From these equivalences, we may say that \( \tilde{w}_\varepsilon \) has no layer, while \( \tilde{u}_\varepsilon \) has a standard boundary layer at \(-1\) and a transmission layer at 0. We also refer to Figure 1 for an illustration of this fact.

**Figure 1.** Exact solutions for \( \varepsilon = 0.1 \) (left) and \( \varepsilon = 0.05 \) (right).
Let us give a more precise result, that will also allow us to underline the fact that the transmission layer at 0 may be seen as a (Dirichlet) boundary layer.

**Theorem 2.1.** For any $\varepsilon \in [0, 1]$, the unique solution $(u_\varepsilon, w_\varepsilon)$ of (2.1) satisfies

$$u_\varepsilon(x) = 1 - \chi^b(x) \exp \left( - \frac{\text{dist}(x, -1)}{\varepsilon} \right) - \chi^i(x) \exp \left( - \frac{\text{dist}(x, 0)}{\varepsilon} \right) + r_\varepsilon(x), \quad \forall x \in ]0, 1[,$$

(2.5)

where $\chi^b$ and $\chi^i$ are the two following cut-off functions:

- $\chi^b = 1$ on $]-1, -1 + \eta[,$
- $\chi^i = 1$ on $] - \eta, \eta[,$
- $\text{supp} \chi^b \cap \text{supp} \chi^i = \emptyset.$

Moreover,

$$\varepsilon \Vert r'_\varepsilon \Vert_{0, -1, 0} + \Vert r_\varepsilon \Vert_{0, -1, 0} + \Vert w_\varepsilon \Vert_{1, 0, 1} \lesssim (\varepsilon e^{-\eta} + 1).$$

(2.6)

**Proof.** Let us define the functions $v^b : x \mapsto - \exp \left( - \frac{\text{dist}(x, -1)}{\varepsilon} \right)$, a solution of

$$-\varepsilon^2 v^b'' + v^b = 0 \quad \text{in } ]-1, 0[, $$
- $v^b(-1) + 1 = 0,$
- $v^b(+\infty) = 0,$

and $v^i : x \mapsto - \exp \left( - \frac{\text{dist}(x, 0)}{\varepsilon} \right)$, a solution of

$$-\varepsilon^2 v^i'' + v^i = 0 \quad \text{in } ]-1, 0[, $$
- $v^i(0) + 1 = 0,$
- $v^i(-\infty) = 0.$

Using these two problems and by substitution of (2.5) in (2.1), we see that $(r_\varepsilon, w_\varepsilon)$ is solution of

$$-\varepsilon^2 r''_\varepsilon + r_\varepsilon = g_\varepsilon \quad \text{in } ]-1, 0[, $$
- $w_\varepsilon - w''_\varepsilon = 0 \quad \text{in } ]0, 1[,$$
- $r_\varepsilon(-1) = 0,$
- $w_\varepsilon(1) = 0,$
- $r_\varepsilon(0) = w_\varepsilon(0),$
- $\varepsilon^2 r'_\varepsilon(0) - w'_\varepsilon(0) = -\varepsilon,$

(2.7)

where

$$g_\varepsilon := \varepsilon^2 \left( [\chi^b; \frac{d^2}{dx^2}] e^{-\frac{x+1}{\varepsilon}} + [\chi^i; \frac{d^2}{dx^2}] e^{\frac{x}{\varepsilon}} \right),$$

the bracket $[\chi^b; \frac{d^2}{dx^2}]$ being defined as usual,

$$[\chi^b; \frac{d^2}{dx^2}] h = \frac{d^2}{dx^2}(\chi^b h) - \chi^b \frac{d^2}{dx^2} h = h \frac{d^2}{dx^2} \chi^b + 2 \frac{d}{dx} \chi^b \frac{d}{dx} h.$$
The variational formulation of this problem is
\[
\int_{-1}^{0} \varepsilon^2 \rho^2 \omega' \, dx + \int_{0}^{1} \rho^2 \omega' \, dx + \int_{-1}^{0} \rho \omega \, dx + \int_{0}^{1} \omega \omega \, dx
= \int_{-1}^{0} g \omega \, dx - \varepsilon w(0), \forall w \in \mathcal{H}_0^1([-1, 1]).
\] (2.8)

Since this left-hand side is trivially coercive on \( \mathcal{H}_0^1([-1, 1]) \), by the Lax-Milgram lemma, this problem has a unique solution \( \rho \in \mathcal{H}^1([-1, 0]) \) and \( \omega \in \mathcal{H}^1([0, 1]) \) such that \( \rho(-1) = \omega(1) = 0 \), and \( \rho(0) = \omega(0) \) (this means that the function \( \rho \) defined by \( \rho \) on \([-1, 0]\) and by \( \omega \) on \([0, 1]\) belongs to \( \mathcal{H}_0^1([-1, 1]) \)). Moreover by taking as test function in (2.8) \( w = k \) we obtain
\[
\varepsilon^2 \| \rho \|_{0, [0, 1]}^2 + \| \rho \|_{0, [1, 1]}^2 + \| \omega \|_{0, [0, 1]}^2 + \| \omega \|_{0, [1, 1]}^2
\leq \| g \|_{0, [-1, 0]} \| \rho \|_{0, [-1, 0]} + \varepsilon |w(0)|.
\] (2.9)

It then remains to estimate the \( L^2 \)-norm of \( g \). The properties of \( \chi^b \) and \( \chi^i \) imply that \( \text{supp } g \subset [-1 + \eta, -\eta] \). Since on this interval \( e^{-\frac{1}{\varepsilon}(2+1)} \leq e^{-\frac{1}{\varepsilon}} \) and \( e^{\varepsilon} \leq e^{-\frac{1}{\varepsilon}} \), we obtain
\[
\| g \|_{0, [-1, 0]} \leq e^{-\frac{1}{\varepsilon}}.
\]

On the other hand, the identities (2.3) and (2.4) imply that
\[
|w(0)| \leq \varepsilon \quad \forall \varepsilon \in [0, 1].
\]

These two estimates in (2.9) yield
\[
\varepsilon^2 \| \rho \|_{0, [-1, 0]}^2 + \| \rho \|_{0, [-1, 0]}^2 + \| \omega \|_{0, [0, 1]}^2 + \| \omega \|_{0, [1, 1]}^2 \leq \varepsilon e^{-\frac{1}{\varepsilon^2}} \| r \|_{0, [-1, 0]}^2 + \varepsilon^2.
\]
The desired estimate (2.6) follows from Young's inequality.

\[ \square \]

Note that the estimate (2.6) is optimal: Direct calculations yield \( \| w \|_{0, [0, 1]} \sim \varepsilon \).

