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EXPONENTIAL ATTRACTORS FOR A CAHN-HILLIARD MODEL IN BOUNDED DOMAINS WITH PERMEABLE WALLS

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Abstract. In a previous article [7], we proposed a model of phase separation in a binary mixture confined to a bounded region which may be contained within porous walls. The boundary conditions were derived from a mass conservation law and variational methods. In the present paper, we study the problem further. Using a Faedo-Galerkin method, we obtain the existence and uniqueness of a global solution to our problem, under more general assumptions than those in [7]. We then study its asymptotic behavior and prove the existence of an exponential attractor (and thus of a global attractor) with finite dimension.

1. Introduction

In this article, we are interested in the asymptotic behavior of a Cahn-Hilliard model that was introduced in Gal [7]. The corresponding equations were studied as an approximate problem of a system of two parabolic equations with dynamical and Wentzell boundary conditions involving two unknowns, namely a temperature $u(x,t)$ at a point x and time t of a substance which can appear in different phases and an order parameter $\phi(t,x)$, which describes the current phase at x and time t. Such models are phase-field equations of Caginalp type. Different versions of Cahn-Hilliard models were studied extensively by many authors in [4, 5, 14, 16, 18] and the references cited there in. For instance, Racke & Zheng [21] show the existence and uniqueness of a global solution to the Cahn-Hilliard equation with dynamic boundary conditions, and later Pruss, Racke & Zheng [21] study the problem of maximal L^p-regularity and asymptotic behavior of the solution and prove the existence of a global attractor to the same Cahn-Hilliard system. Miranville & Zelik [16] prove the existence and uniqueness of a solution under more general assumptions on the potential function (compared to [21, 22]) and construct a robust family of exponential attractors to a regularized version of their problem. We would also like to refer the reader to the papers of Wu & Zheng [25] and Chill, Fasangova and Pruss [3]. They study the problem of convergence to equilibrium of solutions of a Cahn-Hilliard equation with dynamic boundary conditions.

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In [7], we have proposed a model of phase separation in a binary mixture confined to a bounded region Ω which may be contained within porous walls $\Gamma = \partial\Omega$ and there we have proved the existence and uniqueness of global solutions to this model. The course of phase separation in [7] is completely changed from the one described by the Cahn-Hilliard equations with boundary conditions considered previously in the cited papers. Furthermore, the system of equations in [7] was also compared to the models studied earlier by these authors and in some cases, we showed that the solutions of the two different systems resemble each other in suitable Sobolev norms.

In this paper concerns the following system of initial value problems in a bounded domain $\Omega \subset \mathbf{R}^N$, $N = 2, 3$:

$$\partial_t \phi = \Delta\mu \quad \text{in } [0,T] \times \Omega, \tag{1.1}$$

$$\mu = -\Delta\phi + f(\phi) \quad \text{in } [0,T] \times \Omega, \tag{1.2}$$

and

$$\partial_t \phi + b\partial_n \mu + c\mu = 0 \quad \text{on } [0,T] \times \Gamma, \tag{1.3}$$

$$-\alpha\Delta_\Gamma \phi + \partial_n \phi + \beta\phi = \frac{\mu}{b} \quad \text{on } [0,T] \times \Gamma. \tag{1.4}$$

and $\phi\big|_{t=0} = \phi_0$, where ϕ and μ are unknown functions, Δ_Γ is the Laplace-Beltrami operator on the boundary, α, β, c, b are positive constants. Moreover, f is a given nonlinear function that belongs to $C^2(\mathbf{R}, \mathbf{R})$ that satisfies the following assumption:

$$\lim_{|s|\to\infty} \inf f'(s) > 0. \tag{1.5}$$

Compared with the result obtained in Gal [7], these allows us to consider a potential f with arbitrary growth (even in the case $n = 3$). The boundary condition (1.3) is derived from mass conservation laws that include an external mass source (energy density) on Γ. This may be realized, for example, by an appropriate choice of the surface material of the wall, that is, the wall Γ may be replaced by a penetrable permeable membrane. For a complete discussion of (1.3), in the context of heat and wave equations, we refer the reader to Goldstein [10]. The condition (1.4) is similarly derived, since the system tends to minimize its surface free energy, in the presence of surface interactions on Γ. On the other hand, equation (1.4) can be viewed as describing the chemical potential on the walls of the region Ω, and due to (1.4), we assume it to be directly proportional to the driving force which equals the left hand side of (1.4). Therefore, the occurrence of a nontrivial pore surface phase behavior is possible. Let us also mention that this Cahn-Hilliard system is not conservative as discussed in [7]. We note that if the value of $\phi(t)$ is known for some value $t = T$, then the value of the chemical potential $\mu(T)$ can be found from the problem

$$\mu(T) = -\Delta\phi(T) + f(\phi(T)) \quad \text{in } \Omega, \tag{1.6}$$

$$\frac{1}{b}\mu(T) = -\Delta_\Gamma \phi(T) + \partial_n \phi(T) + \beta\phi(T) \quad \text{on } \Gamma, \tag{1.7}$$

Similarly, if $\partial_t\phi(T)$ is known, we can solve for $\mu(T)$ from the following elliptic boundary value problem:

$$-\Delta\mu(T) = -\partial_t\phi(T) \quad \text{in } \Omega, \tag{1.8}$$

$$b\partial_n\mu(T) + c\mu(T) = -\partial_t\phi(T) \quad \text{on } \Gamma. \tag{1.9}$$

Thus, we only need to find the function ϕ. It is in fact more convenient to introduce new unknown functions $\psi(t) := \phi(t)\big|_\Gamma$, $\varpi(t) := \mu(t)\big|_\Gamma$ and to rewrite our system (1.1)–(1.4) as follows:

$$\partial_t\phi = \Delta\mu \quad \text{in } [0,T]\times\Omega, \tag{1.10}$$

$$\partial_t\psi + b\partial_n\mu + c\varpi = 0 \quad \text{on } [0,T]\times\Gamma, \tag{1.11}$$

and

$$\mu = -\Delta\phi + f(\phi) \quad \text{in } [0,T]\times\Omega, \tag{1.12}$$

$$\frac{\varpi}{b} = -\alpha\Delta_\Gamma\psi + \partial_n\phi + \beta\psi \quad \text{on } [0,T]\times\Gamma, \tag{1.13}$$

with $\phi\big|_{t=0} = \phi_0$ and $\psi\big|_{t=0} = \psi_0$. The boundary condition (1.4) is now interpreted as an additional elliptic equation on the boundary Γ. We note that (1.12)–(1.13) still form elliptic boundary value problems in the sense of Agmon & Douglis & Nirenberg [3], Hörmander [14], Peetre [22] or Višik [26].

We organize our paper as follows: in Section 2, we discuss the linear problems associated with our equations and construct the phase spaces necessary for the study of our Cahn-Hilliard system. In Section 3, we derive uniform estimates which are needed to study our problem and in Section 4, we discuss the existence and uniqueness of solutions. Finally, in Section 5, we obtain the existence of global attractors with finite dimension.

2. Preliminary results

The solvability of a similar problem to (1.1) equipped with the boundary condition (1.3) was given in [6, 7]. From now on, through out the paper, we denote the norms on $H^s(\Omega)$ and $H^s(\Gamma)$ by $\|\cdot\|_s$ and $\|\cdot\|_{s,\Gamma}$, respectively. The inner product in these spaces will be denotes by $\langle\cdot,\cdot\rangle_s$ and $\langle\cdot,\cdot\rangle_{s,\Gamma}$, respectively. The spaces H^s, $s>0$ are defined the usual way found in standard textbooks. For example, we can define $H^s(\Gamma)$, using the Laplace-Beltrami operator as follows; let $H^{2m}(\Gamma) = \{f \in L^2(\Gamma) : \Delta_\Gamma^m f \in L^2(\Gamma)\}$ and its norm defined to be the equivalent norm of $C_1\|f\|_{0,\Gamma} + C_2\|\Delta_\Gamma^m f\|_{0,\Gamma}$. It follows that any space $H^s(\Gamma)$, $s>0$ can be defined by interpolation. Furthermore, every product space $H^s(\Omega)\oplus H^s(\Gamma)$ ($s\in\mathbf{N}$) is the completion of $(u\big|_\Omega, u\big|_\Gamma) \in C^s(\Omega)\times C^s(\Gamma)$ under the natural Sobolev norms on H^s. The dynamical boundary condition (1.3) is also related to a Wentzell boundary condition, studied by other authors (see e.g. [8, 11]). We will make this clear later in this section. Let us consider the space $\mathcal{H} = L^2(\overline{\Omega}, dx \oplus \frac{dS}{b})$ with norm

$$\|u\|_{\mathcal{H}} = \Big(\int_\Omega |u(x)|^2 dx + \int_\Gamma |u(x)|^2 \frac{dS_x}{b}\Big)^{1/2}. \tag{2.1}$$

Here we identify $\mathcal{H}$ with $L^2(\Omega, dx)\oplus L^2(\Gamma, \frac{dS}{b})$. If $u \in C(\overline{\Omega})$, we identify u with the vector $U = (u\big|_\Omega, u\big|_\Gamma) \in C(\Omega)\times C(\Gamma)$. We define $\mathcal{H}$ to be the completion of $C(\overline{\Omega})$

with respect to the norm (2.1). For a complete discussion of this space, we refer the reader to [6]. Let us also define, for $s = 0, 1$, the spaces

$$\mathbb{V}_s = \overline{C^s(\overline{\Omega})}^{\|\cdot\|_{\mathbb{V}_s}},$$

where the norms $\|\cdot\|_{\mathbb{V}_s}$ are given by

$$\|(\phi,\psi)\|_{\mathbb{V}_1} = \int_\Omega |\nabla\phi|^2 dx + \int_\Gamma \alpha|\nabla_\Gamma \psi|^2 dS + \int_\Gamma \beta|\psi|^2 dS$$

and

$$\|(\phi,\psi)\|_{\mathbb{V}_0} = \int_\Omega |\phi|^2 dx + \int_\Gamma |\psi|^2 dS,$$

respectively. It easy to see that we can identify $\mathbb{V}_s$ with $H^s(\Omega)\oplus H^s(\Gamma)$ under these norms, when $s = 0, 1$. Moreover, $\mathbb{V}_0 = \mathcal{H}$ up to an equivalent inner product and $\mathbb{V}_s$ is compactly embedded in $\mathbb{V}_{s-1}$ for all $s \geq 1$.

