Electronic Journal of Differential Equations, Vol. 2006(2006), No. 141, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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PERIODIC SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PERIODIC DELAY CLOSE TO ZERO

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Abstract. This paper studies the existence of periodic solutions to the delay differential equation

$$\dot{x}(t) = f(x(t-\mu\tau(t)), \epsilon)\,.$$

The analysis is based on a perturbation method previously used for retarded differential equations with constant delay. By transforming the studied equation into a perturbed non-autonomous ordinary equation and using a bifurcation result and the Poincaré procedure for this last equation, we prove the existence of a branch of periodic solutions, for the periodic delay equation, bifurcating from $\mu = 0$.

1. Introduction

Let us consider the periodic delay differential equations of the form

$$\dot{u}(t) = f(u(t-\mu\tau(t)), \epsilon), \tag{1.1}$$

under the following assumptions:

(H1) $f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2)$, $f(0,\epsilon) = 0$, and $f'_u(0,\epsilon) = \begin{pmatrix} 0 & -\beta_1(\epsilon) \\ \beta_1(\epsilon) & 0 \end{pmatrix}$ where $\beta_1(\epsilon) > 0$ and satisfies $\beta_1(1) = 1$ and $\beta_1'(1) \neq 0$. Moreover, f and its first and second derivatives are bounded so that there is a number $A > 0$, such that $\max(\|f\|_\infty, \|f'_u\|_\infty, \|f''_u\|_\infty) < A$.

(H2) $\tau \in C^1(\mathbb{R}, \mathbb{R}^+)$ is 2π-periodic in t and $\int_0^{2\pi} \tau(s)ds \neq 0$.

(H3) The system

$$\dot{u}(t) = f(u(t), \epsilon)$$

is 3-asymptotically stable for $|\epsilon - 1|$ sufficiently small.

Also we assume that $\mu > 0$ and ϵ are parameters having values in a neighborhood of 0 and 1, respectively.

When the function τ is independent of t (i.e: $\tau(t) = \tau > 0$), system (1.1) is an autonomous equation which is extensively studied in [1, 5, 6, 8, 9, 11]. The aim of this paper is to prove the existence of a branch of a bifurcated periodic solutions for the differential equation (1.1) with periodic delay in the case where μ is small enough.

2000 *Mathematics Subject Classification.* 34K13.
Key words and phrases. Differential equation; periodic delay; bifurcation;
h-asymptotic stability; periodic solution.

Submitted August 25, 2006. Published November 9, 2006.

The problem studied here is local in nature. The work is in line with a previous work by Arino and Hbid [1] in which the delay was assumed to be constant and small. Smallness of the delay was an essential feature which made possible the study of the equation as a perturbation of an ordinary differential equation (ODE). It was also possible to go a little further than the Hopf bifurcation and extend to this situation results obtained previously by Bernfeld and Salvadori [2] on generalized Hopf bifurcation for ODEs. The authors have used a perturbation method to transform the functional differential equation into an ODE. A Poincaré map was constructed in a neighborhood of the bifurcating periodic solutions of the ordinary differential system. The fixed points of this map correspond to periodic solutions of the functional differential equations.

In this paper we proceed in the same general spirit as in [1]. Though our approach could be viewed as a simple adaptation of the one described in [1], some specific features related to the dependence of the delay on the time t are to be mentioned: the main one is that the perturbed method applied here transforms the time dependent delay equation into a non-autonomous ordinary equation. Under an additional hypothesis on h-asymptotic stability (see for instance [3]) the non-autonomous ODE has an attractive bifurcating branch of periodic solutions. A closed bounded convex subset of the space of Lipschitz continuous functions is constructed in the neighborhood of this branch. Finally, we set up a Poincaré map which transforms the convex set into itself. The Poincaré map being eventually compact in the space of Lipschitz continuous functions, has fixed points which yield periodic solutions of the retarded differential equation with periodic delay (1.1).

2. Background

In this section we recall some aspects of bifurcation given in [3] for the periodic ordinary system

$$\dot{z} = g(t, z, \mu, \epsilon) \tag{2.1}$$

where $g \in C^\infty(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g(t, 0, \mu, \epsilon) = 0$ and 2π-periodic in t. μ and ϵ are parameters and have values respectively in a neighborhood of 0 and 1. Because of Floquet theory the Jacobian matrix $f'_z(t, 0, \mu, \epsilon)$ may be assumed without loss of generality to be independent of t and its eigenvalues will be denoted by $\alpha(\mu, \epsilon) \pm i\beta(\mu, \epsilon)$. We will assume that

$$\begin{gathered} \alpha(0, \epsilon) = 0, \quad \beta(0, 1) = 1 \\ \alpha'_\mu(0, 1) \neq 0 \quad \beta'_\epsilon(0, 1) \neq 0. \end{gathered}$$

By a linear transformation of z independent of t, and involving μ, ϵ, Equation (2.1) may be written as

$$\begin{aligned} \dot{x} &= \alpha(\mu, \epsilon)x - \beta(\mu, \epsilon)y + X(t, x, y, \mu, \epsilon) \\ \dot{y} &= \alpha(\mu, \epsilon)y + \beta(\mu, \epsilon)x + Y(t, x, y, \mu, \epsilon), \end{aligned} \tag{2.2}$$

where $X, Y \in C^\infty$ in (x, y, μ, ϵ) and 2π-periodic in t, and X, Y are $O(x^2 + y^2)$.