The above theorem gives an explicit expansion of \( u \), which also shows that \( u \) has two layers (at 0 and \(-1\)). It further says that the natural energy norm of the remainder \( r \) is of order \( \varepsilon \). Finally it says that \( w \) has no layer and that its natural energy norm is of order \( \varepsilon \).

The goal of the next sections is to show similar results for a polygonal domain on the plane.

3. The two-dimensional problem

\[ \text{Figure 2. The domains } \Omega_+ \text{ and } \Omega_- \]
Let $\Omega_+$ and $\Omega_-$ be two polygonal domains of $\mathbb{R}^2$ with respective boundary $\partial \Omega_+$ and $\partial \Omega_-$ having in common a segment $\Sigma = [A,B]$, see Figure 2. Denote by $A_1, A_2, \ldots, A_N$ the vertices of $\partial \Omega_+$ enumerated clockwise and so that $A_1 = A$ and $A_2 = B$. Denote further by $\omega_j$ the interior angle of $\Omega_+$ at the vertex $A_j$, for any $j \in \{1,2,\ldots,N\}$ and let $\varphi_j$ the interior angle of $\Omega_-$ at the vertex $A_j$, $j = 1,2$.

For further purposes we denote by $\Omega = \Omega_+ \cup \Omega_- \cup \Sigma$. Moreover for a function $u$ defined in $\Omega$, we denote by $u_+$ (resp. $u_-$) the restriction of $u$ to $\Omega_+$ (resp. $\Omega_-$).

For $\varepsilon \in ]0,1[$, $f_+ \in C^\infty(\overline{\Omega_+})$ and $h \in C^\infty(\Sigma)$, we consider the transmission problem in $\Omega$: Find $u^\varepsilon$ solution of

\[
\begin{align*}
-\varepsilon^2 \Delta u^\varepsilon_+ + u^\varepsilon_+ &= f_+ \quad \text{in } \Omega_+, \\
-\Delta u^\varepsilon_- + u^\varepsilon_- &= f_- \quad \text{in } \Omega_-, \\
u^\varepsilon_+ &= 0 \quad \text{on } \partial \Omega_+ \setminus \Sigma, \\
u^\varepsilon_- &= 0 \quad \text{on } \partial \Omega_- \setminus \Sigma, \\
u^\varepsilon_+ - \nu^\varepsilon_- &= 0 \quad \text{on } \Sigma, \\
\varepsilon^2 \frac{\partial u^\varepsilon_+}{\partial \nu} - \frac{\partial u^\varepsilon_-}{\partial \nu} &= h \quad \text{on } \Sigma,
\end{align*}
\]

where $\nu$ denotes the outward normal vector along $\Sigma$ oriented outside $\Omega_+$. The variational formulation of this problem consists in finding a unique solution $u^\varepsilon \in H^1_0(\Omega)$ of

\[
\int_{\Omega_+} (\varepsilon^2 \nabla u^\varepsilon_+ \cdot \nabla v_+ + u^\varepsilon_+ v_+) + \int_{\Omega_-} (\nabla u^\varepsilon_- \cdot \nabla v_- + u^\varepsilon_- v_-) = \int_{\Omega_+} f v + \int_{\Sigma} h v, \forall v \in H^1_0(\Omega).
\]

Since this left-hand side is a coercive and continuous bilinear form on $H^1_0(\Omega)$, this problem has a unique solution thanks to the Lax-Milgram lemma.

We now look at the limit of the problem and of $u^\varepsilon$ as $\varepsilon$ goes to zero. As before the formal limit problem is

\[
\begin{align*}
 u^0_+ &= f_+ \quad \text{in } \Omega_+, \\
-\Delta u^0_+ + u^0_+ &= f_- \quad \text{in } \Omega_-, \\
u^0_+ &= 0 \quad \text{on } \partial \Omega_+ \setminus \Sigma, \\
u^0_- &= 0 \quad \text{on } \partial \Omega_- \setminus \Sigma, \\
u^0_+ - \nu^0_- &= 0 \quad \text{on } \Sigma, \\
-\frac{\partial u^0_+}{\partial \nu} &= h \quad \text{on } \Sigma.
\end{align*}
\]

As in dimension 1, in this limit problem $u^0_+$ may be seen as the (unique) solution of a mixed Dirichlet-Neumann problem in $\Omega_-$, and since $f_+$ does not satisfy the Dirichlet boundary condition $f_+ = 0$ on $\partial \Omega_+ \setminus \Sigma$, and $f_+ = u^0_+$ on $\Sigma$, the solution $u^\varepsilon_+$ should develop boundary layers along $\partial \Omega_+$. This will be proved in details in the remainder of this paper. Let us first state a weak convergence.

**Theorem 3.1.** There exists a subsequence of $u^\varepsilon_+$, still denoted by $u^\varepsilon_+$, such that the pair $(u^\varepsilon_+, u^\varepsilon_-)$ converges in $L^2(\Omega_+) \times H^1(\Omega_-)$ to $(u^0_+, u^0_-)$ as $\varepsilon$ goes to 0, where $u^0_+ = f_+$ and $u^0_-$ is the unique variational solution of the mixed Dirichlet-Neumann problem.
Before proving this theorem, let us introduce some notation and give a density result. Let us introduce the following bilinear and linear forms:

\[ a(u, v) = \int_{\Omega^+} \nabla u_+ \cdot \nabla v_+, \]
\[ b(u, v) = \int_{\Omega^-} \nabla u_- \cdot \nabla v_- + \int_{\Omega} u v, \]
\[ F(v) = \int_{\Omega} f v + \int_{\Sigma} h v. \]

Let us define the space

\[ W = \{ w \in L^2(\Omega) : w_- \in H^1(\Omega_-) \text{ and } w_- = 0 \text{ on } \partial \Omega_- \setminus \Sigma \}, \]

which is a Hilbert space, equipped with the norm \( \| w \|_W = b(w, w) \).