We can rewrite the equations (1.10), (1.11) as

$$-\begin{pmatrix}\partial_t\phi\\ \partial_t\psi\end{pmatrix} = A_0\begin{pmatrix}\mu\\ \varpi\end{pmatrix}, \tag{2.2}$$

where A_0 is defined formally as

$$A_0\begin{pmatrix}\mu\\ \varpi\end{pmatrix} = \begin{pmatrix}-\Delta\mu\\ b\partial_n\mu + c\varpi\end{pmatrix}, \tag{2.3}$$

for functions $(\mu,\varpi) \in H^{3/2+\delta}(\Omega)\times L^2(\Gamma)$, for some $\delta > 0$, with $\varpi = \mu\big|_\Gamma$, such that $\Delta\mu \in L^2(\Omega)$. Note that ϖ and $\partial_n\mu$ belong (in the trace sense) to $H^{1+\delta}(\Gamma)$ and $L^2(\Gamma)$, respectively. Here, we have also identified μ with the vector (μ,ϖ). Next, we consider the bilinear form:

$$\Big\langle\begin{pmatrix}\mu\\ \varpi\end{pmatrix}, \begin{pmatrix}\Psi\\ \Psi\big|_\Gamma\end{pmatrix}\Big\rangle_D = \Big\langle A_0\begin{pmatrix}\mu\\ \varpi\end{pmatrix}, \begin{pmatrix}\Psi\\ \Psi\big|_\Gamma\end{pmatrix}\Big\rangle_{\mathcal{H}} = \int_\Omega \nabla\mu\cdot\nabla\Psi dx + \int_\Gamma c\varpi\Psi\big|_\Gamma \frac{dS}{b}, \tag{2.4}$$

for all $\Psi \in H^1(\Omega)$. Note that $\Psi\big|_\Gamma$ is a well defined member of of $H^{1/2}(\Gamma)$ in the trace sense. Define D to be the completion of $C^1(\overline{\Omega})$ with respect to the inner product (2.4). Notice that D is densely injected and continuous in $H^1(\Omega)$ and in fact, D is isometrically isomorphic to $H^1(\Omega)$. It is easy to see that (2.4) defines a closed bilinear form $a(\cdot,\cdot)$ with domain $D(a(\cdot,\cdot)) = H^1(\Omega)\oplus L^2(\Gamma)$ which can be identified with D up to an equivalent inner product. The form is also densely defined (D is dense in $\mathcal{H}$) and nonnegative in $\mathcal{H}$. Then, it is well known (see e.g. [8]) that the bilinear form given by (2.4) defines a strictly positive self-adjoint unbounded operator $A : D(A) = \{(\mu,\varpi)^{\rm tr} \in D : A(\mu,\varpi)^{\rm tr} \in \mathcal{H}\} \to \mathcal{H}$ such that, for all $(\mu,\varpi)^{\rm tr} \in D(A)$ and for any $(\Psi,\Psi\big|_\Gamma)^{\rm tr} \in D$, we have:

$$\Big\langle A\begin{pmatrix}\mu\\ \varpi\end{pmatrix}, \begin{pmatrix}\Psi\\ \Psi\big|_\Gamma\end{pmatrix}\Big\rangle_{\mathcal{H}} = \Big\langle\begin{pmatrix}\mu\\ \varpi\end{pmatrix}, \begin{pmatrix}\Psi\\ \Psi\big|_\Gamma\end{pmatrix}\Big\rangle_D. \tag{2.5}$$

Clearly, we view the operator A as the self-adjoint extension of A_0. Here and everywhere in the paper, the superscript "tr" will denote transposition. The operator A is also a bijection from $D(A)$ into $\mathcal{H}$, since $c > 0$ and $\mathcal{N} := A^{-1} : \mathcal{H} \to \mathcal{H}$ is a linear, self-adjoint and compact operator on $\mathcal{H}$ (see [8]). In other words, we can view the inverse operator $\mathcal{N} : D^* \to D$ by the condition

$$A\Big(\mathcal{N}\begin{pmatrix}\mu\\ \varpi\end{pmatrix}\Big) = \begin{pmatrix}\mu\\ \varpi\end{pmatrix}, \quad \text{for all } \begin{pmatrix}\mu\\ \varpi\end{pmatrix} \in D^*,$$

namely, $\mathcal{N}\binom{\mu_2}{\varpi_2}$ is the solution of the generalized problem

$$-\Delta\mu = \mu_2 \quad \text{in } \Omega,$$
$$b\partial_n\mu + c\mu = \varpi_2 \quad \text{on } \Gamma,$$

hence $\mathcal{N}\binom{\mu_2}{\varpi_2} \in D$ if $(\mu_2, \varpi_2) \in \mathcal{H}$. If in addition, $\mu_2 \in D = H^1(\Omega)$ (thus, by regularity of trace theory, $\varpi_2 = \mu_{2|\Gamma} \in H^{1/2}(\Gamma)$), standard elliptic theory implies that $\mathcal{N}\binom{\mu_2}{\varpi_2} \in H^2(\Omega)$. Furthermore, we infer from standard spectral theory that there exists a complete ortho-normal family of eigenvectors $\{\eta_j\}_j$ with $\eta_j \in D(A)$ and a sequence λ_j, $0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_j \to \infty$ as $j \to \infty$ and $A\eta_j = \lambda_j\eta_j$. Also, by spectral theory, A^{-s}, $s \in \mathbb{N}$ is defined as an infinite series, using the standard spectral decomposition of A. To this end, we define $D(A^{-s})$ to be the completion of $\mathcal{H}$ with respect to the norm

$$\|\Theta\|_{-s}^2 = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^{2s}} |\langle\Theta, \eta_j\rangle_{\mathcal{H}}|^2. \tag{2.6}$$

We have from (2.5) that $D = D(A^{1/2})$. Furthermore, it follows from (2.5) and the definition of the inner products in D and $\mathcal{H}$, that, for any $(\mu, \varpi)^{\mathrm{tr}} \in D(A)$ and $(\Psi, \Psi\big|_\Gamma)^{\mathrm{tr}} \in D$,

$$\int_\Omega (A\mu)\Psi dx + \int_\Gamma (A\mu)\Psi\big|_\Gamma \frac{dS_x}{b} = \int_\Omega \nabla\mu\cdot\nabla\Psi dx + \int_\Gamma c\varpi\Psi\big|_\Gamma \frac{dS}{b}.$$

Integration by parts in the identity above yields for $\mu(= (\mu, \varpi)) \in D$, that $A\mu = -\Delta\mu \in \mathcal{H}$ and

$$\int_\Gamma -(\Delta\mu)\Psi\big|_\Gamma \frac{dS_x}{b} = \int_\Gamma b\partial_n\mu\Psi\big|_\Gamma \frac{dS}{b} + \int_\Gamma c\varpi\Psi\big|_\Gamma \frac{dS}{b},$$

holds for all $\Psi\big|_\Gamma \in H^{1/2}(\Gamma)$. Thus, the following boundary condition holds in $H^{-1/2}(\Gamma)$ (that is, the dual of $H^{1/2}(\Gamma)$):

$$(\Delta\mu + b\partial_n\mu + c\mu)\big|_\Gamma = 0.$$

This is a Wentzell boundary condition for μ. Such boundary conditions have been considered in many papers (see e.g. [6]-[10], [12], [25]). Similarly, it follows that the eigenvector η_j satisfies $A\eta_j = \lambda_j\eta_j$, that is,

$$-\Delta\eta_j = \lambda_j\eta_j \quad \text{in } \Omega,$$
$$(\Delta\eta_j + b\partial_n\eta_j + c\eta_j)\big|_\Gamma = 0,$$

where this boundary condition may be replaced by the eigenvalue dependent boundary condition:

$$b\partial_n\eta_j + (c - \lambda_j)\eta_j = 0 \quad \text{on } \Gamma.$$

Such problems with explicit eigenparameter dependence in the boundary condition were widely considered in the literature, therefore we will not dwell on this issue any further (see [6] for further references). Furthermore, let

$$W = \{u \in H^2(\Omega) : (\Delta u)\big|_\Gamma \in L^2(\Gamma, \frac{dS}{b}), \quad (\Delta u + b\partial_n u + cu)\big|_\Gamma = 0\},$$

where the space W is endowed with the natural norm

$$\|u\|_W^2 = \|u\|_2^2 + \|(\Delta u)\big|_\Gamma\|_{L^2(\Gamma, \frac{dS}{b})}^2. \tag{2.7}$$

Let us also note that the embeddings $W \subset D \subset \mathcal{H} = \mathcal{H}^* \subset D^* \subset W^*$ are dense and continuous. We can then consider D^* endowed with the norm, for $(v, \xi) \in \mathcal{H}$,

$$\|(v,\xi)\|_{D^*}^2 = \|\mathcal{N}^{1/2}\begin{pmatrix} v \\ \xi \end{pmatrix}\|_{\mathcal{H}}^2 = \langle \mathcal{N}\begin{pmatrix} v \\ \xi \end{pmatrix}, \begin{pmatrix} v \\ \xi \end{pmatrix}\rangle_{\mathcal{H}}, \tag{2.8}$$

which can be defined in terms of the spectral decomposition of A (see (2.6)). It follows that the following relations

$$\langle A\begin{pmatrix} \mu \\ \varpi \end{pmatrix}, \mathcal{N}\begin{pmatrix} \phi \\ \psi \end{pmatrix}\rangle_D = \langle \begin{pmatrix} \mu \\ \varpi \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix}\rangle_D, \tag{2.9}$$

$$\langle \begin{pmatrix} \mu \\ \varpi \end{pmatrix}, \mathcal{N}\begin{pmatrix} \phi \\ \psi \end{pmatrix}\rangle_{\mathcal{H}} = \langle \begin{pmatrix} \mu \\ \varpi \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix}\rangle_{D^*} \tag{2.10}$$

hold, for all $(\mu, \varpi) \in D$, $(\phi, \psi) \in D^*$.

Having established this framework, we introduce the phase space for our problem (1.10)–(1.13):

$$\begin{aligned} \mathbb{Y} := \big\{(\phi,\psi) \in H^2(\Omega) \times H^2(\Gamma) : \mu \in H^1(\Omega), \\ \varpi \in L^2(\Gamma, \frac{cdS}{b}), \phi\big|_\Gamma = \psi,\ \mu\big|_\Gamma = \varpi\big\}, \end{aligned} \tag{2.11}$$

with the obvious norm

$$\|(\phi,\psi)\|_{\mathbb{Y}}^2 := \|\phi\|_2^2 + \|\psi\|_{2,\Gamma}^2 + \|\nabla\mu\|_0^2 + \frac{c}{b}\|\varpi\|_{0,\Gamma}^2. \tag{2.12}$$

We recall that (μ, ϖ) is computed from (ϕ, ψ) via (1.6), (1.7) or (1.8), (1.9).

Definition 2.1. Let us consider $T > 0$ be fixed, but otherwise arbitrary. By a solution of (1.10)–(1.13) we mean a pair of functions $(\phi(t), \psi(t)) \in L^\infty([0,T], \mathbb{Y})$ with $\partial_t\phi \in L^2([0,T], H^1(\Omega))$ and $\partial_t\psi \in L^2([0,T], H^1(\Gamma))$ which satisfy the equations in the average sense of the spaces $L^2([0,T], L^2(\Omega))$ and $L^2([0,T], L^2(\Gamma))$. Moreover, since $\Omega \subset \mathbf{R}^N$, $N = 2, 3$, we have the embedding $H^2 \subset C$, therefore the nonlinearity f in (1.12) is well defined and belongs to the space $C([0,T], L^2(\Omega))$. Also, by regularity theory, since $(\partial_t\phi, \partial_t\psi) \in L^2([0,T], H^1(\Omega) \times H^1(\Gamma))$, we get $(\mu, \varpi) \in L^2([0,T], H^2(\Omega) \times H^{3/2}(\Gamma))$ and thus the boundary conditions are well defined.

We close this section with the definition of the weak energy space $\mathbb{X} := D^*$ for our problem (1.10)–(1.13) through the norm given by

$$\|(\phi,\psi)\|_{\mathbb{X}} := \|(\phi,\psi)\|_{D^*} \tag{2.13}$$

where D^* is endowed with the inner product given by (2.8).

3. Uniform a priori estimates

In this section, we derive several estimates for the solutions of the problem (1.10)–(1.13) which are necessary for the study of the asymptotic behavior. In the first step, we obtain dissipative estimates for solutions in the spaces $\mathbb{X}$ and $\mathbb{Y}$.