We remark that there exist a neighborhood N of $(x, y) := (0, 0)$ and three positive numbers a, b, ω such that for any $t_0 \in \mathbb{R}$, $(x_0, y_0) \in N$, $\mu < b$, $|\epsilon - 1| < a$ the solution $(x(t), y(t))$ of (2.2) through $(t_0, x_0, y_0, \mu, \epsilon)$ exists in the interval $[t_0, t_0 + 2\pi]$ and for the corresponding angle $\theta(t)$ we have $|\dot{\theta}(t)| > \tau$. We denote by $(x(t,t_0,c,\mu,\epsilon),y(t,t_0,c,\mu,\epsilon))$ the solution of (2.2) such that $x_0=c>0, y_0=0, (c,0)\in N$.

Lemma 2.1 ([3]). *There exist three positive numbers $\overline{a},\overline{b},\overline{c},(\overline{c},0)\in\mathbb{N}$, and a function $\overline{\epsilon}\in C^\infty(\mathbb{R}\times[0,\overline{c}]\times[-\overline{b},\overline{b}],[1-\overline{a},1+\overline{a}])$, $\overline{\epsilon}(t_0,0,0)=1$, such that for any $t_0\in\mathbb{R}, c\in[0,\overline{c}], \mu\in[0,\overline{b}]$, and $|\epsilon-1|<\overline{a}$ the equation $y(t_0+2\pi,t_0,c,\mu,\epsilon)=0$ is satisfied if and only if $\epsilon=\overline{\epsilon}(t_0,c,\mu)$.*

Consider now the function $V\in C^\infty(\mathbb{R}\times[0,\overline{c}]\times[-\overline{b},\overline{b}],\mathbb{R})$ defined by

$$V(t_0,c,\mu)=x(t_0+2\pi,t_0,c,\mu,\overline{\epsilon}(t_0,c,\mu))-c. \tag{2.3}$$

Clearly the 2π-periodic solutions of (2.2) relative to any triplet (c,μ,ϵ) for which $c\in[0,\overline{c}], |\mu|\in[0,\overline{b}]$, and $|\epsilon-1|\in[0,\overline{a}]$ correspond to the zeros of $V(t_0,c,\mu)$. We will call V the displacement function. The following theorem holds.

Theorem 2.2 ([3]). *Suppose that $\overline{a},\overline{b},\overline{c}$ are sufficiently small. Assume that there exist two functions $\mu^*\in C^\infty(\mathbb{R}\times[0,\overline{c}],[-\overline{b},\overline{b}]), \epsilon^*\in C^\infty(\mathbb{R}\times[0,\overline{c}],[1-\overline{a},1+\overline{a}])$ such that if $t_0\in\mathbb{R}, c\in[0,\overline{c}]$, $|\mu^*|\in[0,\overline{b}], |\epsilon-1|\in[0,\overline{a}]$. Then the solution $(x(t,t_0,c,\mu,\epsilon),y(t,t_0,c,\mu,\epsilon))$ of (2.2) is 2π-periodic if and only if $\mu=\mu^*(t_0,c), \epsilon=\epsilon^*(t_0,c)$. Moreover $\epsilon^*(t_0,c)=\overline{\epsilon}(t_0,c,\mu^*(t_0,c))$.*

We will assume also that the functions X and Y are independent of t when $\mu=0$. Then system (2.2) may be written as

$$\begin{aligned}\dot x&=\alpha(\mu,\epsilon)x-\beta(\mu,\epsilon)y+\tilde X(x,y,\epsilon)+\mu X^*(t,x,y,\mu,\epsilon)\\ \dot y&=\alpha(\mu,\epsilon)y+\beta(\mu,\epsilon)x+\tilde Y(x,y,\epsilon)+\mu Y^*(t,x,y,\mu,\epsilon).\end{aligned} \tag{2.4}$$

which for $\mu=0$ has the form

$$\begin{aligned}\dot x&=-\beta(0,\epsilon)y+\tilde X(x,y,\epsilon)\\ \dot y&=\beta(0,\epsilon)x+\tilde Y(x,y,\epsilon).\end{aligned} \tag{2.5}$$

Definition 2.3 ([3]). Let $h\in\mathbb{N}$. The solution $\xi=0$ of system (2.5) is said to be h-asymptotically stable (resp. h-completely unstable) if the following conditions are satisfied:

1. For all $\tau_1,\tau_2\in C(\mathbb{R}^2,\mathbb{R})$ of order h, the solution 0 of system
$$\begin{aligned}\dot x&=-\beta(0,\epsilon)y+\tilde X(x,y,\epsilon)+\tau_1(x,y)\\ \dot y&=\beta(0,\epsilon)x+\tilde Y(x,y,\epsilon)+\tau_2(x,y).\end{aligned}$$
is asymptotically stable (resp.unstable).
2. h is the smallest integer such that the property (1) above is satisfied.

We have the following equivalence between h-asymptotic stability and the existence of an appropriate polynomial in (x,y). This polynomial may be determined by an algebraic procedure due to Poincaré.

Proposition 2.4 ([3]). *The origin of (2.5) is h-asymptotically stable if and only if h is odd and there exists a polynomial in (x,y), $F(x,y,\epsilon)$, of degree $h+1$ having the form*

$$F(x,y,\epsilon)=x^2+y^2+f_3(x,y,\epsilon)+\dots+f_{h+1}(x,y,\epsilon)$$

(f_i is homogeneous of degree i in (x,y)) such that the derivative along the solutions of (2.5) is given by

$$\dot F(x,y,\epsilon)=G_{h+1}(\epsilon)(x^2+y^2)^{(h+1)/2}+o((x^2+y^2)^{(h+1)/2}).$$

Here $G_{h+1}(\epsilon) < 0$ is a constant.