**Lemma 3.2.** \( H^1_0(\Omega) \) is dense in \( W \).

**Proof.** Let \( w \in W \). Since \( w_- \in \tilde{H}^{1/2}(\Sigma) \), by \([3, \text{Theorem 1.5.2.3}]\) (trace Theorem), there exists \( \tilde{w}_+ \in H^1(\Omega_+) \) such that

\[ \tilde{w}_+ = w_- \text{ on } \Sigma, \]
\[ \tilde{w}_+ = 0 \text{ on } \partial \Omega_+ \setminus \Sigma. \]

Since \( w_+ - \tilde{w}_+ \) belongs to \( L^2(\Omega_+) \) and since \( H^1_0(\Omega_+) \) is dense in \( L^2(\Omega_+) \), there exists a sequence of functions \( w_+^n \in H^1_0(\Omega_+), n \in \mathbb{N} \) such that

\[ \| w_+^n - (w_+ - \tilde{w}_+) \|_{0, \Omega_+} \to 0 \text{ as } n \to \infty. \] (3.6)

For all positive integer \( n \), we introduce the function \( \tilde{w}^n \) defined in \( \Omega \) as follows

\[ \tilde{w}^+_n = w_+^n + \tilde{w}_+, \]
\[ \tilde{w}^-_n = w_- \cdot \]

From the boundary condition satisfied by \( \tilde{w}_+, \tilde{w}_n \) belongs to \( H^1_0(\Omega) \). Moreover from the definition of \( \tilde{w}^n \) and owing to (3.6), we have

\[ \| \tilde{w}^n - w \|_W = \| w_+^n - (w_+ - \tilde{w}_+) \|_{0, \Omega_+} \to 0. \]

**Proof of Theorem 3.1.** From (3.2) and the definition of 0, we see that \( u^\varepsilon \in H^1_0(\Omega) \) and \( u^0 \in W \) are the respective solution of

\[ \varepsilon^2 a(u^\varepsilon, v) + b(u^\varepsilon, v) = F(v), \forall v \in H^1_0(\Omega), \] (3.7)
\[ b(u^0, w) = F(w), \forall w \in W. \] (3.8)

**Step 1.** \( u^\varepsilon \) is weakly convergent to \( u^0 \) in \( W \). We first remark that

\[ \| u^\varepsilon \|_W^2 = b(u^\varepsilon, u^\varepsilon) \leq b(u^\varepsilon, u^\varepsilon) + \varepsilon^2 a(u^\varepsilon, u^\varepsilon). \]
Now taking $v = u^\varepsilon$ in (3.7) and $w = u^\varepsilon$ in (3.8) we obtain
\[ \varepsilon^2 a(u^\varepsilon, u^\varepsilon) + b(u^\varepsilon, u^\varepsilon) = b(u^0, u^\varepsilon). \] (3.9)

Using Cauchy-Schwarz’s inequality, we directly have
\[ |b(u^0, u^\varepsilon)| \leq \|u^0\|_W \|u^\varepsilon\|_W. \]

These three properties imply that
\[ \|u^\varepsilon\|_W \leq \|u^0\|_W. \] (3.10)

Therefore, there exists $w \in W$ and a subsequence of $u^\varepsilon$, still denoted by $u^\varepsilon$, weakly convergent to $w$ in $W$.

Now for any fixed $v \in H^{1,0}_0(\Omega)$, using successively (3.9) and (3.10) we may write
\[ |a(u^\varepsilon, v)| \leq \|\nabla u^\varepsilon\|_{0,\Omega_+} \|\nabla v\|_{0,\Omega_+} \]
\[ \leq \varepsilon^{-1}(\varepsilon^2 \|\nabla u^\varepsilon\|_{0,\Omega_+}^2 + b(u^\varepsilon, u^\varepsilon))^{\frac{1}{2}} \|\nabla v\|_{0,\Omega_+} \]
\[ = \varepsilon^{-1} b(u^0, u^\varepsilon)^{\frac{1}{2}} \|\nabla v\|_{0,\Omega_+} \]
\[ \leq \varepsilon^{-1} \|u^0\|_{W}^{\frac{1}{2}} \|u^\varepsilon\|_{W}^{\frac{1}{2}} \|\nabla v\|_{0,\Omega_+} \]
\[ \leq \varepsilon^{-1} \|u^0\|_W \|\nabla v\|_{0,\Omega_+}. \]

This last estimate implies that
\[ \lim_{\varepsilon \to 0} \varepsilon^2 a(u^\varepsilon, v) = 0, \quad \forall v \in H^1_0(\Omega). \]

Therefore, passing to the limit in (3.7), we obtain
\[ \lim_{\varepsilon \to 0} b(u^\varepsilon, v) = F(v) = b(u^0, v), \quad \forall v \in H^1_0(\Omega). \]

Since $H^1_0(\Omega)$ is dense in $W$, we conclude that
\[ b(u^0, v) = b(w, v), \quad \forall v \in W. \]

Since $b(\cdot, \cdot)$ is the inner product of $W$, we deduce that $u^0 = w$.

**Step 2.** $u^\varepsilon$ is strongly convergent to $u^0$ in $W$.
\[ \|u^\varepsilon - u^0\|_W^2 = b(u^\varepsilon - u^0, u^\varepsilon - u^0) \]
\[ = b(u^\varepsilon, u^\varepsilon) - b(u^0, u^\varepsilon) - b(u^\varepsilon - u^0, u^0). \]

Taking into account (3.9), we obtain
\[ \|u^\varepsilon - u^0\|_W^2 \leq -b(u^\varepsilon - u^0, u^0). \]

Then we have the conclusion, by the weak convergence in $W$ of $u^\varepsilon$ to $u^0$. \( \square \)

From this Theorem we may see $u^0$ as the first term of the outer expansion of $u^\varepsilon$.