But before we derive our estimates, it is convenient to rewrite our system of equations in a different manner, as follows:

$$-\begin{pmatrix} \Delta\phi \\ b\alpha\Delta_\Gamma\psi + b\partial_n\phi + b\beta\psi \end{pmatrix} + \begin{pmatrix} f(\phi) \\ 0 \end{pmatrix} = \begin{pmatrix} \mu \\ \varpi \end{pmatrix}, \tag{3.1}$$

with the obvious coupling $\psi(t) := \phi(t)\big|_{\Gamma}$, $\varpi(t) := \mu(t)\big|_{\Gamma}$, where the vector $(\mu, \varpi)^{\rm tr}$ is given by (2.2) via the operator $\mathcal{N} = A^{-1}$, that is,

$$\begin{pmatrix} \mu \\ \varpi \end{pmatrix} = -\mathcal{N} \begin{pmatrix} \partial_t \phi \\ \partial_t \psi \end{pmatrix}. \tag{3.2}$$

Then (3.1) becomes the following functional equation:

$$\mathcal{N} \begin{pmatrix} \partial_t \phi \\ \partial_t \psi \end{pmatrix} - \begin{pmatrix} \Delta \phi \\ b\alpha \Delta_\Gamma \psi + b\partial_n \phi + b\beta \psi \end{pmatrix} + \begin{pmatrix} f(\phi) \\ 0 \end{pmatrix} = 0, \quad \psi(t) = \phi(t)\big|_{\Gamma}. \tag{3.3}$$

Let us define the following function $F(v) = \int_0^v f(s)ds$. Without loss of generality, we let $\alpha = \beta = 1$ for the rest of this section. We have the following result.

Proposition 3.1. *Let the nonlinearity f satisfy* (1.5) *and let $(\phi(t), \psi(t))$ be a given solution of* (3.3)*. Then*

$$\begin{aligned} &\|(\phi(t), \psi(t))\|_{\mathbb{X}}^2 + \int_t^{t+1} (\|\phi(s)\|_1^2 + \|\psi(s)\|_{1,\Gamma}^2) ds + \int_t^{t+1} \|F(\phi(s))\|_{L^1(\Omega)} ds \\ &\leq C_1 \|(\phi(0), \psi(0))\|_{\mathbb{X}}^2 e^{-\rho t} + C_2, \end{aligned} \tag{3.4}$$

where C_1, C_2, ρ are positive constants independent of t.

Proof. First, we take the inner product in $\mathcal{H}$ of (3.3) with the vector $(\phi(t), \psi(t))^{\rm tr}$, and use the fact that $-(\mu(t), \varpi(t))^{\rm tr} = \mathcal{N}(\partial_t \phi(t), \partial_t \psi(t))^{\rm tr}$. Then, relation (2.8) and integration by parts yield the following equation:

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} [\|(\phi(t), \psi(t))\|_{D^*}^2] + \|\nabla \phi(t)\|_0^2 + \|\nabla_\Gamma \psi(t)\|_{0,\Gamma}^2 \\ &+ \|\psi(t)\|_{0,\Gamma}^2 + \langle f(\phi(t)), \phi(t) \rangle_0 = 0. \end{aligned} \tag{3.5}$$

Due to assumption (1.5), we have

$$\frac{1}{2} |f(v)|(1 + |v|) \leq f(v)v + C_f, \tag{3.6}$$

for each $v \in \mathbf{R}$. Here C_f is a positive, sufficiently large constant. Moreover, by the obvious inequality $\|\phi\|_1^2 \leq C(\|\nabla \phi\|_0^2 + \|\psi\|_{1,\Gamma}^2)$, we obtain from (3.5):

$$\begin{aligned} &\frac{d}{dt} \big[\|(\phi(t), \psi(t))\|_{D^*}^2\big] + \rho(\|\phi(t)\|_1^2 + \|\psi(t)\|_{1,\Gamma}^2) + \|\nabla \phi(t)\|_0^2 \\ &+ \|\nabla_\Gamma \psi(t)\|_{0,\Gamma}^2 + \langle f(\phi(t)), \phi(t) \rangle_0 = 0. \end{aligned} \tag{3.7}$$

Consequently, the inequality $\|\phi\|_1^2 + \|\psi\|_{1,\Gamma}^2 \geq C\|(\phi, \psi)\|_{\mathcal{H}}^2 \geq \widetilde{C}\|(\phi, \psi)\|_{D^*}^2$, (3.6) and (3.7) yield

$$\begin{aligned} &\|(\phi(t), \psi(t))\|_{D^*}^2 + \int_t^{t+1} (\|\phi(s)\|_1^2 + \|\psi(s)\|_{1,\Gamma}^2) ds + \int_t^{t+1} (|f(\phi(s))|, (1 + |\phi(s)|))_0 ds \\ &\leq C_1 \|(\phi(0), \psi(0))\|_{D^*}^2 e^{-\rho t} + C_2. \end{aligned} \tag{3.8}$$

To deduce (3.4) from (3.8), we observe that the assumption (1.5) also implies that $|F(v)| - C \leq |f(v)|(1 + |v|)$, for some positive constant C and all $v \in \mathbf{R}$. The proof is complete. □

Proposition 3.2. *Let the assumptions of Proposition 3.1 hold and let* $(\phi(t), \psi(t))$ *be a solution of* (3.3)*. Then, the following estimate holds:*

$$\|(\phi(t), \psi(t))\|_{\mathbb{Y}}^2 + \int_0^t (\|\partial_t \phi(s)\|_1^2 + \|\partial_t \psi(s)\|_{1,\Gamma}^2) ds \le Q(\|(\phi(0), \psi(0))\|_{\mathbb{Y}}) e^{Mt}, \quad (3.9)$$

where the positive constant M *and the monotonic function* Q *are independent of* t.

Proof. We give a formal derivation of (3.9), which can be justified by a standard regularization of the solution, that is, we can define $\widehat{\phi}(t) := \int_0^\infty K_r(t-s)\phi(s)ds$, where K_r is smooth and $\operatorname{supp} K_r \subset [0, r]$, $\int_0^\infty K_r(s)ds = 1$. Then passing to the limit $r \to 0$, we have $\widehat{\phi}(t) \to \phi(t)$. Therefore, without loss of generality, we can (and do) differentiate (3.3) and define

$$(u(t), p(t), v(t), q(t)) := (\partial_t \phi(t), \mu_t(t), \partial_t \psi(t), \varpi_t(t)).$$

Then, we have

$$\mathcal{N}\begin{pmatrix} u_t(t) \\ v_t(t) \end{pmatrix} - \begin{pmatrix} \Delta u(t) \\ b\alpha \Delta_\Gamma v(t) + b\partial_n u(t) + b\beta v(t) \end{pmatrix} + \begin{pmatrix} f'(\phi(t)) u(t) \\ 0 \end{pmatrix} = 0, \quad (3.10)$$

where $u(t)\big|_\Gamma = v(t)$ and $p(t)\big|_\Gamma = q(t)$. The identity (3.2) implies $-(p(t), q(t))^{\mathrm{tr}} = \mathcal{N}(u_t(t), v_t(t))^{\mathrm{tr}}$. Taking the inner product in $\mathcal{H}$ of (3.10) with $(u, v)^{\mathrm{tr}}$ and integrating by parts again (as in (3.5)), we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big[\|(u(t), v(t))\|_{\mathbb{X}}^2\big] + \|\nabla u(t)\|_0^2 + \|\nabla_\Gamma v(t)\|_{0,\Gamma}^2 \\ &+ \|v(t)\|_{0,\Gamma}^2 + \langle f'(\phi(t)) u(t), u(t)\rangle_0 = 0. \end{aligned} \quad (3.11)$$

Due to assumption (1.5), we have $f'(v) \ge -M$, for some positive constant M and each $v \in \mathbf{R}$. Consequently, applying Gronwall's inequality on (3.11) and then using the interpolation inequality $\|(u, v)\|_{\mathcal{H}}^2 \le C\|(u, v)\|_{\mathbb{X}} \|u\|_{H^1(\Omega)}$, we obtain,

$$\begin{aligned} &\|(u(t), v(t))\|_{\mathbb{X}}^2 + \int_0^t (\|\nabla u(s)\|_0^2 + \|\nabla_\Gamma v(s)\|_{0,\Gamma}^2 + \|v(s)\|_{0,\Gamma}^2) ds \\ &\le C_3 \|(u(0), v(0))\|_{\mathbb{X}}^2 e^{M_2 t}, \end{aligned} \quad (3.12)$$

where C_3 and M_2 are positive constants independent of t. Recall that $(u(t), v(t))^{\mathrm{tr}} = -A(\mu(t), \varpi(t))^{\mathrm{tr}}$, since $u(t) = \partial_t \phi(t)$ and $v(t) = \partial_t \psi(t)$. Thus, using the relations (2.4)–(2.10), we can rewrite (3.12) as

$$\begin{aligned} &\|\nabla \mu(t)\|_0^2 + \frac{c}{b}\|\varpi(t)\|_{0,\Gamma}^2 + \int_0^t (\|\partial_t \phi(s)\|_1^2 + \|\partial_t \psi(s)\|_{1,\Gamma}^2) ds \\ &\le C_3 e^{M_2 t} (\|\nabla \mu(0)\|_0^2 + \frac{c}{b}\|\varpi(0)\|_{0,\Gamma}^2). \end{aligned} \quad (3.13)$$

To obtain estimate (3.9), it remains to deduce an estimate for $\|\phi\|_2^2$ and $\|\psi\|_{2,\Gamma}^2$. The required estimates for the H^2-norms of ϕ and ψ can be obtained by rewriting our problem (1.10)–(1.13) as a second order nonlinear elliptic problem where the chemical potentials are considered as external forces. We have

$$-\Delta \phi + f(\phi) = g_1(t) := \mu(t) \quad \text{in } \Omega, \quad \phi\big|_\Gamma = \psi \quad (3.14)$$

$$-\Delta_\Gamma \psi + \beta \psi + \partial_n \phi = g_2(t) := \frac{\varpi(t)}{b} \quad \text{in } \Gamma, \quad \mu\big|_\Gamma = \varpi. \quad (3.15)$$

We note that the estimates (3.12) and (3.13) imply

$$\|g_1(t)\|_0^2 + \|g_2(t)\|_{0,\Gamma}^2 \le C_3\|(\phi(0),\psi(0))\|_{\mathbb{Y}}^2 e^{M_2 t}. \tag{3.16}$$

Applying now the maximum principle [17, Lemma A.2] to the problem (3.14), (3.15), we obtain

$$\begin{aligned}\|\phi(t)\|_\infty^2 + \|\psi(t)\|_{\infty,\Gamma}^2 &\le C_f + \|g_1(t)\|_0^2 + \|g_2(t)\|_{0,\Gamma}^2\\ &\le C_4(1+\|(\phi(0),\psi(0))\|_{\mathbb{Y}}^2)e^{M_2 t}.\end{aligned}$$

Finally, applying the above estimate combined with a H^2-regularity theorem [17, Lemma A.1] to the elliptic boundary value problem above, but with the nonlinearity f acting as an external force, we easily deduce that

$$\|\phi(t)\|_2^2 + \|\psi(t)\|_{2,\Gamma}^2 \le Q(\|(\phi(0),\psi(0))\|_{\mathbb{Y}})e^{M_2 t},$$

where Q is a monotonic function independent of t. The proof is complete. □

In the second step, we state additional smoothing properties for the solutions of (1.10)–(1.13).