The explicit constant $G_{h+1}(\epsilon)$, called Lyapunov constant, can be obtained for each odd integer h by the following proposition

Proposition 2.5 ([11]). *Given an even integer $k > 2$, the Poincaré constant $G_k(\epsilon)$ is given in a unique way by*

$$G_k(\epsilon) = \frac{p_k + p_{k-2}/(k-1)\beta(0,\epsilon) + \sum_{s=1}^{k/2-1} c_s d_s}{\frac{k}{2(k-1)\beta(0,\epsilon)} + \sum_{s=1}^{k/2-1} c_s \frac{C_{s+1}^{k/2}}{(k-2s+1)\beta(0,\epsilon)} + 1},$$

where $c_s = \frac{3\times 5\times 7\cdots\times(2s+1)}{(k-1)\times(k-3)\cdots\times(k-2s+1)}$, $d_s = \frac{p_{k-2s-2}}{(k-2s-1)}$ for all $s \in \{1, \dots, \frac{k}{2}-1\}$. and the terms $p_j, j = 0..k$ are given by

$$\rho_k(\xi_1,\xi_2) = \sum_{j=0}^{k} p_j \xi_1^{k-j}\xi_2^j$$

with $\rho_j(\xi_1,\xi_2)$ is the homogeneous part of degree j of the function $\rho(\xi_1,\xi_2)$ given by

$$\rho(\xi_1,\xi_2) = \tilde{X}(\xi_1,\xi_2,\epsilon)\frac{\partial}{\partial\xi_1}(\sum_{l=3}^{j-1} f_l(\xi_1,\xi_2,\epsilon)) + \tilde{Y}(\xi_1,\xi_2,\epsilon)\frac{\partial}{\partial\xi_2}(\sum_{l=3}^{j-1} f_l(\xi_1,\xi_2,\epsilon)).$$

We have the following results.

Theorem 2.6 ([3]). *Suppose there exists an odd integer $h \geq 3$ such that the origin of (2.5) is h-asymptotically stable for every $\epsilon \in [1-\overline{a}, 1+\overline{a}]$. Then if $\alpha'(0,1) < 0$ (resp $\alpha'(0,1) > 0$) the bifurcating 2π-periodic solutions of (2.4) occur for $\mu > 0$ (resp $\mu < 0$). Moreover the positive numbers $\overline{a}$, $\overline{b}$, $\overline{c}$ of Theorem 2.2 can be chosen such that for any $t_0 \in \mathbb{R}$ and $\mu \in [0,\overline{b}]$ (resp $\mu \in [-\overline{b}, 0]$) there exists one and only one $c \in [0,\overline{c}]$ such that $\mu = \mu^*(t_0, c)$.*

3. Main result

In the sequel, we transform equation (1.1) into a periodic ODE perturbed by a small time dependent-delay term.

We define $\tau_\infty := \sup\{|\tau(t)| : t \in [0, 2\pi]\}$ and C the space of continuous functions from $[-\mu\tau_\infty, 0]$ to $\mathbb{R}^2$, then we have the following result

Proposition 3.1. *Under hypothesis (H2), the periodic delay system (1.1) can be written in the form*

$$\dot{u}(t) = g(t, u(t), \mu, \epsilon) + H(t, u_t, \mu, \epsilon),$$

where

$$g(t, u, \mu, \epsilon) := \left[I + \mu\tau(t) f'_u(u,\epsilon)\right]^{-1} f(u,\epsilon)$$

and H a function defined on $\mathbb{R} \times C^1 \times \mathbb{R} \times \mathbb{R}$ and satisfies

$$\begin{aligned} &H(t, u_t, \mu, \epsilon) \\ &= \left[I + \mu\tau(t) f'_u(u(t),\epsilon)\right]^{-1} \int_{-\mu\tau(t)}^{0} [f'_u(u(t),\epsilon)\dot{u}(t) - f'_u(\dot{u}(t+\sigma),\epsilon)\dot{u}(t+\sigma)]d\sigma \end{aligned}$$

for all solutions u of system (1.1).

Proof. Let u be a solution of equation (1.1). We have

$$f(u(t-\mu\tau(t)),\epsilon) = f(u(t),\epsilon) - \int_{-\mu\tau(t)}^{0} f'_u(u(t+\sigma),\epsilon)\dot{u}(t+\sigma)d\sigma$$

Then

$$\begin{aligned} f(u(t-\mu\tau(t)),\epsilon) &= f(u(t),\epsilon) - \mu\tau(t)f'_u(u(t),\epsilon)\dot{u}(t) \\ &\quad + \int_{-\mu\tau(t)}^{0} [f'_u(u(t),\epsilon)\dot{u}(t) - f'_u(u(t+\sigma),\epsilon)\dot{u}(t+\sigma)]d\sigma. \end{aligned}$$

Since u is a solution of equation (1.1), we obtain

$$\begin{aligned} &\big(I + \mu\tau(t)f'_u(u(t),\epsilon)\big)f(u(t-\mu\tau(t),\epsilon) \\ &= f(u(t),\epsilon) + \int_{-\mu\tau(t)}^{0} [f'_u(u(t),\epsilon)\dot{u}(t) - f'_u(u(t+\sigma),\epsilon)\dot{u}(t+\sigma)]d\sigma. \end{aligned}$$