Let us now pass to the description of the boundary layers.
4. Boundary layers

In the sequel let \( L_\varepsilon \) denote the operator \( L_\varepsilon = I - \varepsilon^2 \Delta \). In this section, we define in \( \Omega_+ \), the boundary layer \( v^b_j \) along \( \Gamma_j = [A_{j-1}, A_j], \) \( j = 2, 3, \ldots, N \) and the interface layer \( v^i \) along \( \Sigma \), such that if \( V_j \) denote a small neighbourhood of \( \Gamma_j \), we have

\[
L_\varepsilon (u_+^\varepsilon - f_+ - v^b_j) = \varepsilon^2 O(\varepsilon) \quad \text{in} \quad V_j \cap \Omega_+,
\]

\[
f_+ + v^b_j = 0 \quad \text{on} \quad \Gamma_j
\]

and

\[
L_\varepsilon (u_+^\varepsilon - f_+ - v^i) = \varepsilon^2 O(\varepsilon) \quad \text{in} \quad V_1 \cap \Omega_+,
\]

\[
f_+ + v^i = u_0^- \quad \text{on} \quad \Sigma,
\]

when \( O(\varepsilon) \) denote as usual a function of \( \varepsilon \) bounded in a neighbourhood of \( \varepsilon = 0 \).

Note that the situation is not the same along \( \Sigma \) due to the lack of regularity of \( u_0^- \) (see below).

4.1. Some notation and definitions. We denote by \((x, y)\) the Cartesian coordinates of the plane with origin at \( A_1 \) and such that \( \Gamma_1 \subset \{(x, 0), x > 0\} \). Similarly we denote by \((x_j, y_j)\) the Cartesian coordinates of the plane with origin at \( A_j \) and such that \( \Gamma_j \subset \{(x_j, 0), x_j > 0\} \).

We now fix two cut-off functions \( \chi^1_j, \chi^2_j \in C^\infty(\mathbb{R}) \) satisfying \( \text{supp} \chi^1_j \subset [-a_j, a_j] \), and

\[
\chi^1_j(x) = 1 \quad \text{on} \quad ]0, l_j[,
\]

where \( l_j \) is the length of \( \Gamma_j \), and \( \text{supp} \chi^2_j \subset [-b, b] \), as well as

\[
\chi^2_j(y) = 1 \quad \text{on} \quad ]-\frac{b}{2}, \frac{b}{2}[,
\]

for a sufficiently small fixed \( b > 0 \).

Now we can introduce the cut-off function along \( \Gamma_j \) by

\[
\chi^b_j(x, y) = \chi^1_j(x) \chi^2_j(y).
\]

We finally take \( \chi^i = \chi^1_1 \).

Now we assume that \( f_+ \) is the restriction to \( \Omega_+ \) of a smooth function \( \tilde{f}_+ \in C^\infty(\mathbb{R}^2) \) and that \( \Omega_+ \) is convex, i.e., \( 0 < \omega_j < \pi, \) for all \( j = 1, \ldots, N \). This last assumption is simply made to simplify the construction of corner layers. Using the method of [11], we probably can treat the non convex case.

4.2. Construction of \( v^b_j \). They are standard, see for instance [4, 5]. For \( j = 2, \ldots, N \), \( v^b_j \) is the unique solution of the problem

\[
v^b_j - \varepsilon^2 v^b_j'' = 0 \quad \text{in} \quad y_j > 0,
\]

\[
v^b_j = -\tilde{f}_+(x_j, \cdot) \quad \text{at} \quad y_j = 0,
\]

\[
v^b_j = 0 \quad \text{at} \quad y_j = +\infty.
\]

It is explicitly given by

\[
v^b_j(x_j, y_j) = -\tilde{f}_+(x_j, 0) e^{-y_j/\varepsilon}.
\]
Since $\omega_j < \pi$, the function $\chi_j^b$ is well defined in $\Omega_+$ and satisfies the conditions (4.1). Moreover it has the regularity $C^\infty(\Omega_+)$ and for any $(x_j, y_j) \in \Omega_+$,

$$L(\chi_j^b)(x_j, y_j) = (I - \varepsilon^2 \Delta_{(x_j, y_j)})(\chi_j^b)(x_j, y_j)$$

$$= \varepsilon^2 \chi_j^b e^{-\frac{y}{\varepsilon}} (\partial_{x_j})^2 \bar{f}^j(x_j, 0) + \varepsilon^2 [\chi_j^b \Delta_{(x_j, y_j)}] v^i,$$

where we recall that $[\chi; \Delta_{(x_j, y_j)}] v := \chi \Delta_{(x_j, y_j)} v - \Delta_{(x_j, y_j)} (\chi v)$. Since

$$|\chi_j^b e^{-\frac{y}{\varepsilon}} (\partial_{x_j})^2 \bar{f}^j(x_j, 0)| \leq 1,$$

$$|[\chi_j^b ; \Delta_{(x_j, y_j)}] v^i| \leq \frac{1}{\varepsilon} e^{\frac{y}{\varepsilon}},$$

we deduce that

$$\|L(\chi_j^b)(x_j, y_j)\|_{0, \Omega_+} \leq \varepsilon^2 + \varepsilon^{1/2}. \quad (4.5)$$

4.3. **Construction of $v^i$.** In general the solution $u^0_\varepsilon$ of problem (3.4) has only the regularity $u^0_\varepsilon \in H^1(\Omega_-)$. Consequently if we proceed as in the previous subsection, namely if we take

$$v^i(x_1, y_1) = (\bar{u}^0_\varepsilon - \bar{f}^j)(x_1, 0) e^{-y_i/\varepsilon},$$

the regularity of $v^i$ is not sufficient to obtain an estimate similar to (4.5). To overcome this difficulty, we shall use the decomposition of $u^0_\varepsilon$ into a regular part and singular one.

For $j = 1, 2$, we recall that the singular exponents associated with the mixed Dirichlet-Neumann problem near $A_j$ are given by (see [3, 2])

$$\Lambda_j = \{\lambda_k = \frac{\pi}{2\varphi_j} + \frac{k\pi}{\varphi_j}, k \in \mathbb{Z}\}.$$

Let $(r_j, \theta_j)$ be the polar coordinates centred at $A_j$ and such that $\theta_j = 0$ on $\Sigma$, and $\theta_j = -\omega_j$ on the other edge of $\Omega_-$. For $\lambda_k \in \Lambda_j$, we denote

$$S_{j, \lambda_k}(r_j, \theta_j) = r_j^{\lambda_k} \sin \lambda_k(\varphi_j + \theta_j), \quad -\varphi_j < \theta_j < \omega_j. \quad (4.6)$$