Proposition 3.3. *Let the assumptions of Proposition 3.1 hold and let $(\phi(t),\psi(t))$ be a solution of (1.10)–(1.13). Then, we have the estimate*

$$\|(\phi(t),\psi(t))\|_{\mathbb{Y}}^2 \le C_{t_0} Q(\|(\phi(0),\psi(0))\|_{\mathbb{X}}^2), \tag{3.17}$$

for every $t\in[t_0,1]$, where t_0 is a fixed value in $(0,1)$ and the positive constant C_{t_0} and Q are independent of t, but depend on t_0.

Proof. We take the inner product in $\mathcal{H}$ of (3.3) with $(\partial_t\phi(t),\partial_t\psi(t))^{\mathrm{tr}}$, then using the expressions for $\mu(t)$ and $\varpi(t)$, and integrating by parts, we obtain

$$\begin{aligned}\|(\partial_t\phi(t),\partial_t\psi(t))\|_{\mathbb{X}}^2 &= -\langle\begin{pmatrix}\mu(t)\\ \varpi(t)\end{pmatrix},\begin{pmatrix}\partial_t\phi(t)\\ \partial_t\psi(t)\end{pmatrix}\rangle_{\mathcal{H}}\\ &= -\frac{1}{2}\frac{d}{dt}\big[\|\phi(t)\|_1^2 + \|\psi(t)\|_{1,\Gamma}^2 + 2\langle F(\phi(t)),1\rangle_0\big].\end{aligned}$$

It follows that

$$\frac{d}{dt}\big[\frac{1}{2}\|\phi(t)\|_1^2 + \frac{1}{2}\|\psi(t)\|_{1,\Gamma}^2 + \langle F(\phi(t)),1\rangle_0\big] + 2\|(\partial_t\phi(t),\partial_t\psi(t))\|_{\mathbb{X}}^2 = 0.$$

Multiplying the above identity by t and integrating over $[0,t]$, $t\in[0,1]$, we have

$$\begin{aligned}&t[\|\phi(t)\|_1^2 + \|\psi(t)\|_{1,\Gamma}^2 + \langle F(\phi(t)),1\rangle_0] + 2\int_0^t s\|(\partial_t\phi(s),\partial_t\psi(s))\|_{\mathbb{X}}^2 ds\\ &= \int_0^t \big[\|\phi(s)\|_1^2 + \|\psi(s)\|_{1,\Gamma}^2 + 2\langle F(\phi(s)),1\rangle_0\big]ds.\end{aligned}$$

We estimate the term on the right hand side of the above equation, using the estimate (3.4) and obtain

$$\int_0^t s\|(\partial_t\phi(s),\partial_t\psi(s))\|_{\mathbb{X}}^2 ds \le C\|(\phi(0),\psi(0))\|_{\mathbb{X}}^2 + C_*,$$

where C, C_* are independent of the solution (ϕ,ψ). It is an easy consequence of the above, that there exists $t\in(t_0/2,t_0)$ such that we have

$$\|(\partial_t\phi(t),\partial_t\psi(t))\|_{\mathbb{X}}^2 \le \frac{C}{t_0}[1+\|(\phi(0),\psi(0))\|_{\mathbb{X}}^2]. \tag{3.18}$$

Arguing exactly as in the derivation of (3.4), we can verify that (3.18) holds for every $t \in [t_0, 1]$. Using now the relations (2.8) – (2.10) and (3.2), we can rewrite the left hand side of (3.18) and have

$$\|\mu(t)\|_1^2 \leq C\big(\|\nabla\mu(t)\|_0^2 + \frac{c}{b}\|\varpi(t)\|_{0,\Gamma}^2\big) \leq \frac{C}{t_0}\big[1 + \|(\phi(0), \psi(0))\|_{\mathbb{X}}^2\big]. \tag{3.19}$$

Now, we may proceed as in (3.14)–(3.15) to obtain the L^∞ bound for the solution $(\phi(t), \psi(t))$ which together with the H^2-elliptic estimate of [17, Lemma A.1] yields

$$\|\phi(t)\|_2^2 + \|\psi(t)\|_{2,\Gamma}^2 \leq \frac{C}{t_0} Q\big(\|(\phi(0), \psi(0))\|_{\mathbb{X}}^2\big), \tag{3.20}$$

for a suitable monotonic function Q. Finally, the estimates (3.19) and (3.20) yield the conclusion (3.17). This concludes the proof. □

The next theorem follows as consequence of the estimate (3.9) for $t \leq 1$ and estimates (3.4), (3.17) for $t \geq 1$.

Theorem 3.4. *Let the assumptions of Proposition 3.1 hold. Then, every solution of* (1.10)–(1.13) *satisfies the following dissipative estimate*

$$\|(\phi(t), \psi(t))\|_{\mathbb{Y}}^2 \leq Q\big(\|(\phi(0), \psi(0))\|_{\mathbb{Y}}^2\big)e^{-\rho t} + C_4, \tag{3.21}$$

where the positive constants C_4, ρ and the monotonic function Q are independent of t.

We close this section with a theorem that gives uniform bounds for solutions (ϕ, ψ) and μ of our problem in $H^3(\Omega) \times H^3(\Gamma)$ and W respectively.

Theorem 3.5. *Let the assumptions of Proposition 3.1 hold and let $\gamma \in [0, 1/2)$. Then, every solution of* (1.10)–(1.13) *satisfies the following dissipative estimates:*

$$\|\phi(t)\|_{3+\gamma}^2 + \|\psi(t)\|_{3+\gamma,\Gamma}^2 \leq \frac{1}{t} Q_1(\|(\phi(0), \psi(0))\|_{\mathbb{Y}}^2) + C_5, \quad t \geq t_0, \tag{3.22}$$

$$\|\mu(t)\|_W^2 \leq \frac{1}{t} Q_1(\|(\phi(0), \psi(0))\|_{\mathbb{Y}}^2), \quad t \geq t_0, \tag{3.23}$$

where C_5, $\rho > 0$, $t_0 > 0$ and the monotonic function Q_1 are independent of t, $\phi(t)$, $\psi(t)$ and $\mu(t)$, $\varpi(t)$.

Proof. Taking the inner product in $\mathcal{H}$ of (3.3) with $(\partial_t\phi(t), \partial_t\psi(t))^{\mathrm{tr}}$, we obtain

$$\frac{d}{dt}\big[\|\phi(t)\|_1^2 + \|\psi(t)\|_{1,\Gamma}^2 + \langle F(\phi(t)), 1\rangle_0\big] + \|(\partial_t\phi(t), \partial_t\psi(t))\|_{\mathbb{X}}^2 = 0. \tag{3.24}$$

Integrating over $[0, t]$, and using (3.21), we deduce

$$\int_0^t \|(\partial_t\phi(s), \partial_t\psi(s))\|_{\mathbb{X}}^2 ds \leq Q_1\big(\|(\phi(0), \psi(0))\|_{\mathbb{Y}}^2\big), \tag{3.25}$$

for a suitable monotonic function Q_1 independent of t and the solution $(\phi(t), \psi(t))$. Furthermore, multiplying (3.24) by t and then integrating over $[0, t]$, we deduce that

$$\begin{aligned} &t[\|\phi(t)\|_1^2 + \|\psi(t)\|_{1,\Gamma}^2 + \langle F(\phi(t)), 1\rangle_0] + \int_0^t s\|(\partial_t\phi(s), \partial_t\psi(s))\|_{\mathbb{X}}^2 ds \\ &= \int_0^t \big[\|\phi(s)\|_1^2 + \|\psi(s)\|_{1,\Gamma}^2 + \langle F(\phi(s)), 1\rangle_0\big] ds. \end{aligned}$$

Using (3.4) to estimate the right hand side term in the above relation, we obtain

$$\int_0^t s\|(\partial_t\phi(s), \partial_t\psi(s))\|_{\mathbb{X}}^2 ds \le C\|(\phi(0), \psi(0))\|_{\mathbb{X}}^2 e^{-\rho t} + C, \tag{3.26}$$

for some positive constant C and ρ. Recall that by (3.10), we have

$$\mathcal{N}\begin{pmatrix} u_t(t) \\ v_t(t) \end{pmatrix} - \begin{pmatrix} \Delta u(t) \\ b\alpha\Delta_\Gamma v(t) + b\partial_n u(t) + b\beta v(t) \end{pmatrix} + \begin{pmatrix} f'(\phi(t))u(t) \\ 0 \end{pmatrix} = 0, \tag{3.27}$$

where $u(t)\big|_\Gamma = v(t)$ and $p(t)\big|_\Gamma = q(t)$. Taking the inner product of (3.27) with $(u(t), v(t))^{\rm tr}$ in $\mathcal{H}$ and integrating by parts again, we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(u(t), v(t))\|_{\mathbb{X}}^2 + \|\nabla u(t)\|_0^2 + \|\nabla_\Gamma v(t)\|_{0,\Gamma}^2 \\ &+ \|v(t)\|_{0,\Gamma}^2 + \langle f'(\phi(t))u(t), u(t)\rangle_0 = 0. \end{aligned} \tag{3.28}$$

Due to assumption (1.5), we have $f'(v) \ge -N$, for some $N > 0$ and $v \in \mathbf{R}$. We estimate the last term in (3.28) as follows:

$$\begin{aligned} |\langle f'(\phi(t))u(t), u(t)\rangle_0| &\le N\|u(t)\|_0^2 \le N\|(u(t), v(t))\|_{\mathcal{H}}^2 \\ &\le C\|(u(t), v(t))\|_{\mathbb{X}}\|u(t)\|_1 \\ &\le \frac{C}{2}\|(u(t), v(t))\|_{\mathbb{X}}^2 + \frac{1}{2}\|u(t)\|_1^2, \end{aligned} \tag{3.29}$$

Here, we have used the fact that $\|u\|_0^2 \le C\|(u,v)\|_{\mathcal{H}}^2 \le \widetilde{C}\|u\|_1\|(u,v)\|_{\mathbb{X}}$ and $D = H^1(\Omega)$. Multiplying (3.28) by t, integrating (3.28) over $[0,t]$, and using the above estimates together with (3.25), (3.26), we obtain

$$\int_0^t s[\|u(s)\|_1^2 + \|v(s)\|_{1,\Gamma}^2]ds + t\|(u(t), v(t))\|_{\mathbb{X}}^2 \le Q_2(\|(\phi(0), \psi(0))\|_{\mathbb{Y}}^2), \tag{3.30}$$

for $t \in [0,T]$, where Q_2 is independent of t.