For μ small enough, the matrix $\big(I + \mu\tau(t)f'_u(u(t),\epsilon)\big)$ is invertible and we can write

$$f(u(t-\mu\tau(t),\epsilon) = g(t,u(t),\mu,\epsilon) + H(t,u_t,\mu,\epsilon)$$

with g and H as defined above. □

In the sequel, the equation under study is

$$\dot{u}(t) = g(t,u(t),\mu,\epsilon) + H(t,u_t,\mu,\epsilon). \tag{3.1}$$

In what follows we look for periodic solutions of the following 2-dimensional system

$$\dot{w}(t) = g(t,w(t),\mu,\epsilon). \tag{3.2}$$

Theorem 3.2. *Suppose (H1)–(H3) hold. Then there exists a sufficiently small positive numbers $\overline{a},\overline{b},\overline{c}$, and there exist two functions μ^* in $C^\infty(\mathbb{R}\times[0,\overline{c}],[0,\overline{b}])$, and ϵ^* in $C^\infty(\mathbb{R}\times[0,\overline{c}],[1-\overline{a},1+\overline{a}])$ such that if $t_0\in\mathbb{R}$, $\mu\in[0,\overline{b}]$, $|\epsilon-1|\in[0,\overline{a}]$, then there exists one and only one $c\in[0,\overline{c}]$, such that the solution*

$$(w_1(t,t_0,c,\mu,\epsilon), w_2(t,t_0,c,\mu,\epsilon))$$

of (3.2) is 2π-periodic if and only if $\mu=\mu^*(t_0,c), \epsilon=\epsilon^*(t_0,c)$. Moreover the family of the bifurcating solutions are of amplitude of order $\sqrt{\mu}$.

Proof. We first show the conditions imposed in [3]: We have $g(t,u(t),0,\epsilon) := f(u(t),\epsilon)$, and the eigenvalues of the Jacobian matrix $g'_w(t,0,\mu,\epsilon)$ have the form $\alpha(\mu,\epsilon)\pm i\beta(\mu,\epsilon)$, where $\alpha(0,\epsilon)=0$ and $\beta(0,\epsilon)=\beta_1(\epsilon)$, then from hypothesis (H1) we have $\alpha(0,\epsilon)=0, \beta(0,1)=1$ and $\beta'_\epsilon(0,1)\neq 0$, it remains to prove that $\alpha'_\mu(0,1)\neq 0$, however, $\lambda(\mu) := \alpha(\mu,1)+i\beta(\mu,1)$ is the characteristic exponent of the Jacobian matrix $g'_w(t,0,\mu,1) = [I+\mu\tau(t)f'_u(0,1)]^{-1}f'_u(0,1)$, and with a few computations, we obtain that

$$\frac{\partial}{\partial\mu}g'_w(t,0,\mu,1) = -\tau(t)[[I+\mu\tau(t)f'_u(0,1)]^{-1}f'_u(0,1)]^2,$$

then

$$\big[\frac{\partial}{\partial\mu}g'_w(t,0,\mu,1)\big]_{\mu=0} = -\tau(t)\beta_1^2(1)I,$$

and because of the regularity of $\lambda(.)$, we deduce that $\lambda'_\mu(0)$ is the characteristic exponent of the matrix $-\tau(t)\beta_1^2(1)I$, that's $\alpha'_\mu(0,1) = \frac{\beta_1^2(1)}{2\pi}\int_0^{2\pi}\tau(s)ds$, and by hypothesis (H2) we have that $\alpha'_\mu(0,1) > 0$. Then the first part of theorem is a consequence of Theorems 2.2 and 2.6. If the hypothesis (H3) is satisfied then the origin of (3.2) is 3−asymptotically stable, consequently for any $t_0 \in \mathbb{R}$ we have (see the proof of Theorem 2.6 in [3]):

$$\frac{\partial V}{\partial c}(t_0,0,0) = \frac{\partial^2 V}{\partial c^2}(t_0,0,0) = 0 \quad \text{and} \quad \frac{\partial^3 V}{\partial c^3}(t_0,0,0) < 0,$$

Moreover, we have

$$\frac{\partial^2 \mu^*}{\partial c^2}(t_0,0) = -\frac{1}{6\pi\alpha'_\mu(0,1)}\frac{\partial^3 V}{\partial c^3}(t_0,0,0) > 0,$$

so by developing the function $c \mapsto \mu^*(t_0,c)$ in a neighborhood of zero, we obtain

$$\mu^*(t_0,c) = \frac{1}{2}\frac{\partial^2 \mu^*}{\partial c^2}(t_0,0)c^2 + o(c^2).$$

Then $\mu^*(t_0,c)$ is of order c^2, however, the first part of the theorem tells us that the map $c \mapsto \mu^*(t_0,c)$ is injective, consequently the inverse function $c(t_0,\mu)$ of $\mu^*(t_0,.)$ is of order $\sqrt{\mu}$. This shows the second part of the theorem. □

Remark 3.3. According to the above theorem, for a given μ and t_0 there is one and only one periodic solution of (3.2). Precisely, this periodic solution is obtained by assuming $\epsilon = \epsilon_1(t_0,\mu)$, where $\epsilon_1(t_0,\mu) = \epsilon^*(t_0, c(t_0,\mu))$, in (3.2).