Recall that this function satisfies

$$\Delta S_{j, \lambda_k} = 0,$$

$$S_{j, \lambda_k}(r_j, -\varphi_j) = 0,$$

$$\frac{\partial}{\partial \theta_j} S_{j, \lambda_k}(r_j, 0) = 0. \quad (4.7)$$

According to [3, Corollary 4.4.3.8], the solution $u^0_\varepsilon \in H^1(\Omega_-)$ of (3.4) admits the decomposition

$$u^0_\varepsilon = u^0_{\varepsilon, r} + \sum_{j=1,2} \sum_{\lambda_k \in \Lambda_j, \varepsilon \lambda_k < 2} C_{j, \lambda_k} S_{j, \lambda_k}, \quad (4.8)$$

where $u^0_{\varepsilon, r} \in H^3(\Omega_- \cap \mathcal{V}_1)$, $C_{j, \lambda_k}$ are real constants and $\eta_j$ is a (radial) cut-off function equal to 1 in neighbourhood of $A_j$ and equal to zero outside another neighbourhood of $A_j$, $j = 1, 2$. Using this expansion, we can define

$$v^i = v_r^i + v_s^i \quad (4.9)$$

where

$$v_r^i(x, y) = (\bar{u}^0_{\varepsilon, r} - \bar{f}^j)(x, 0) e^{-y_i/\varepsilon},$$

$$v_s^i(x, y) = \sum_{j=1,2} \sum_{\lambda_k \in \Lambda_j, \varepsilon \lambda_k < 2} C_{j, \lambda_k} S_{j, \lambda_k} e^{-y_i/\varepsilon}. \quad (4.10)$$
where \( \tilde{u}^0_r(\cdot) \) is an extension to the real line of \( u^0_r(\cdot, 0) \). Since \( u^0_r(\cdot, 0) \) belongs to \( H^{\frac{3}{2}}(\Sigma) \), this extension may be chosen in \( H^{\frac{3}{2}}(\mathbb{R}) \) and by the Sobolev embedding Theorem, \( v^i_r \in C^2(\bar{\Omega}_+) \). Therefore, as in the previous subsection, we have

\[
\| \mathcal{L}_\varepsilon (\chi^i v^i_r) \|_{0, \Omega_+} \lesssim \varepsilon^2 + \varepsilon e^{-\frac{b}{\varepsilon}}. \tag{4.11}
\]

On the other hand, Leibniz’s rule yields

\[
\Delta (\chi^i \eta_j S_{j, \lambda_k} e^{-y/\varepsilon}) = \chi^i \eta_j S_{j, \lambda_k} \Delta e^{-y/\varepsilon} + 2 \nabla (\chi^i \eta_j S_{j, \lambda_k}) \cdot \nabla e^{-y/\varepsilon} + \Delta (\chi^i \eta_j S_{j, \lambda_k}) e^{-y/\varepsilon}
\]
\[
eq e^{-y/\varepsilon} \left( \frac{1}{\varepsilon^2} \chi^i \eta_j S_{j, \lambda_k} - \frac{2}{\varepsilon} \partial (\chi^i \eta_j S_{j, \lambda_k}) + \chi^i \eta_j \Delta S_{j, \lambda_k} - [\chi^i \eta_j; \Delta] S_{j, \lambda_k} \right),
\]

and therefore (reminding \( \Delta S_{j, \lambda_k} = 0 \))

\[
\mathcal{L}_\varepsilon (\chi^i \eta_j S_{j, \lambda_k} e^{-y/\varepsilon}) = \varepsilon^2 e^{-y/\varepsilon} \left( \frac{2}{\varepsilon} \partial (\chi^i \eta_j S_{j, \lambda_k}) + [\chi^i \eta_j; \Delta] S_{j, \lambda_k} \right).
\]

From this identity, we deduce that

\[
\| \mathcal{L}_\varepsilon (\chi^i v^i_r) \|_{0, \Omega_+} \lesssim \varepsilon^{1+\lambda} + \varepsilon e^{-b/(2\varepsilon)}, \tag{4.12}
\]

where \( \lambda = \min_{k=1,2} \min \{ \lambda_k : \lambda_k \in \Lambda_k \} \). As \( v^i = v^i_+ + v^i_- \), the estimates (4.11) and (4.12) lead to

\[
\| \mathcal{L}_\varepsilon (\chi^i v^i) \|_{0, \Omega_+} \lesssim \varepsilon^{1+\lambda} + \varepsilon e^{-\frac{b}{\varepsilon}}. \tag{4.13}
\]

At this stage if we set

\[
U_+ := f_+ + \sum_{j=2}^N \chi_j b v^b_j + \chi^i v^i \quad \text{in } \Omega_+,
\]

then we may write (since \( \mathcal{L}_\varepsilon u^+ = f_+ \))

\[
\mathcal{L}_\varepsilon (u^+ - U_+) = \varepsilon^2 \Delta f_+ - \mathcal{L}_\varepsilon \left( \sum_{j=2}^N \chi_j b v^b_j \right) - \mathcal{L}_\varepsilon (\chi^i v^i).
\]

And by (4.9) and (4.13), we arrive at

\[
\| \mathcal{L}_\varepsilon (u^+ - U_+) \|_{0, \Omega_+} \lesssim \varepsilon^{1+\lambda} + \varepsilon e^{-\frac{b}{\varepsilon}}. \tag{4.15}
\]

At this stage we can say that \( U_+ \) approaches \( u^+ \) in the interior of \( \Omega_+ \), satisfies the Dirichlet boundary condition in the interior of \( \Gamma_j, j = 2, \ldots, N \) and the correct interface condition in the interior of \( \Sigma \). But the correct boundary/interface conditions are not satisfied near the corners \( A_j \). Therefore, corner correctors have to be introduced.