Finally, taking the inner product in $\mathcal{H}$ of (3.27) with $(u_t(t), v_t(t))^{\rm tr}$, then multiplying the resulting equation by t^2, we have

$$\begin{aligned} &t^2\|(u_t(t), v_t(t))\|_{\mathbb{X}}^2 + \frac{1}{2}\frac{d}{dt}(t^2\|u(t)\|_1^2 + t^2\|v(t)\|_{1,\Gamma}^2) + t^2\langle f'(\phi(t))u(t), u_t(t)\rangle_0 \\ &= 2t[\|u(t)\|_1^2 + \|v(t)\|_{1,\Gamma}^2]. \end{aligned} \tag{3.31}$$

Estimating the last term on the right hand side of (3.30) as in (3.29), we obtain

$$\begin{aligned} &t^2|\langle f'(\phi(t))u(t), u_t(t)\rangle_0| \\ &\le Ct^2\|f'(\phi(t))u(t)\|_1\|(u_t(t), v_t(t))\|_{\mathbb{X}} \\ &\le \frac{t^2}{2}\|(u_t(t), v_t(t))\|_{\mathbb{X}}^2 + \frac{Ct^2}{2}Q_3(\|\phi(t)\|_2^2)\|u(t)\|_1^2, \end{aligned} \tag{3.32}$$

for a suitable function Q_3 independent of t. Integrating (3.31) now over $[0,t]$, and inserting relation (3.32), we obtain

$$\begin{aligned} &t^2[\|u(t)\|_1^2 + \|v(t)\|_{1,\Gamma}^2] + \int_0^t s^2\|(u_t(s), v_t(s))\|_{\mathbb{X}}^2 ds \\ &\le Ct\int_0^t Q_3(\|\phi(s)\|_2^2)s\|u(s)\|_1^2 ds + \int_0^t s[\|u(s)\|_1^2 + \|v(s)\|_{1,\Gamma}^2]ds. \end{aligned} \tag{3.33}$$

Furthermore, estimating the terms on the right hand side of (3.33) using (3.17) and (3.30), and the fact that $u(t) = \partial_t \phi(t)$, $v(t) = \partial_t \psi(t)$, we deduce that

$$\|\partial_t \phi(t)\|_1^2 + \|\partial_t \psi(t)\|_{1,\Gamma}^2 \leq \frac{t+1}{t^2} Q_4(\|(\phi(0), \psi(0))\|_{\mathbb{Y}}^2), \tag{3.34}$$

for $t > 0$ and a suitable monotonic function Q_4 independent of t.

To deduce estimate (3.23), it remains to write (1.10), (1.11) as an elliptic boundary value problem for the chemical potential μ, that is, we have

$$\begin{gathered} -\Delta\mu = -\partial_t \phi \quad \text{in } \Omega, \\ b\partial_n \mu + c\varpi = -\partial_t \psi \quad \text{on } \Gamma, \quad \varpi = \mu\big|_\Gamma. \end{gathered} \tag{3.35}$$

Thus, we have the estimate

$$\|\mu(t)\|_2^2 \leq C\big(\|\partial_t \phi(t)\|_0^2 + \|\mu(t)\|_1^2 + \|\partial_t \psi(t)\|_{\frac{1}{2},\Gamma}^2\big). \tag{3.36}$$

Consequently, a classical trace theorem and estimate (3.36), imply

$$\|\mu(t)\|_2^2 + \|\varpi(t)\|_{\frac{3}{2},\Gamma}^2 \leq \frac{t+1}{t^2} Q_4(\|(\phi(0), \psi(0))\|_{\mathbb{Y}}^2). \tag{3.37}$$

As in the proof of Proposition 3.2, we now rewrite problem (1.12), (1.13) as an elliptic boundary-value problem:

$$\begin{gathered} -\Delta\phi = g_1(t) := \mu(t) - f(\phi) \quad \text{in } \Omega, \quad \phi\big|_\Gamma = \psi \\ -\Delta_\Gamma \psi + \beta\psi + \partial_n \phi = g_2(t) := \frac{\varpi(t)}{b} \quad \text{in } \Gamma,\ \mu\big|_\Gamma = \varpi. \end{gathered} \tag{3.38}$$

Applying the H^s-elliptic estimate of [17, Lemma A.1], with $s \in \mathbf{R}$, $s + 1/2 \notin \mathbf{N}$, but with the nonlinear term f acting as an external force, we deduce from known embedding theorems:

$$\|\phi(t)\|_{3+\gamma}^2 + \|\psi(t)\|_{3+\gamma,\Gamma}^2 \leq C(\|\mu(t)\|_{1+\gamma}^2 + \|\varpi(t)\|_{1+\gamma,\Gamma}^2 + \|f(\phi)\|_{1+\gamma}^2).$$

Combining the estimate (3.37) and the fact that $H^{3/2}(\Gamma) \subset H^{1+\gamma}(\Gamma)$, $H^2(\Omega) \subset H^{1+\gamma}(\Omega)$, for $\gamma \in [0, 1/2)$, $H^2 \subset C$ together with (2.2), (3.19), we easily verify our conclusion. Thus Theorem 3.5 is proven. □

4. Existence and uniqueness of solutions

The existence of solutions to our problem (1.10)–(1.13) or equivalently (3.3) can be proved in a standard way, based on the a priori estimates derived in Section 3 and on a standard Faedo-Galerkin approximation scheme. To this end, let us consider the operator $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ given formally by

$$\mathcal{B}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} -\Delta\phi \\ -b\alpha\Delta_\Gamma \psi + b\partial_n \phi + b\beta\psi \end{pmatrix}.$$

Then, according to [8], [21], $\mathcal{B}$ defines a positive self-adjoint operator on $\mathcal{H}$ such that $D(\mathcal{B}) = \{H^2(\Omega) \times H^2(\Gamma) : \phi\big|_\Gamma = \psi\}$. Thus, for $i \in \mathbf{N}$, we take a complete system of eigenfunctions $\{\Phi_i = (\phi_i, \psi_i)\}_i$ of the problem $\mathcal{B}\Phi_i = \lambda_i \Phi_i$ in $\mathbb{V}_1^*$ with $\Phi_i \in D(\mathcal{B})$. According to the general spectral theory, the eigenvalues λ_i can be increasingly ordered and counted according to their multiplicities in order to form a real divergent sequence. Moreover, the respective eigenvectors turn out to form an orthogonal basis both in $\mathbb{V}_1$ and $\mathbb{V}_0 = \mathcal{H}$ and may be assumed to be normalized in the norm of $\mathbb{X}$. At this point, we set the spaces

$$K_n = \operatorname{span}\{\Phi_1, \Phi_2, \dots, \Phi_n\}, \quad K_\infty = \cup_{n=1}^\infty K_n.$$

Clearly, K_∞ is a dense subspace of both $\mathbb{V}_1$ and $\mathbb{V}_2$. For any, $n \in \mathbf{N}$, we look for functions of the form

$$\Phi = \Phi_n = \sum_{i=1}^{n} c_i(t)\Phi_i \tag{4.1}$$

solving the approximate problem that we will introduce below. Note that $\Lambda_n = (\mu_n, \varpi_n)$ can be found in terms of Φ_n from (1.6) – (1.7). That is, as mentioned previously, it is enough to solve for Φ_n. Note that in the definition of Φ_n, $c_i(t)$ are sought to be suitably regular real valued functions. As approximations for the initial data $\Phi_0 = (\phi_0, \psi_0)$, we take

$$\Phi_{n0} \in \mathbb{Y} \quad \text{such that } \lim_{n\to\infty} \Phi_{n0} = \Phi_0 \text{ in } \mathbb{Y}.$$

The problem that we must solve is given by (P_n), for any $n \geq 1$,

$$\langle \partial_t(\mathcal{N}\Phi_n), \overline{\Phi}\rangle_{\mathcal{H}} + \langle \mathcal{B}\Phi_n, \overline{\Phi}\rangle_{\mathcal{H}} + \langle \mathcal{F}(\Phi_n), \overline{\Phi}\rangle_{\mathcal{H}} = 0, \tag{4.2}$$

and

$$\langle \Phi_n(0), \overline{\Phi}\rangle_{\mathcal{H}} = \langle \Phi_{n0}, \overline{\Phi}\rangle_{\mathcal{H}},$$

for all $\overline{\Phi} = (\overline{\phi}, \overline{\psi}) \in K_n$. Here the operator $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is given by $\mathcal{F}(\Phi) = (f(\phi), 0)^{\mathrm{tr}}$.

We aim to apply the standard existence theorems for ODE's. For this purpose, if n is fixed, let us choose $\overline{\Phi} = \Phi_j$, $1 \leq j \leq n$ and substitute the expressions (4.1) to the unknowns Φ_n in (4.2). Performing direct calculations, we actually derive the equation:

$$\sum_{i=1}^{n} \langle \Phi_i, \Phi_j\rangle_{\mathbb{X}} \frac{dc_i(t)}{dt} + \sum_{i=1}^{n} \langle \mathcal{B}\Phi_i, \Phi_j\rangle_{\mathcal{H}} c_i(t) + \widetilde{\mathcal{F}}_j(c_i(t)) = 0, \tag{4.3}$$

for $1 \leq j \leq n$, where

$$\widetilde{\mathcal{F}}_j(c_i(t)) = \langle \mathcal{F}(\sum_{i=1}^{n} c_i(t)\Phi_i), \Phi_j\rangle_{\mathcal{H}} = \langle f(\sum_{i=1}^{n} c_i(t)\Phi_i), \phi_j\rangle_0.$$

Note that the matrix coefficient of $c'(t)$ in (4.3) is symmetric and positive-definite, hence, non-singular. Since the bilinear form $\langle \mathcal{B}\Phi_i, \Phi_j\rangle_{\mathcal{H}} = \langle \Phi_i, \Phi_j\rangle_{\mathbb{V}_1} = \lambda_i \langle \Phi_i, \Phi_j\rangle_{\mathcal{H}}$ is $\mathbb{V}_1$-coercive and $f \in C^2(\mathbf{R})$, applying Cauchy's theorem for ODE's, we find a small time $t_n \in (0, T)$ such that (4.3) holds for all $t \in [0, t_n]$. This gives the desired local solution Φ to our problem (4.2), since Φ_n satisfies (4.3). Now, based on the uniform a priori estimates with respect to t, derived for the solution Φ of (3.3), we obtain, in particular, that any local solution is a actually a global solution that is defined on the whole interval $[0, T]$. It remains then to pass to the limit as $n \to \infty$.

According to the a priori estimates derived in Section 3, we have

$$\|\Phi_n\|_{L^\infty([0,T];\mathbb{V}_2)} + \|\Phi_n\|_{L^2([0,T];\mathbb{V}_3)} + \|\partial_t \Phi_n\|_{L^2([0,T];\mathbb{V}_1)} \leq C,$$

and for $\Lambda_n = (\mu_n, \varpi_n)$, $n \in \mathbf{N}$,

$$\|\Lambda_n\|_{L^\infty([0,T];H^1(\Omega)\times H^{1/2}(\Gamma))} + \|\Lambda_n\|_{L^2([0,T];W\times H^{3/2}(\Gamma))} \leq C,$$

where C depends on Ω, Γ, T, Φ_0, but is independent of n and t. From this point on, all convergence relations will be intended to hold up to the extraction of suitable subsequences, generally not labelled. Thus, we observe that weak and weak star compactness results applied to the above sequences Φ_n and Λ_n entail that there exist $\Phi = (\phi, \psi)$ and $\Lambda = (\mu, \varpi)$ such that as $n \to \infty$, the following properties hold:

$$\Phi_n \to \Phi \quad \text{weakly star in } L^\infty([0,T]; \mathbb{V}_2),$$
$$\Phi_n \to \Phi \quad \text{weakly in } L^\infty([0,T]; \mathbb{V}_3),$$
$$\partial_n \Phi_n \to \partial_t \Phi \quad \text{weakly in } L^2([0,T]; \mathbb{V}_1)$$

and

$$\Lambda_n \to \Lambda \quad \text{weakly star in } L^\infty([0,T]; H^1(\Omega) \times H^{1/2}(\Gamma)),$$
$$\Lambda_n \to \Lambda \quad \text{weakly in } L^2([0,T]; W \times H^{3/2}(\Gamma)),$$

where recall that $\|u\|_W = \|u\|_2 + \|(1/b)(\Delta u)\big|_\Gamma\|_{0,\Gamma}$. Then, standard interpolation (for instance, $H^{2-\delta} \subset C$, for $\delta \in (0, 1/2)$, since $\Omega \subset \mathbf{R}^N$ with $N \leq 3$) and compact embedding results for vector valued functions [7, Lemma 10] ensure that

$$\Phi_n \to \Phi \quad \text{strongly in } C([0,T]; C(\Omega) \times C(\Gamma)). \tag{4.4}$$

Standard arguments and (4.4) imply that $\Phi(0) = \Phi_0$. By the Lipschitz continuity of f, the converges above allows us to infer that

$$\mathcal{F}(\Phi_n) \to \mathcal{F}(\Phi) \quad \text{strongly in } C([0,T]; \mathcal{H}).$$

Thus, passing to the limit in (4.2) and using the above convergence properties, we immediately have that the solution Φ satisfies (3.3) in the sense introduced in Definition 2.1, Section 2.