In the sequel, we let $t_0 = 0$ and we assume that $\epsilon = \epsilon_1(0,\mu)$ for any μ in equation (1.1). Denote by $y(\mu) := (y_1(\mu), 0)$ the initial data of the bifurcating periodic solutions of (3.2). From theorem 3.2 we see that there exists a constant $C > 0$ such that $\|y(\mu)\| \le C\mu^{1/2}$. Let $u(\phi)$ be the solution of (1.1) with initial data $u_0 = \phi$. From lemma 2.1 and remark 3.3 one can find a solution $w^*$of (3.2) such that $w^*(0) = \phi(0)$, $w_2^*(2\pi) = 0$ and $w_1^*(2\pi) > 0$.

To state the nest proposition, we introduce the subset

$$\mathcal{B}(\mu) := \{\phi \in C^1 : \|\phi(s) - y(\mu)\| \le C\mu^{3/2}\}.$$

Proposition 3.4. *Under the hypothesis (H1)–(H3), there exists a constant $C_1 > 0$, such that for a given $T > 0$ and μ close to zero, we have*

$$\|u(\phi)(t)\| \le C_1\mu^{1/2}$$

for all $\phi \in \mathcal{B}(\mu)$ and $t \in [0,T]$.

Proof. let $\tau_0 := \inf\{\tau(t) : t \in [0,2\pi]\}$, $t \in [0,\mu\tau_0]$, then $t - \mu\tau(t) \le 0$, and

$$u(\phi)(t) = \phi(0) + \int_0^t f(\phi(s - \mu\tau(s)))ds$$

it follows that

$$\|u(\phi)(t)\| \le \|\phi\|_\infty + \mu\tau_0 A\|\phi\|_\infty \le C(1 + \mu\tau_0 A)\mu^{1/2}.$$

In a similar manner, we show by iteration that for $t \in [0, k\mu\tau_0]$, we have

$$\|u(\phi)(t)\| \le C(1 + \mu\tau_0 A)^k\mu^{1/2}.$$

Let k the unique natural integer such that $k\mu\tau_0 \le T < (k+1)\mu\tau_0$, then for $t \in [0,T]$ we have

$$\|u(\phi)(t)\| \le C(1 + \mu\tau_0 A)^k\mu^{1/2} \le Ce^{\mu\tau_0 A(k+1)}\mu^{1/2} \le Ce^{(\mu\tau_0 A + AT)}\mu^{1/2}.$$

Finally, for μ close to zero, one obtain a constant C_1 independent of μ such that

$$\|u(\phi)(t)\| \leq C_1\mu^{1/2}.$$

□

Proposition 3.5. *Under the hypothesis (H1)–(H3), there exist positive constant C_2 such that for μ close to zero and $\phi \in \mathcal{B}(\mu)$, we have*

$$\|H(t, u_t, \mu, \epsilon)\| \leq C_2\mu^{5/2} \quad \text{for } t \in [3\mu\tau_\infty, T].$$

Moreover

$$\|H(t, u_t, \mu, \epsilon)\| \leq C_2\mu^{3/2} \quad \text{for } t \in [0, 3\mu\tau_\infty].$$

Proof. Note that from theorem 3.1, for $t \in [3\mu\tau_\infty, T]$, we have

$$\begin{aligned} &H(t, u_t, \mu, \epsilon) \\ &= [I + \mu\tau(t)f'_u(u(t), \epsilon)]^{-1} \int_{-\mu\tau(t)}^{0} [f'_u(u(t), \epsilon)\dot{u}(t) - f'_u(u(t+\sigma), \epsilon)\dot{u}(t+\sigma)]d\sigma. \end{aligned}$$

Using the inequality

$$\begin{aligned} &\|f'_u(u(t), \epsilon)\dot{u}(t) - f'_u(u(t+\sigma), \epsilon)\dot{u}(t+\sigma)\| \\ &\leq \|f'_u(u(t), \epsilon)\dot{u}(t) - f'_u(u(t+\sigma), \epsilon)\dot{u}(t)\| \\ &\quad + \|f'_u(u(t+\sigma), \epsilon)\dot{u}(t) - f'_u(u(t+\sigma), \epsilon)\dot{u}(t+\sigma)\|, \end{aligned}$$

we obtain

$$\begin{aligned} &\|f'_u(u(t), \epsilon)\dot{u}(t) - f'_u(u(t+\sigma), \epsilon)\dot{u}(t+\sigma)\| \\ &\leq A\|u(t+\sigma) - u(t)\|\|\dot{u}(t)\| + A\|\dot{u}(t+\sigma) - \dot{u}(t)\|. \end{aligned}$$

On the other hand, for $t \in [3\mu\tau_\infty, T]$ and $\sigma \in [-\mu\tau_\infty, 0]$, we have

$$\begin{aligned} \|u(t+\sigma) - u(t)\| &\leq -\sigma \sup_{s\in[t,t+\sigma]} \|\dot{u}(s)\| \\ &= -\sigma \sup_{s\in[t,t+\sigma]} \|f(u(s - \mu\tau(s))\| \\ &\leq -\sigma A \sup_{s\in[t,t+\sigma]} \|u(s)\| \\ &\leq -\sigma AC\mu^{1/2} := -\sigma A_1\mu^{1/2}, \end{aligned}$$