4.4. **Corner correctors.** For all \( j = 1, \ldots, N \) consider polar coordinates \( (r_j, \theta_j) \) centered at \( A_j \) and such that \( \Gamma_j \subset \{(r_j, 0), r_j > 0\} \) and therefore

\[
\Gamma_{j-1} \subset \{(r_j \cos \omega_j, r_j \sin \omega_j), r_j > 0\}
\]

(here and below the index are considered modulo \( N \), i.e. \( 0 = N \)). Denote

\[
S_j = \{(r_j, \theta_j), r_j > 0, 0 < \theta_j < \omega_j\},
\]
\[
\bar{\Gamma}_{j-1} = \{(r_j, \omega_j), r_j > 0\},
\]
\[
\tilde{\Gamma}_j = \{(r_j \cos \omega_j, r_j \sin \omega_j), r_j > 0\}.
\]
and let \( R_j > 0 \) be fixed sufficiently small so that
\[
\text{supp } \chi_j^{b-1} \cap \text{supp } \chi_j^b \cap S_j \subset B(A_j, \frac{R_j}{2}),
\]
\[
B(A_j, R_j) \cap B(A_k, R_k) = \emptyset \quad \text{if } k \neq j.
\]
To each vertex \( A_j \) we associate a radial cut-off function \( \chi_j^b \) such that
\[
\chi_j^b(r) = \begin{cases} 1 & \text{if } r < \frac{R_j}{2}, \\ 0 & \text{if } r > R_j. \end{cases}
\]
In the sector \( S_j \), according to the definition of the function \( U_+ \) we may write
\[
U_+(x, y) = f_+(x, y) + \chi_{j-1}^b(x_{j-1}, y_{j-1})v_{j-1}^b(x_{j-1}, y_{j-1}) + \chi_j^b(x_j, y_j)v_j^b(x_j, y_j),
\]
where for shortness we write \( v_j^b = v^i, \chi_j^b = \chi^i \). By construction of the boundary layers \( v_j^b \), we then have
\[
U_+|_{\partial S_j} = \begin{cases} \chi_j^b v_j^b & \text{on } \tilde{\Gamma}_{j-1}, \\ \chi_{j-1}^b v_{j-1}^b & \text{on } \Gamma_j. \end{cases}
\]
Now we introduce the changes of coordinates
\[
\Psi_j : (r_j, \theta_j) \longrightarrow (x_j, y_j) = (r_j \cos \theta_j, r_j \sin \theta_j),
\]
\[
\Phi_j : (x_j, y_j) \longrightarrow (x_{j-1}, y_{j-1}).
\]
Using the fact that \( \tilde{\Gamma}_{j-1} \) (resp. \( \tilde{\Gamma}_j \)) is parametrized by \((x_j, y_j) = (r_j \cos \omega_j, r_j \sin \omega_j)\) (resp. \((x_j, y_j) = (r_j, 0)\)) and using the definition of \( v_j^b \) and \( v^i \), we see that
\[
U_+|_{\partial S_j} = \begin{cases} g_1^j(r_j) \exp \left(-\frac{r_j \sin \omega_j}{\varepsilon}\right) & \text{on } \tilde{\Gamma}_{j-1}, \\ g_2^j(r_j) \exp \left(-\frac{r_j \sin \omega_j}{\varepsilon}\right) & \text{on } \tilde{\Gamma}_j, \end{cases}
\]
where, except in the case \( j = k = 1 \) and \( j = k = 2 \), the functions \( g_j^k \) are smooth, while in the exceptional case, due to (4.9) and (4.10), we have
\[
g_1^j(r_1) = g_{1, r}^1(r_1) + g_{1, s}^1(r_1), \quad (4.17)
g_2^j(r_2) = g_{2, r}^2(r_2) + g_{2, s}^2(r_2), \quad (4.18)
g_1^j, r(r_1) = \chi^i \circ \Psi_1(r_1, \omega_1)v^i_1(r_1 \cos \omega_1, 0), \quad (4.19)
g_1^j, s(r_1) = \chi^i \circ \Psi_1(r_1, \omega_1) \eta_1(r_1) \sum_{\lambda_k \in A_1, 0 < \lambda_k < 2} C_{1, \lambda_k} S_{1, \lambda_k}(r_1, \omega_1),
g_2^j, r(r_2) = \chi^i \circ \Phi_2^{-1} \circ \Psi_2(r_2, \omega_2)v^i_1(-r_2 \cos \omega_2 + \ell_1, 0),
g_2^j, s(r_2) = \chi^i \circ \Phi_2^{-1} \circ \Psi_2(r_2, \omega_2) \eta_2(r_2) \sum_{\lambda_k \in A_2, 0 < \lambda_k < 2} C_{2, \lambda_k} S_{2, \lambda_k}(r_2, \omega_2).
\]
The boundary condition imposed at \( v_j^b \) on \( \Gamma_j \) implies \( v_j^b(A_j) = v_{j-1}^b(A_j) = -f_+(A_j), j = 3, \ldots, N \). On the other hand \( u_0^b \in H^1(\Omega_-) \) and satisfies the Dirichlet condition on \( \partial \Omega_- \setminus \Sigma \). By the continuity of \( u_0^b \) (due to the expansion (4.8)) we get
\[
u_0^b(A_1) = u_0^b(A_2) = 0, \quad \text{and consequently } v_1^b(A_j) = -f_+(A_j), j = 1, 2. \quad \text{All together the next compatibility conditions are satisfied}
\]
\[
g_1^j(0) = g_2^j(0) \quad \forall j = 1, \ldots, N.
\]
(4.20)
Moreover, so that it fulfills $\sigma$. Since the term $g_{1,r}^1$ and $g_{2,r}^2$ are sufficiently smooth (namely $H^{5/2}$), they can be treated as the functions $g_j^k$, for $j > 2$. As a consequence we split $u_j^c = u_{j,r}^c + u_{j,s}^c$, where $u_{j,s}^c = 0$ for $j \neq 1,2$ and

$$u_{1,s}^c (r_1, \theta_1) = \begin{cases} 0 & \text{if } \theta_1 = 0, \\ -g_{1,s}^1 (r_1) & \text{if } \theta_1 = \omega_1, \end{cases} \quad (4.21)$$

$$u_{2,s}^c (r_2, \theta_2) = \begin{cases} 0 & \text{if } \theta_2 = \omega_2, \\ -g_{2,s}^2 (r_2) & \text{if } \theta_2 = 0, \end{cases} \quad (4.22)$$

and

$$u_{j,r}^c = -\hat{g}_j^1 \text{ on } \hat{\Gamma}_{j-1}, \quad (4.23)$$

$$u_{j,r}^c = -\hat{g}_j^2 \text{ on } \hat{\Gamma}_j, \quad (4.24)$$

where $\hat{g}_j^1 = g_j^1$ except if $j = k = 1$ and $j = k = 2$; in that last cases, we take $\hat{g}_1^1 = g_1^1$, $\hat{g}_2^2 = g_2^2$.