Thus, we have the following result on the solvability of our problem (1.10)–(1.13). Let $T > 0$ be fixed, but otherwise arbitrary.

Theorem 4.1. *Let $(\phi_0, \psi_0) \in \mathbb{X}$ and suppose that the nonlinearity f satisfies assumption* (1.5). *Then, the problem* (1.10)–(1.13) *has a unique solution in the sense of the Definition 2.1 in Section* 2. *Moreover, the solution $(\phi(t), \psi(t))$ belongs to the space $C([0,T], \mathbb{X}) \cap L^\infty_{loc}((0,T], \mathbb{Y})$.*

Proof. It remains to verify only the uniqueness. Suppose that $(\phi_1(t), \psi_1(t))$ and $(\phi_1(t), \psi_1(t))$ are two solutions of (3.3) with same initial data. We set

$$(u(t), p(t)) := (\phi_1(t) - \phi_2(t), \mu_1(t) - \mu_2(t))$$
$$(v(t), q(t)) := (\psi_1(t) - \psi_2(t), \varpi_1(t) - \varpi_2(t))$$

These functions satisfy the equation

$$\mathcal{N}\begin{pmatrix} u_t(t) \\ v_t(t) \end{pmatrix} - \begin{pmatrix} \Delta u(t) \\ b\alpha\Delta_\Gamma v(t) + b\partial_n u(t) + b\beta v(t) \end{pmatrix} + \begin{pmatrix} f(\phi_1(t)) - f(\phi_2(t)) \\ 0 \end{pmatrix} = 0, \tag{4.5}$$

where $\psi(t) = \phi(t)\big|_\Gamma$. Taking the inner product in $\mathcal{H}$ of (4.5) with $(u(t), v(t))^{\text{tr}}$ and using the relations (2.2)–(2.10), we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[\|(u(t), v(t))\|^2_{\mathbb{X}}] + \|\nabla u(t)\|^2_0 + \|\nabla_\Gamma v(t)\|^2_{0,\Gamma} \\ &+ \|v(t)\|^2_{0,\Gamma} + \langle f(\phi_1(t)) - f(\phi_2(t)), u(t)\rangle_0 = 0. \end{aligned} \tag{4.6}$$

Due to assumption (1.5), we have $f'(v) \geq -M$, for some positive constant M. Consequently, (4.6) implies

$$\begin{aligned} &\frac{d}{dt}[\|(u(t), v(t))\|^2_{\mathbb{X}}] + 2\|\nabla u(t)\|^2_0 + 2\|\nabla_\Gamma v(t)\|^2_{0,\Gamma} \\ &+ 2\|v(t)\|^2_{0,\Gamma} \leq 2M\|u(t)\|^2_0. \end{aligned} \tag{4.7}$$

Using now the interpolation inequality $\|(u,v)\|_{\mathcal{H}}^2 \le C\|(u,v)\|_{\mathbb{X}}\|(u,v)\|_D$, the obvious inequality $\|u\|_0 \le C\|(u,v)\|_{\mathcal{H}}$ and the fact $D = H^1(\Omega)$ up to an equivalent norm $\|u\|_1$, in order to estimate the term on the right hand side of (4.7), and applying Gronwall's inequality, we obtain

$$\|(u(t),v(t))\|_{\mathbb{X}}^2 \le C\|(u(0),v(0))\|_{\mathbb{X}}^2 e^{M_3 t}, \tag{4.8}$$

where the positive constants C, M_3 are independent of t and the norm of initial data. This finishes the proof of the uniqueness.

As we mentioned already, the existence of solutions in the phase space $\mathbb{Y}$ (in the sense of Definition 2.1) can be verified in a standard way, whenever $(\phi(0),\psi(0)) \in \mathbb{Y}$. Thus, problem (1.10)–(1.13) generates a semiflow

$$S(t) : \mathbb{Y} \to \mathbb{Y},$$

such that

$$S(t)(\phi(0),\psi(0)) = (\phi(t),\psi(t)),$$

where $(\phi(t),\psi(t))$ is the unique solution of (1.10)–(1.13) with initial data in $\mathbb{Y}$. Moreover, by estimate (4.8), we have the Lipschitz continuity (with respect to the initial data) in the $\mathbb{X}$-norm:

$$\|S(t)(\phi_1,\psi_1) - S(t)(\phi_2,\psi_2)\|_{\mathbb{X}} \le Ce^{Mt}\|(\phi_1-\phi_2,,\psi_1-\psi_2)\|_{\mathbb{X}}, \tag{4.9}$$

for all $(\phi_i,\psi_i) \in \mathbb{Y}$, $i = 1,2$. In what follows, we extend this semigroup, which we still denote by $S(t)$, in a unique way by continuity such that it maps $\mathbb{X}$ into $\mathbb{Y}$. For this purpose, let $(\phi(0),\psi(0)) \in \mathbb{X}$. By the obvious dense injection $\mathbb{Y} \hookrightarrow \mathbb{X}$, we can construct a sequence $(\phi_n,\psi_n) \in \mathbb{Y}$ such that $(\phi_n,\psi_n) \to (\phi(0),\psi(0))$ in the norm of $\mathbb{X}$. Therefore, we extend $S(t)$ to the semigroup

$$S(t)(\phi(0),\psi(0)) := \lim_{n\to\infty} S(t)(\phi_n,\psi_n), \tag{4.10}$$

where the convergence takes place in $\mathbb{X}$. Since the solutions $(\phi_n,\psi_n) \in C([0,T],\mathbb{Y})$, (and clearly, $\mathbb{Y} \subset \mathbb{X}$), we get that the limit solution $(\phi(t),\psi(t)) = S(t)(\phi(0),\psi(0))$ belongs to the space $C([0,T],\mathbb{X})$, and it satisfies the same estimates as in Section 3. Thus, passing to the limit as $n \to \infty$, for each $t > 0$, we have $S(t) : \mathbb{X} \to \mathbb{Y}$ and $(\phi(t),\psi(t))$ satisfies the equations (1.10)–(1.13) in the sense defined in Section 2. The proof is complete. □

5. Exponential attractors

In this section, we shall prove the existence of global attractors for the semiflow T in $\mathbb{Y}$ and moreover, the existence of a semiflow and of a global attractor in $\mathbb{X}$ will also be obtained as a consequence. The existence of a global attractor to our problem $(1.8)-(1.11)$ can be deduced as a result of the uniform and dissipative estimates of our semiflow obtained in Section 3.

Recall that the compact set $\mathcal{A} \subset V$ is called the global attractor for the semigroup $S(t)$ on V if it is invariant by $S(t)$, that is,

$$S(t)\mathcal{A} = \mathcal{A}, \quad \text{for } t \ge 0 \tag{5.1}$$

and it attracts the bounded subsets of V as $t \to \infty$, that is, for every bounded $B \subset V$,

$$\lim_{t\to\infty} \operatorname{dist}_V(S(t)B,\mathcal{A}) = 0,$$

where dist_V is the Hausdorff semi-distance in V.

According to the abstract attractor existence theorem in [2], [22], it suffices to verify that the operators $S(t) : \mathbb{Y} \to \mathbb{Y}$ are continuous for each $t \geq 0$ and that the semigroup $S(t)$ possesses an attracting set $\mathbb{B}$ in $\mathbb{Y}$ and that the orbits are precompact in $\mathbb{Y}$.

To this end, we introduce the following ball $\mathbb{B}$ with sufficiently large radius R in the space $H^3(\Omega) \times H^3(\Gamma)$:

$$\mathbb{B}_R = \{(\phi, \psi) \in H^3(\Omega) \times H^3(\Gamma) : \|(\phi, \psi)\|_{H^3(\Omega)\times H^3(\Gamma)} \leq R\}. \tag{5.2}$$

Then obviously, $\mathbb{B}_R \subset \mathbb{Y}$ by (3.22), (3.23). Moreover, due to the dissipative estimates (3.21), (3.22) and the smoothing property (3.17), there exist sufficiently large $\overline{R}$ ($\geq C_6$) and $T_0 = T_0(\rho, \overline{R})$ such that $\mathbb{B} := \mathbb{B}_{\overline{R}}$ is an absorbing set for the semigroup $S(t)$ acting on $\mathbb{Y}$ and $S(t)(\mathbb{B}) \subset \mathbb{B}$ for $t \geq T_0$. The continuity of the semigroup $S(t)$ was actually verified in Theorem 5.1. It remains to prove the relative compactness of orbits (ϕ, ψ) in $\mathbb{Y}$. This property follows thanks to the estimates of Theorem 3.5 and the compact embedding $H^3 \subset H^2$. Thus, the semigroup $S(t)$ possesses a compact global attractor $\mathcal{A} \subset \mathbb{B} \subset \mathbb{Y}$. Due to the parabolic nature of the problem (1.10)–(1.13), we have the standard smoothing property for its solutions as given by Proposition 3.1, Theorem 3.4, 3.5 and 5.1 and the concrete choice of the space $\mathbb{Y}$ is not essential and can be replaced by the weak energy space $\mathbb{X}$. In fact, in Section 4, we have extended the unique solution $(\phi, \psi)(t) = S(t)(\phi_0, \psi_0)$, for every $(\phi_0, \psi_0) = (\phi(0), \psi(0)) \in \mathbb{Y}$ by continuity, to the semigroup $S(t) : \mathbb{X} \to \mathbb{X}$ which possesses the smoothing property $S(t) : \mathbb{X} \to \mathbb{Y}$ for each fixed $t > 0$. Consequently, we have proved the following result.

Theorem 5.1. *The semigroup $S(t)$ defined by* (4.10) *possesses a compact global attractor $\mathcal{A} \subset \mathbb{Y}$ which has the following structure*

$$\mathcal{A} = \mathcal{I}_0\mathcal{K},$$

where $\mathcal{K}$ denotes the set of all complete bounded trajectories of the semigroup $S(t)$, that is,

$$\mathcal{K} = \big\{(\phi, \psi) \in C_b(\mathbb{R}, \mathbb{X}) : S(t)(\phi, \psi) = (\phi(t + h), \psi(t + h)) \\ \text{for } t \in \mathbb{R},\ h \geq 0,\ \|(\phi, \psi)\|_{\mathbb{X}} \leq C_{\phi,\psi}\big\},$$

and $\mathcal{I}_0(\phi, \psi) \equiv (\phi(0), \psi(0))$.