and

$$\begin{aligned} \|\dot{u}(t+\sigma) - \dot{u}(t)\| &\leq -\sigma \sup_{s\in[t,t+\sigma]} \|\ddot{u}(s)\| \\ &= -\sigma \sup_{s\in[t,t+\sigma]} \|f'_u(u(s - \mu\tau(s)), \epsilon)\dot{u}(s - \mu\tau(s))(1 - \mu\dot{\tau}(s))\| \\ &\leq -\sigma A^2C\mu^{1/2}(1 + \mu \sup_{s\in[0,2\pi]} \|\dot{\tau}(s)\| \leq -\sigma A_2\mu^{1/2}, \end{aligned}$$

for some constant $A_2 > 0$. This implies

$$\|f'_u(u(t), \epsilon)\dot{u}(t) - f'_u(u(t+\sigma), \epsilon)\dot{u}(t+\sigma)\| \leq -\sigma AA_1^2\mu - \sigma AA_2\mu^{1/2},$$

it follows that there exists a constant $C_2^* > 0$ such that for μ close to zero we have

$$\|\int_{-\mu\tau(t)}^{0} [f'_u(u(t), \epsilon)\dot{u}(t) - f'_u(u(t+\sigma), \epsilon)\dot{u}(t+\sigma)]d\sigma\| \leq \frac{C_2^*}{2}\mu^{5/2}.$$

On the other hand, for μ close to zero such that $\mu A\tau^{\infty} < \frac{1}{2}$ we obtain

$$\|[I+\mu\tau(t)f'_u(u(t),\epsilon)]^{-1}\| \leq 2.$$

Which prove the first inequality of the proposition.

Now let t be in the interval $[0, 3\mu\tau_{\infty}]$ and $u(t) = u(\phi)(t)$ for some $\phi \in \mathcal{B}(\mu)$. We have

$$g(t,u(t),\mu,\epsilon) = [I+\mu\tau(t)f'_u(u(t),\epsilon)]^{-1}f(u(t),\epsilon) = f(u(t),\epsilon) + \mu O(u(t)).$$

Using proposition 3.4 we obtain

$$g(t,u(t),\mu,\epsilon) = f(u(t),\epsilon) + O(\mu^{3/2}).$$

Then

$$H(t,u_t,\mu,\epsilon) = f(u(t-\mu\tau(t)),\epsilon) - f(u(t),\epsilon) + O(\mu^{3/2})$$

and

$$\|H(t,u_t,\mu,\epsilon)\| \leq \|f(u(t-\mu\tau(t)),\epsilon) - f(u(t),\epsilon)\| + O(\mu^{3/2}).$$

Since $f$is a smooth function, we deduce that

$$\begin{aligned}
\|H(t,u_t,\mu,\epsilon)\| &\leq A\|u(t-\mu\tau(t)) - u(t)\| + O(\mu^{3/2}) \\
&\leq A\int_{t-\mu\tau_\infty}^{t} \|\dot{u}(s)\|ds + O(\mu^{3/2}) \\
&\leq A\int_{t-\mu\tau_\infty}^{t} \|f((u(s-\mu\tau(s)),\epsilon)\|ds + O(\mu^{3/2}) \\
&\leq A^2\int_{t-\mu\tau_\infty}^{t} \|u(s-\mu\tau(s))\|ds + O(\mu^{3/2})
\end{aligned}$$

Using once more proposition 3.4 we deduce that there exist $C_2^{**} > 0$ such that

$$\|H(t,u_t,\mu,\epsilon)\| \leq C_2^{**}\mu^{3/2}$$

which completes the proof with $C_2 := \min(C_2^{*}, C_2^{**})$. □

As a result of proposition 3.1 and the above theorem, the equation (1.1) can be written as a perturbation of an ordinary differential equation by a small term.

We are now in position to give an estimation of the difference between $u(\phi)$ and w^*.

Lemma 3.6. *There exist a positive constant C_3 such that for all $\phi \in \mathcal{B}(\mu)$ and $t \in [0, 2\pi]$ we have*

$$\|u(\phi)(t) - w^*(t)\| \leq C_3\mu^{5/2}.$$

Proof. Let $\phi \in \mathcal{B}(\mu)$, we have

$$\frac{d}{dt}[\dot{u}(\phi)(t) - \dot{w}^*(t)] = g(t,u(\phi)(t),\mu,\epsilon) + H(t,u_t(\phi),\mu,\epsilon) - g(t,w^*(t),\mu,\epsilon),$$

then from hypothesis (H1) and (H2) and using the inner product in $\mathbb{R}^2$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u(\phi)(t) - w^*(t)\|^2 \\
&\leq 2A^2\|u(\phi)(t) - w^*(t)\|^2 + \|u(\phi)(t) - w^*(t)\|\|H(t,u_t(\phi),\mu,\epsilon)\|,
\end{aligned}$$

form which it follows that

$$D^+\|u(\phi)(t) - w^*(t)\| \leq 2A^2\|u(\phi)(t) - w^*(t)\| + \|H(t,u_t(\phi),\mu,\epsilon)\|,$$

where D^+ denotes the derivative from the right. Using the Gronwall's inequality and in view of $u(\phi)(0) = \phi(0) = w^*(0)$, we obtain

$$\|u(\phi)(t) - w^*(t)\| \leq \int_0^t e^{2A^2(t-s)}\|H(s, u_s(\phi), \mu, \epsilon)\|,$$

then

$$\begin{aligned}
&\|u(\phi)(t) - w^*(t)\| \\
&\leq \int_0^{3\mu\tau_\infty} e^{2A^2(t-s)}\|H(s, u_s(\phi), \mu, \epsilon)\| + \int_{3\mu\tau_\infty}^t e^{2A^2(t-s)}\|H(s, u_s(\phi), \mu, \epsilon)\|.
\end{aligned}$$