For our purpose, we introduce the functions

$$\sigma_{j,\lambda_k} (r_j, \theta_j) = \begin{cases} \frac{\sin(\lambda_k (r_j + \omega_j))}{\omega_j} & \text{if } \sin(\lambda_k \omega_j) = 0, \\ \frac{\sigma_j (r_j, \omega_j)}{\sin(\lambda_k \omega_j)} & \text{if } \sin(\lambda_k \omega_j) \neq 0, \end{cases}$$

so that it fulfills $\sigma_{j,\lambda_k} (r_j,0) = 0$ and $\sigma_{j,\lambda_k} (r_j, \omega_j) = \sigma_{j,\lambda_k} (r_j, \omega_j)$. Note that the first choice is also valid in the (generic) case $\sin(\lambda_k \omega_j) \neq 0$, but in this case the second choice gives rise to a harmonic function.

**Lemma 4.1.** Let

$$u_{1,s}^c (r_j, \theta_j) = -\chi^i \circ \Psi_j (r_1, \omega_1) \eta_1 (r_1) \sum_{\lambda_k \in \Lambda_1, 0 < \lambda_k < 2} C_{1,\lambda_k} \sigma_{1,\lambda_k}, \quad (4.25)$$

$$u_{2,s}^c (r_j, \theta_j) = -\chi^i \circ \Phi_j^{-1} \circ \Psi_j (r_2, \omega_2) \eta_2 (r_2) \sum_{\lambda_k \in \Lambda_2, 0 < \lambda_k < 2} C_{2,\lambda_k} \sigma_{2,\lambda_k}. \quad (4.26)$$

Then they respectively satisfy (4.21) and (4.22) and by setting $\alpha_j = \sin \omega_j$,

$$\|e^{-\frac{\alpha_j}{\omega_j}} u_{j,s}^c \|_{0,S_j} + \|e^{-\frac{\alpha_j}{\omega_j}} \nabla u_{j,s}^c \|_{0,S_j} \lesssim \varepsilon^{1+\lambda}, \quad j = 1,2. \quad (4.27)$$

Moreover

$$\Delta (\chi^i e^{-\frac{\alpha_j^i}{\omega_j}} u_{i,\theta_j}^c) \in L^p (S_j),$$

for all $p \in [1, \frac{2}{2-p}]$, where $\lambda = \min_{k=1,2} \min \{ \lambda_k : \lambda_k \in \Lambda_k \}.$

**Proof.** For simplicity, let us set

$$\chi^i (r_1) = \chi^i \circ \Psi_1 (r_1, \omega_1) \eta_1 (r_1) \text{ if } j = 1,$$

$$\chi^i (r_2) = \chi^i \circ \Phi_2^{-1} \circ \Psi_2 (r_2, \omega_2) \eta_2 (r_2) \text{ if } j = 2.$$
The estimate (4.27) follows directly from (4.28) and (4.29). The regularity of $u^\varepsilon$ follows. The second assertion is proved similarly. □

We are now ready to formulate the main result of this paper.

**Lemma 4.2.** There exists $u^\varepsilon_{j,r} \in H^1(S_j)$ satisfying (4.23) and (4.24) and such that

$$
\| \chi_j e^{-\alpha r} u^\varepsilon_{j,r}\|_{0,S_j} + \varepsilon \| \chi_j e^{-\alpha r} \nabla u^\varepsilon_{j,r} \|_{0,S_j} \lesssim \varepsilon.
$$

Moreover

$$
\Delta(\chi_j e^{-\alpha r} u^\varepsilon_{j,r}(r_j,\theta_j)) \in L^p(S_j),
$$

for all $p \in [1,2)$.

**Proof.** We simply take

$$
u^\varepsilon_{j,r}(r,\theta) = \left( \hat g^1_j(r) - \hat g^2_j(r) \right) \frac{\theta}{\omega_j} + \hat g^2_j(r),
$$

which clearly satisfies (4.23) and (4.24). As $\hat g^1_j \in \tilde H^2(\Gamma_j^0)$, $\hat g^2_j \in \tilde H^2(\Gamma_j^0)$ and are equal to zero for $r > R$, we deduce that $\chi^j_{j,r} u^\varepsilon_{j,r}, \chi^j_{j,r} \frac{\partial u^\varepsilon_{j,r}}{\partial r} \in L^\infty(S_j)$ and

$$
\chi_j \frac{1}{r} \frac{\partial u^\varepsilon_{j,r}}{\partial \theta} = \chi_j \left( \frac{\hat g^1_j(r) - \hat g^2_j(0)}{r} \frac{1}{\omega_j} - \frac{\hat g^2_j(r) - \hat g^2_j(0)}{r} \frac{1}{\omega_j} \right) \in L^\infty(S_j).
$$

Consequently it holds

$$
\| \chi_j e^{-\alpha r} u^\varepsilon_{j,r}\|_{0,S_j} + \varepsilon \| \chi_j e^{-\alpha r} \nabla u^\varepsilon_{j,r} \|_{0,S_j} \lesssim e^{-\alpha r}\varepsilon \|_{0,S_j}.
$$

By the change of variable $\rho = \frac{r}{\omega_j}$, one has $\| e^{-\alpha r}\varepsilon \|_{0,S_j} \lesssim \varepsilon$ and the estimate (4.30) follows. The second assertion is proved similarly. □

### 5. The Full Decomposition

We are now ready to formulate the main result of this paper.

**Theorem 5.1.** Assume that $f_+$ is the restriction to $\Omega_+$ of a smooth function $\tilde f_+ \in \mathcal{C}^\infty(\mathbb{R}^2)$ and that $\Omega_+$ is convex. Write for shortness

$$
U_+ = \sum_{k=1}^N \chi_k e^{-\sin \omega_k z} u_+^k.
$$

Then the unique solution $u^\varepsilon \in H^1_0(\Omega)$ of (3.4) admits the splitting

$$
u^\varepsilon_+ = f_+ + \sum_{j=2}^N \chi_j v^j_+ + \chi v^i + U_+ + r^\varepsilon_+ \quad \text{in } \Omega_+,
$$

$$
u^\varepsilon_- = u^0_- + r^\varepsilon_- \quad \text{in } \Omega_-,
$$

Since the function $e^{-\frac{\varepsilon}{\alpha_j} r} \sigma_{j,k}$ behaves like $e^{-\frac{\varepsilon}{\alpha_j} r} \lambda_k - |\gamma|$ at 0 and at $\infty$, we have

$$
\| \chi e^{-\frac{\varepsilon}{\alpha_j} r} \sigma_{j,k}\|_{0,S_j} \lesssim \| \chi e^{-\frac{\varepsilon}{\alpha_j} r} \lambda_k - |\gamma| \|_{0,S_j}.
$$