Remark 5.2. Recall that $\Omega \subset \mathbf{R}^n$, $n = 2, 3$. Then Theorem 5.1 and the continuous embedding of $\mathbb{V}_2 \subset C(\overline{\Omega})$ imply that for each $t > 0$, we have the following regularity:

$$S(t) : \mathbb{X} \to C(\overline{\Omega}) \quad \text{and} \quad \mathcal{A} \subset C(\overline{\Omega}).$$

The fact that this global attractor $\mathcal{A}$ has finite fractal dimension in the topology of $H^2(\Omega)\times H^2(\Gamma)$ will be a consequence of the existence of the exponential attractor below (see Proposition 5.3). But first, let us recall that a compact set $\mathcal{M} \subset V$ is called an exponential attractor for the semigroup on V, if it is semi-invariant, that is,

$$S(t)\mathcal{M} \subset \mathcal{M}, \quad \text{for } t \geq 0 \tag{5.3}$$

and it attracts exponentially all bounded subsets of V, that is, there is a constant $\eta > 0$, such that for every bounded $B \subset V$, we have

$$\lim_{t\to\infty} |\operatorname{dist}_V(S(t)B, \mathcal{M}) \leq Q(\|B\|_V)e^{-\eta t},$$

and it has finite fractal dimension in V, that is, $d(\mathcal{M}, V) < \infty$.

Since, we lose the invariance for the semigroup, that is, the assumption (5.3) instead of (5.1), then the exponential attractor is not necessarily unique. However, we always have $\mathcal{A} \subset \mathcal{M}$. The following proposition gives sufficient conditions for the existence of an exponential attractor in Banach spaces (see [4]).

Proposition 5.3. *Let H and H_1 be two Banach spaces and H_1 compactly embedded in H. Let E be a bounded subset of H. We consider a nonlinear map $L : E \to E$ such that L can be decomposed in the sum of two maps*

$$L = L_0 + L_1 \quad \text{with} \quad L_i : E \to H \quad (i = 0, 1),$$

where L_0 is a contraction, that is,

$$\|L_0(x_1) - L_0(x_2)\|_H \le k\|x_1 - x_2\|_H, \tag{5.4}$$

for any $x_1, x_2 \in E$ with $k \le 1/2$ and L_1 satisfies the condition

$$\|L_1(x_1) - L_1(x_2)\|_{H_1} \le C\|x_1 - x_2\|_H, \tag{5.5}$$

for all $x_1, x_2 \in E$. Then the map L possesses an exponential attractor $\mathcal{M}^*$.

To verify the conditions (that is the assumption (5.4) and (5.5)) of Proposition 5.3, we need to decompose the solution in a sum of two components. To this end, we decompose the vector function $(\phi(t), \psi(t)) = (\widehat{\phi}(t), \widehat{\psi}(t)) + (\widetilde{\phi}(t), \widetilde{\psi}(t))$ into the sum of an exponentially decaying and a smoothing part, where the vector functions $(\widehat{\phi}(t), \widehat{\psi}(t))$ and $(\widetilde{\phi}(t), \widetilde{\psi}(t))$ satisfy the equations:

$$\begin{gathered}\mathcal{N}\begin{pmatrix}\widehat{\phi}_t(t)\\ \widehat{\psi}_t(t)\end{pmatrix} - \begin{pmatrix}\Delta\widehat{\phi}(t)\\ b\alpha\Delta_\Gamma\widehat{\psi}(t) + b\partial_n\widehat{\phi}(t) + b\beta\widehat{\psi}(t)\end{pmatrix} = 0,\\ \big(\widehat{\phi}(0), \widehat{\psi}(0)\big) = \big(\phi(0), \psi(0)\big),\end{gathered} \tag{5.6}$$

and

$$\mathcal{N}\begin{pmatrix}\widetilde{\phi}_t(t)\\ \widetilde{\psi}_t(t)\end{pmatrix} - \begin{pmatrix}\Delta\widetilde{\phi}(t)\\ b\alpha\Delta_\Gamma\widetilde{\psi}(t) + b\partial_n\widetilde{\phi}(t) + b\beta\widetilde{\psi}(t)\end{pmatrix} + \begin{pmatrix}f(\widehat{\phi}(t) + \widetilde{\phi}(t))\\ 0\end{pmatrix} = 0, \tag{5.7}$$

with $(\widetilde{\phi}(0), \widetilde{\psi}(0)) = (0, 0)$, respectively.

We prove our required estimates in the next lemma. Recall that $S(t)(\mathbb{B}) \subset \mathbb{B}$ for $t \ge T_0$, where $\mathbb{B}$ was introduced in (5.2).

Lemma 5.4. *Let $(\phi_1(0), \psi_1(0))$ and $(\phi_2(0), \psi_2(0))$ belong to $\mathbb{B}$. The corresponding solutions of equation (5.6) and (5.7) satisfy the following two estimates:*

$$\begin{aligned}&\|\widehat{\phi}_1(t) - \widehat{\phi}_2(t)\|_1^2 + \|\widehat{\psi}_1(t) - \widehat{\psi}_2(t)\|_{1,\Gamma}^2\\ &\le C_1\frac{e^{-\alpha t}}{t}\|\phi_1(0) - \phi_2(0), \psi_1(0) - \psi_2(0)\|_{\mathbb{X}}^2,\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}&\|\widetilde{\phi}_1(t) - \widetilde{\phi}_2(t)\|_1^2 + \|\widetilde{\psi}_1(t) - \widetilde{\psi}_2(t)\|_{1,\Gamma}^2\\ &\le C_2\frac{e^{Kt}}{t}\|\phi_1(0) - \phi_2(0), \psi_1(0) - \psi_2(0)\|_{\mathbb{X}}^2,\end{aligned} \tag{5.9}$$

for $t > 0$, where $C_1, C_2 > 0$ are independent of b, t.

Proof. We first note that, due to the estimates (3.20) and (5.2), we have

$$\|\phi_i(t)\|_2 + \|\psi_i(t)\|_{2,\Gamma} \le C, \tag{5.10}$$

for $t \ge 0$, $i = 0, 1$, where the C is independent of t and depends at most on $\overline{R}$. Here, $\phi_0 := \widehat{\phi}$, $\phi_1 := \widetilde{\phi}$ and $\psi_0 := \widehat{\psi}$, $\psi_1 := \widetilde{\psi}$ respectively. Due to the continuous embedding $H^2 \subset C$, we have analogous estimates for L^∞ norms of the solution (ϕ_i, ψ_i), which are necessary in order to handle the nonlinear term f. We now set $(\Phi(t), \Psi(t))^{\rm tr} := (\widehat{\phi}_1(t) - \widehat{\phi}_2(t), \widehat{\psi}_1(t) - \widehat{\psi}_2(t))^{\rm tr}$. Then this vector-valued function $(\Phi(t), \Psi(t))^{\rm tr}$ satisfies (5.6). Taking the inner product of (5.6) with the vector $(\Phi(t), \Psi(t))^{\rm tr}$ in $\mathcal{H}$ and integrating by parts, we deduce

$$\frac{1}{2}\frac{d}{dt}[\|\Phi(t), \Psi(t)\|_{\mathbb{X}}^2] + \|\Phi(t)\|_1^2 + \|\Psi(t)\|_{1,\Gamma}^2 = 0. \tag{5.11}$$

Consequently, the inequality $\|\Phi\|_1^2 + \|\Psi\|_{1,\Gamma}^2 \ge C\|(\Phi, \Psi)\|_{\mathcal{H}}^2 \ge \widetilde{C}\|(\Phi, \Psi)\|_{\mathbb{X}}^2$ and relation (5.11) yield

$$\|\Phi(t), \Psi(t)\|_{\mathbb{X}}^2 + \int_0^t \big(\|\Phi(s)\|_1^2 + \|\Psi(s)\|_{1,\Gamma}^2\big) ds \le Ce^{-\rho t}\|\Phi(0), \Psi(0)\|_{\mathbb{X}}^2, \tag{5.12}$$

for some positive constants ρ, C independent of t. Similarly, we take the inner product in $\mathcal{H}$ of (5.6) with $t(\Phi_t(t), \Psi_t(t))^{\rm tr}$ and integrate by parts. Consequently, we obtain

$$t\|\Phi_t(t), \Psi_t(t)\|_{\mathbb{X}}^2 + \frac{1}{2}\frac{d}{dt}[t(\|\Phi(t)\|_1^2 + \|\Psi(t)\|_{1,\Gamma}^2)] = \frac{1}{2}(\|\Phi(t)\|_1^2 + \|\Psi(t)\|_{1,\Gamma}^2). \tag{5.13}$$

Integrating now (5.13) over $[0, t]$ and using estimate (5.12), we deduce

$$t(\|\Phi(t)\|_1^2 + \|\Psi(t)\|_{1,\Gamma}^2) + \int_0^t s\|\Phi_t(s), \Psi_t(s)\|_{\mathbb{X}}^2 ds \le Ce^{-\rho t}\|\Phi(0), \Psi(0)\|_{\mathbb{X}}^2.$$

This last estimate yields our conclusion (5.8). In order to verify estimate (5.9), we will use (5.10). We define

$$(\overline{\Phi}(t), \overline{\Psi}(t))^{\rm tr} := (\widetilde{\phi}_1(t) - \widetilde{\phi}_2(t), \widetilde{\psi}_1(t) - \widetilde{\psi}_2(t))^{\rm tr}.$$

This vector-valued function satisfies (5.7). Arguing as in the derivation of the estimate (5.12), we similarly deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}[\|\overline{\Phi}(t), \overline{\Psi}(t)\|_{\mathbb{X}}^2] + \|\overline{\Phi}(t)\|_1^2 + \|\overline{\Psi}(t)\|_{1,\Gamma}^2 \\ &= -\langle f(\widehat{\phi}_1(t) + \widetilde{\phi}_1(t)) - f(\widehat{\phi}_2(t) + \widetilde{\phi}_2(t)), \overline{\Phi}(t)\rangle_0.\end{aligned} \tag{5.14}$$

In contrast to the proof of Proposition 3.1, we can not estimate the nonlinear term f using assumption (1.5), so then instead we use the uniform estimate (5.10) and the analogous L^∞−estimates. We have

$$-\langle f(\widehat{\phi}_1(t) + \widetilde{\phi}_1(t)) - f(\widehat{\phi}_2(t) + \widetilde{\phi}_2(t)), \overline{\Phi}(t)\rangle_0 \le C\|\overline{\Phi}(t)\|_0^2. \tag{5.15}$$

Using now the obvious interpolation inequality

$$\|\overline{\Phi}(t)\|_0^2 \le C\|(\overline{\Phi}(t), \overline{\Psi}(t))\|_{\mathcal{H}}^2 \le \widehat{C}\|(\overline{\Phi}(t), \overline{\Psi}(t))\|_D\|(\overline{\Phi}(t), \overline{\Psi}(t))\|_{\mathbb{X}} \tag{5.16}$$

and the fact $D = H^1(\Omega)$ (up to an equivalent norm), we deduce from (5.14), (5.15) and Gronwall's inequality that

$$\begin{aligned}&\|(\overline{\Phi}(t), \overline{\Psi}(t))\|_{\mathbb{X}}^2 + \int_0^t (\|\overline{\Phi}(s)\|_1^2 + \|\overline{\Psi}(s)\|_{1,\Gamma}^2) ds \\ &\le e^{Kt} \|(\phi_1(0) - \phi_2(0), \psi_1(0) - \psi_2(0))\|_{\mathbb{X}}^2.\end{aligned} \tag{5.17}$$