From proposition 3.5, we have

$$\int_0^{3\mu\tau_\infty} e^{2A^2(t-s)}\|H(s, u_s(\phi), \mu, \epsilon)\| \leq \int_0^{3\mu\tau_\infty} e^{2A^2(t-s)}O(\mu^{3/2}) \leq \frac{C_3}{2}\mu^{5/2}$$

and

$$\int_{3\mu\tau_\infty}^t e^{2A^2(t-s)}\|H(s, u_s(\phi), \mu, \epsilon)\| \leq \int_{3\mu\tau_\infty}^t e^{2A^2(t-s)}O(\mu^{5/2}) \leq \frac{C_3}{2}\mu^{5/2}$$

for some constant $C_3 > 0$. Thus

$$\|u(\phi)(t) - w^*(t)\| \leq C_3\mu^{5/2}$$

□

Lemma 3.7. *For any $t \in [2\pi - \mu\tau_\infty, 2\pi]$ and any $\phi \in \mathcal{B}(\mu)$, we have*

$$\|u(\phi)(t) - u(\phi)(2\pi)\| \leq C_4\mu^{3/2}$$

where C_4 is a positive constant independent of μ.

Proof. Let $t \in [2\pi - \mu\tau_\infty, 2\pi]$, μ close to zero such that $\mu < \frac{2\pi}{\tau_\infty}$ and $\phi \in \mathcal{B}(\mu)$, we have

$$\begin{aligned}
\|u(\phi)(t) - u(\phi)(2\pi)\| &\leq \mu\tau_\infty \sup_{s\in[2\pi-\mu\tau_\infty, 2\pi]} \|\frac{d}{ds}u(\phi)(s)\| \\
&\leq \mu\tau_\infty \sup_{s\in[2\pi-\mu\tau_\infty, 2\pi]} \|f(u(s - \mu\tau(s))\| \\
&\leq \mu\tau_\infty A \sup_{\sigma\in[0,2\pi]} \|u(\sigma)\| \leq \mu\tau_\infty AC_1\mu^{1/2} := C_4\mu^{3/2}.
\end{aligned}$$

Which completes the proof. □

Proposition 3.8. *Assume (H1)–(H3) are satisfies, then there exists $K_1 > 0$ such that for μ close to zero and $\phi \in \mathcal{B}(\mu)$, we have*

$$\|w^*(2\pi) - y(\mu)\| \leq C[1 - K_1|y(\mu)|^2]\mu^{3/2}.$$

Proof. Put $c' := w_1^*(0)$ and $c := y_1(\mu)$, we have

$$\|w^*(2\pi) - y(\mu)\| = |V(0, c', \mu(c))) + c' - c|.$$

On the other hand we have

$$V(0, c', \mu(c)) = V(0, c, \mu(c)) + \frac{\partial}{\partial c}V(0, \eta, \mu(c)(c' - c)$$

for some $\eta \in]\min(c, c'), \max(c, c')[$ and

$$\frac{\partial}{\partial c}V(0, \eta, \mu(c)) = \frac{\partial}{\partial c}V(0, \eta, 0) + \frac{\partial^2 V(0, \eta, v_0\mu(c))}{\partial\mu\partial c}\mu(c)$$

for some $v_0 \in]0,1[$. However, we have $V(0,c,\mu(c)) = 0$ since $c := y(\mu)$ the initial data of the bifurcating periodic solutions of (3.2), then

$$V(0,c',\mu(c)) = [\frac{\partial}{\partial c}V(0,\eta,0) + \frac{\partial^2 V(0,\eta,v_0\mu(c))}{\partial\mu\partial c}\mu(c)](c'-c).$$

According to the 3-asymptotic stability, we have

$$\frac{\partial V}{\partial c}(0,0,0) = \frac{\partial^2 V}{\partial c^2}(0,0,0) = 0 \quad \text{and} \quad \frac{\partial^3 V}{\partial c^3}(0,0,0) < 0.$$

Moreover, we have

$$\mu^*(0,0) = 0, \frac{\partial \mu^*}{\partial c}(0,0) = 0 \quad \text{and} \quad \frac{\partial^2 \mu^*}{\partial c^2}(0,0) = -\frac{1}{3}\frac{\partial^3 V}{\partial c^3}(0,0,0)/\frac{\partial^2}{\partial\mu\partial c}V(0,0,0),$$

then

$$\frac{\partial V}{\partial c}(0,\eta,0) = \frac{1}{2!}\frac{\partial^3 V}{\partial c^3}(0,0,0)\eta^2 + o(\eta^2)$$

and

$$\begin{aligned}\frac{\partial^2 V(0,\eta,v_0\mu(c))}{\partial\mu\partial c}\mu(c) &= \frac{1}{2!}\frac{\partial^2 V(0,0,0)}{\partial\mu\partial c}\frac{\partial^2\mu^*}{\partial c^2}(0,0)c^2 + o(c^2)\\ &= -\frac{1}{6}\frac{\partial^3 V}{\partial c^3}(0,0,0)c^2 + o(c^2),\end{aligned}$$

it follows that

$$\begin{aligned}&\frac{\partial}{\partial c}V(0,\eta,0) + \frac{\partial^2 V(0,\eta,v_0\mu(c))}{\partial\mu\partial c}\mu(c)\\ &= \frac{1}{2!}\frac{\partial^3 V}{\partial c^3}(0,0,0)\eta^2 - \frac{1}{6}\frac{\partial^3 V}{\partial c^3}(0,0,0)c^2 + o(\eta^2) + o(c^2).\end{aligned}$$