For $|\gamma| \leq 1 < \lambda_k + 1$, by the scaling $\rho_j = \frac{r}{\varepsilon}$, we obtain

$$
\| \chi e^{-\frac{\varepsilon}{\alpha_j} r} \lambda_k - |\gamma| \|_{0,S_j} \lesssim \int_0^\infty r^{2(\lambda_k - |\gamma|)} e^{-2\frac{\varepsilon}{\alpha_j} r} dr
$$

$$
= \varepsilon^{2(\lambda_k - |\gamma|)} \int_0^\infty \rho^{2(\lambda_k - |\gamma|)} e^{-2\rho} \rho d\rho
$$

$$
\lesssim \varepsilon^{2(\lambda_k - |\gamma|)}.
$$

The estimate (4.27) follows directly from (4.28) and (4.29). The regularity of $\Delta(\chi e^{-\frac{\varepsilon}{\alpha_j} r} u^\varepsilon_{1,r}(r_j,\theta_j)) \in L^p(S_j)$ is proved in a similar manner.
where \( r^ε \in H^1_0(\Omega) \) is the variational solution of
\[
\int_{\Omega_+} (\varepsilon^2 \nabla r^ε_+ \cdot \nabla v_+ + r^ε_+ v_+) + \int_{\Omega_-} (\nabla r^ε_- \cdot \nabla v_- + r^ε_- v_-) = \int_{\Omega_+} f^ε v + \int_{\Omega_+} (\varepsilon^2 \nabla U_c \cdot \nabla v_+ + U_c v_+), \quad \forall v \in H^1_0(\Omega),
\]
where \( f^ε = L_ε(u^ε + U_+) \) and \( h^ε = \varepsilon^2 \frac{\partial}{\partial ν}(f^ε + U_c) \). Moreover,
\[
\varepsilon \|\nabla r^ε_+\|_{0,\Omega_+} + \|r^ε_+\|_{0,\Omega_+} + \|r^ε_-\|_{1,\Omega_-} \lesssim \varepsilon.
\]

**Proof.** By construction, \( r^ε \) clearly belongs to \( H^1_0(\Omega) \), and satisfies \( \Delta r^ε_\pm \in L^p(\Omega_\pm) \), for some \( p \in (1,2) \). Therefore applying \cite{3} Theorem 1.5.3.11, we may write
\[
\int_{\Omega_+} \mathcal{L}_ε r^ε_+ v_+ + \int_{\Omega_-} (\varepsilon^2 \nabla U_c \cdot \nabla v_+ + U_c v_+) = \int_{\Omega_+} \nabla r^ε_+ \cdot \nabla v_+ + \int_{\Omega_-} (\varepsilon^2 \nabla U_c \cdot \nabla v_+ + U_c v_+ - \varepsilon^2 \frac{\partial}{\partial ν}(f^ε + U_c)) H^1_0(\Omega_\pm) - H^1_0(\Sigma) \), \quad \forall v \in \mathcal{D}(\Omega).
\]

We remark that the splitting \((5.1)\) means that
\[r^ε = u^ε + U_+ - U_c.\]

Since Lemmas \cite{4.1} and \cite{4.2} guarantees that \( U_c \in H^1(\Omega_+) \) and \( \Delta U_c \in L^p(\Omega_+) \), for some \( p \in (1,2) \), again the application of \cite{3} Theorem 1.5.3.11 yields
\[
\int_{\Omega_+} \mathcal{L}_ε U_c v_+ = \int_{\Omega_+} (\varepsilon^2 \nabla U_c \cdot \nabla v_+ + U_c v_+) - (\varepsilon^2 \frac{\partial}{\partial ν} U_c + v) H^1_0(\Omega_\pm) - H^1_0(\Sigma) \), \quad \forall v \in \mathcal{D}(\Omega).
\]

Inserting this expression in \((5.4)\), we obtain \((5.2)\) since \( \mathcal{D}(\Omega) \) is dense in \( H^1_0(\Omega) \).

Now taking \( v = r^ε \) in \((5.2)\), applying Cauchy-Schwarz’s inequality and a trace theorem (in \( \Omega_- \)), we get
\[
\varepsilon \|\nabla r^ε_+\|_{0,\Omega_+} + \|r^ε_+\|_{0,\Omega_+} + \|r^ε_-\|_{1,\Omega_-} \lesssim \|f^ε\|_{0,\Omega_+} + \|h^ε\|_{0,\Sigma} + \varepsilon \|\nabla U_c\|_{0,\Omega_+} + \|U_c\|_{0,\Omega_+}.
\]

The estimate \((5.3)\) follows from this one if we can show that each term of this right-hand side is bounded by \( \varepsilon \). The first term is estimate with the help of \((4.15)\).

For the second term, due to \((4.14)\), we may write
\[
h^ε = -\varepsilon^2 \frac{\partial}{\partial ν}(f^ε + \chi^b_{N} v^b_{N} + \chi^i v^i).
\]

Now by \((4.7)\) we remark that
\[
\left| \frac{\partial}{\partial ν} f^ε \right| \lesssim 1,
\]
\[
\left| \frac{\partial}{\partial ν} (\chi^b_{N} v^b_{N}) \right| = \left| \frac{\partial \chi^b_{N}}{\partial ν} v^b_{N} + \frac{\partial v^b_{N}}{\partial ν} \chi^b_{N} \right| \lesssim \frac{1}{\varepsilon},
\]
\[
\left| \frac{\partial}{\partial ν} (\chi^i v^i) \right| = \left| \frac{\partial \chi^i}{\partial ν} v^i + \frac{\partial v^i}{\partial ν} \chi^i \right| \lesssim \frac{1}{\varepsilon}.
\]

These estimates lead to \( \|h^ε\|_{0,\Sigma} \lesssim \varepsilon \). Finally for the last terms of the right-hand side, using \((4.27)\), \((4.30)\) and Leibniz’s rule, we get
\[
\varepsilon \|\nabla U_c\|_{0,\Omega_+} + \|U_c\|_{0,\Omega_+} \lesssim \varepsilon.
\]

□
References


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