However, we have $|c-\eta| \le |c-c'|$ and

$$|c-c'| = \|w^*(0) - y(\mu)\| = \|\phi(0) - y(\mu)\| \le C\mu(c)^{3/2},$$

then

$$\lim_{c\longrightarrow 0}\frac{1}{c^2}[\frac{\partial}{\partial c}V(0,\eta,0) + \frac{\partial^2 V(0,\eta,v_0\mu(c))}{\partial\mu\partial c}\mu(c)] = \frac{1}{3}\frac{\partial^3 V}{\partial c^3}(0,0,0) < 0.$$

Consequently, there exists a constant $K_1 > 0$ such that for μ close to zero we have

$$\frac{1}{c'-c}V(0,c',\mu(c)) \le -K_1 c^2,$$

hence

$$\begin{aligned}|V(0,c',\mu(c)) + c' - c| &= |c'-c||1 + \frac{1}{c'-c}V(0,c',\mu(c))|\\ &\le |c'-c|(1-K_1c^2)\\ &\le C\mu^{3/2}(1-K_1c^2),\end{aligned}$$

which implies

$$\|w^*(2\pi) - y(\mu)\| \le C[1 - K_1|y(\mu)|^2]\mu^{3/2}.$$

The proof is complete. $\square$

Proposition 3.9. *For each $\phi \in \mathcal{B}(\mu)$, we have $u_{2\pi}(\phi) \in \mathcal{B}(\mu)$.*

Proof. From Lemma 3.6, 3.7 and Proposition 3.8, we have

$$\begin{aligned}
&\|u(\phi)(t) - y(\mu)\| \\
&\le \|u(\phi)(t) - u(\phi)(2\pi)\| + \|u(\phi)(2\pi) - w^*(2\pi)\| + \|w^*(2\pi) - y(\mu)\| \\
&\le C_4\mu^{3/2} + C_3\mu^{5/2} + C[1 - K_1|y(\mu)|^2]\mu^{3/2}.
\end{aligned}$$

Then

$$\|u(\phi)(t) - y(\mu)\| \le [C_4 + C_3\mu + (1 - K_1|y(\mu)|^2)C]\mu^{3/2},$$

from which we conclude that $u_{2\pi}(\phi) \in \mathcal{B}(\mu)$ for μ close to zero. □

Theorem 3.10. *Under hypotheses (H1)–(H3), equation* (1.1) *has at least one nontrivial periodic solution for μ close to zero.*

Proof. Define the Poincaré operator

$$\mathcal{P} : \mathcal{B}(\mu) \to C([-\mu\tau_\infty, 0], \mathbb{R}^2)$$

such that for $\phi \in \mathcal{B}(\mu)$, $\mathcal{P}\phi := u_{2\pi}(\phi)$. Proposition 3.9 shows that $\mathcal{P}$ is defined from $\mathcal{B}(\mu)$, (which is a convex bounded set) into itself and that $\mathcal{P}$ as continuous and compact (see [7]). So using the second Schauder fixed point theorem (see, for example [4]) we conclude that $\mathcal{P}$ has at least one fixed point which corresponds to a periodic solution of the retarded equation (1.1). Sine $\mathcal{B}(\mu)$ does not contain zero, the obtained periodic solutions are nontrivial. □

4. Examples

Consider the system of equations

$$\begin{aligned}
\frac{d}{dt}x_1(t) &= \epsilon x_2(t-\mu\tau(t)) + a_1x_1^2(t-\mu\tau(t)) + b_1x_2^2(t-\mu\tau(t)) \\
&\quad + O(x_1^3(t-\mu\tau(t)), x_2^3(t-\mu\tau(t))) \\
\frac{d}{dt}x_2(t) &= -\epsilon x_1(t-\mu\tau(t)) + a_2x_1^2(t-\mu\tau(t)) + b_2x_2^2(t-\mu\tau(t)) \\
&\quad + O(x_1^3(t-\mu\tau(t)), x_2^3(t-\mu\tau(t)))
\end{aligned} \tag{4.1}$$

where $a_1, a_2, b_1, b_2, \mu, \epsilon$ are real parameters. Here $\mu > 0, \epsilon$ has values respectively in a neighborhood of 0 and 1. $\tau \in C^1(\mathbb{R}, \mathbb{R}^+)$ is 2π-periodic in t and $\int_0^{2\pi} \tau(s)ds \neq 0$.

Thus, applying the formulas given in [11] for the computation of the Lyapunov constant we obtain the expression of $G_4(\epsilon)$ (see proposition 2.5)

$$G_4(\epsilon) = -\frac{1}{2}\frac{4b_1b_2 + 7a_1b_2 + a_2a_1}{\epsilon}.$$

which implies that system (4.1) with $\mu = 0$ is 3-asymptotically stable if $(4b_1b_2 + 7a_1b_2 + a_2a_1)\epsilon > 0$.

From theorem 3.10 we have the following proposition

Proposition 4.1. *If $(4b_1b_2 + 7a_1b_2 + a_2a_1)\epsilon > 0$, then for any μ sufficiently small there exists a periodic solution for system (4.1).*

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