Arguing as in (5.13), we deduce

$$\begin{aligned}&t\|(\overline{\Phi}_t(t), \overline{\Psi}_t(t))\|_{\mathbb{X}}^2 + \frac{1}{2}\frac{d}{dt}[t(\|\overline{\Phi}(t)\|_1^2 + \|\overline{\Psi}(t)\|_{1,\Gamma}^2)] \\ &= -\langle f(\widehat{\phi}_1(t) + \widetilde{\phi}_1(t)) - f(\widehat{\phi}_2(t) + \widetilde{\phi}_2(t)), t\overline{\Phi}_t(t)\rangle_0 \\ &\quad + \frac{1}{2}(\|\Phi(t)\|_1^2 + \|\Psi(t)\|_{1,\Gamma}^2).\end{aligned} \tag{5.18}$$

We estimate the first term on the right hand side of (5.18), using (5.10), as follows:

$$\begin{aligned}&- \langle f(\widehat{\phi}_1(t) + \widetilde{\phi}_1(t)) - f(\widehat{\phi}_2(t) + \widetilde{\phi}_2(t)), t\overline{\Phi}_t(t)\rangle_0 \\ &\le Ct\|\overline{\Phi}(t)\|_1^2 + \frac{1}{2}t\|(\overline{\Phi}_t(t), \overline{\Psi}_t(t))\|_{\mathbb{X}}^2,\end{aligned} \tag{5.19}$$

where we have used again the interpolation inequality (5.16) and the L^∞-norm estimates for $\overline{\Phi}$ and Φ. Gronwall's lemma and the estimates (5.17)–(5.19) then yield the final conclusion (5.9), which finishes the proof of the lemma. □

Let us now fix $t^* \ge T_0$. It is left to observe that for every $(\phi_1(0), \psi_1(0))$ and $(\phi_2(0), \psi_2(0))$ belonging to $\mathbb{B}$, the functions $\overline{\Phi}$ and $\overline{\Psi}$ (defined in the proof of Lemma 5.4 satisfy the following smoothing estimate:

$$\|\overline{\Phi}(t^*)\|_1^2 + \|\overline{\Psi}(t^*)\|_{1,\Gamma}^2 \le C_T \|\phi_1(0) - \phi_2(0), \psi_1(0) - \psi_2(0)\|_{\mathbb{X}}^2, \tag{5.20}$$

where C_T depends on T. Obviously, $\mathbb{V}_1 = H^1(\Omega) \times H^1(\Gamma)$ is compactly embedded in $\mathbb{X}$, since $\mathbb{V}_1$ is compactly injected in $\mathcal{H}$ and the definition of the norm of $\mathbb{X}$ is based on that of $\mathcal{H}$ (see (2.6), (2.7)). It follows that the assumption (5.5) of Proposition 5.3 is verified. Moreover, due to the decaying estimate (5.8), the functions Φ and Ψ satisfy

$$\|(\Phi(t^*), \Psi(t^*))\|_{\mathbb{X}}^2 \le k\|\phi_1(0) - \phi_2(0), \psi_1(0) - \psi_2(0)\|_{\mathbb{X}}^2, \tag{5.21}$$

where $k < 1/2$ (note that it is possible to do this, thanks to the estimate (5.8) and obvious inequality $\widetilde{C}\|(\cdot,\cdot)\|_{\mathbb{X}} \le \|(\cdot,\cdot)\|_{\mathcal{H}} \le C\|(\cdot,\cdot)\|_{H^1(\Omega)\times H^1(\Gamma)}$). Thus, assumption (5.4) is also verified.

Thus, according to the conclusion of Proposition 5.3, the operator $L = S(t^*)$ possesses an exponential attractor $\mathcal{M}^*$ on $\mathbb{B}$. Since, $\mathbb{B}$ is an absorbing set for this semigroup on $\mathbb{X}$, then these attractors attract exponentially all the bounded subsets of $\mathbb{X}$ with respect to the metric of $\mathbb{X}$. Moreover, the fractal dimension of $\mathcal{M}^*$ is finite, that is,

$$\dim_F(\mathcal{M}^*, \mathbb{X}) < \infty.$$

Next, using a standard formula, we define the desired exponential attractor by

$$\mathcal{M} = \cup_{t\in[0,T]} S(t)\mathcal{M}^*. \tag{5.22}$$

The semi-invariance (5.3) is then an immediate consequence of the semi-invariance of $\mathcal{M}^*$ and the definition (5.22). The exponential attraction

$$\lim_{t\to\infty} \operatorname{dist}_{\mathbb{X}}(S(t)B, \mathcal{M}) \le Q(\|B\|_{\mathbb{X}})e^{-\eta t}, \ \ t \ge 0,$$

(for every bounded subset $B \subset \mathbb{X}$) follows from the fact that $\mathbb{B}$ defined by (5.2) is an absorbing set for $S(t)$ (due to (3.12)) and from the uniform Lipschitz continuity (with respect to the initial data) due to (4.9). Thus, there only remains to verify that $\mathcal{M}$ has finite fractal dimension. Nevertheless, since the map $S(t)$ is uniformly Hölder continuous on $[0,T] \times \mathbb{B}$ in the norm of $\mathbb{X}$, it follows that $\mathcal{M}$ is still a compact set with finite fractal dimension that will be exponentially attracting for the semiflow $S(t)$ on $\mathbb{B}$. The Lipschitz continuity with respect to the initial data was verified in (4.9) and the Hölder continuity with respect to t follows from the fact that $(\partial_t\phi(t), \partial_t\psi(t))$ is in $\mathbb{X}$ if $(\phi(t), \psi(t))$ is in $\mathbb{Y}$. This result is given below in the following corollary.

Corollary 5.5. *The semigroup $S(t)$ defined in* (4.10) *is Holder continuous on $[0,T] \times \mathbb{B}$ in the topology of $\mathbb{X}$, ($\mathbb{B}$ is defined in* (5.2)*), that is,*

$$\begin{aligned} &\|S(t_2)(\phi_0^1, \psi_0^1) - S(t_1)(\phi_0^2, \psi_0^2)\|_{\mathbb{X}} \\ &\le C(R,T)[\|(\phi_0^1 - \phi_0^2, \psi_0^1 - \psi_0^2)\|_{\mathbb{X}} + |t_2 - t_1|^{1/2}], \end{aligned}$$

for $(\phi_0^i, \psi_0^i) \in \mathbb{B}$ and $t_i \in [0,T]$. Moreover, the constant $C(R,T)$ is independent of t.

Proof. The uniform Lipschitz continuity of S with respect to the initial data in the metric of $\mathbb{X}$ was actually verified in (4.9). It is left to verify the Hölder continuity with respect to t. Arguing as in the proof of Theorem 5.1, (3.25), we obtain

$$\int_T^{T+1} \|(\partial_t\phi(s), \partial_t\psi(s))\|_{\mathbb{X}}^2 ds \le Q(\|(\phi(0), \psi(0))\|_{\mathbb{Y}}^2) \le C(R), \tag{5.23}$$

(for a suitable monotonic function Q) if $(\phi_0, \psi_0) \in \mathbb{B}$. Moreover, for every t_1, $t_2 \in [0,T]$,

$$\begin{aligned} \|(\phi,\psi)(t_1) - (\phi,\psi)(t_2)\|_{\mathbb{X}} &= \|\int_{t_1}^{t_2} \partial_t(\phi(s), \psi(s)) ds\|_{\mathbb{X}} \\ &\le \int_{t_1}^{t_2} \|(\partial_t\phi(s), \partial_t\psi(s))\|_{\mathbb{X}} ds \\ &\le \Big(\int_{t_1}^{t_2} \|(\partial_t\phi(s), \partial_t\psi(s))\|_{\mathbb{X}}^2 ds\Big)^{1/2} |t_2 - t_1|^{1/2} \\ &\le \sqrt{C(R)} |t_2 - t_1|^{1/2}. \end{aligned}$$

□

Thus, we have constructed the exponential attractor $\mathcal{M}$ for the metric of $\mathbb{X}$ such that we have

$$\dim_F(\mathcal{M}, \mathbb{X}) \le \dim_F(\mathcal{M}^*, \mathbb{X}) + 2.$$

To obtain the attractor $\mathcal{M}$ with the respect of the metric in $H^2(\Omega) \times H^2(\Gamma)$, it is enough to note that the semigroup $S(t)$ possesses smoothing properties according to Theorem 3.4 and 3.5 and use the following interpolation inequality:

$$\|(\phi,\psi)\|_{H^2(\Omega)\times H^2(\Gamma)} \le C\|(\phi,\psi)\|_{\mathbb{X}}^{1/6} \|(\phi,\psi)\|_{\mathbb{B}}^{5/6}, \tag{5.24}$$

for some positive constant C independent of the solution. Thus, arguing as in [5], [16], [17], we can extend the above result to the case when the norm of $\mathbb{X}$ is replaced by $H^2(\Omega) \times H^2(\Gamma)$. We summarize this final result of this section in the following theorem on the existence of exponential attractors for problem (1.10)–(1.13).

Theorem 5.6. *There exists a compact set $\mathcal{M} \subset \mathbb{Y}$ which satisfies the following properties:*

(1) *The fractal dimension of $\mathcal{M}$ is finite, that is,* $\dim_F(\mathcal{M}, H^2(\Omega) \times H^2(\Gamma)) < \infty$.
(2) *Semi-invariance: $S(t)(\mathcal{M}) \subset \mathcal{M}$, $t \geq 0$.*
(3) *There exists a positive constant ρ and a monotonic function Q, such that for every bounded subset B of $\mathbb{X}$, we have:*
$$\lim_{t\to\infty} \operatorname{dist}_{H^2(\Omega)\times H^2(\Gamma)}(S(t)B, \mathcal{M}) \leq Q(\|B\|_{\mathbb{X}})e^{-\rho t}, \qquad t \geq 0.$$

6. Final remarks and open questions

In this section, let us consider a conserved version of the Cahn-Hilliard equation (1.1)–(1.4), that is,

$$\partial_t\phi = \Delta\mu \quad \text{in } [0,T]\times\Omega, \tag{6.1}$$
$$\mu = -\Delta\phi + f(\phi) \quad \text{in } [0,T]\times\Omega, \tag{6.2}$$

and

$$\partial_t\psi + b\partial_n\mu = 0 \quad \text{on } [0,T]\times\Gamma, \tag{6.3}$$
$$-\alpha\Delta_\Gamma\psi + \partial_n\phi + \beta\psi = \frac{\varpi}{b} \quad \text{on } [0,T]\times\Gamma, \tag{6.4}$$

with $\psi = \phi\big|_\Gamma$, $\varpi = \mu\big|_\Gamma$ and initial conditions $\phi(0,x) = \phi_0(x)$, $\psi(0,x) = \psi_0(x)$. We note that due to the divergence theorem and equations (6.1), (6.3), we have

$$\frac{d}{dt}\Big(\int_\Omega \phi(t,x)dx + \int_\Gamma \psi(t,x)\frac{dS}{b}\Big) = 0,$$

hence the total mass

$$\int_\Omega \phi(t,x)dx + \int_\Gamma \psi(t,x)\frac{dS}{b} = \int_\Omega \phi_0(x)dx + \int_\Gamma \psi_0(x)\frac{dS}{b} \tag{6.5}$$

is conserved for all time $t > 0$. We also observe that the boundary condition (6.3) is the same as (1.3), if $c = 0$. This case becomes more difficult to treat since the operator defined by (2.4)–(2.5) may not be invertible on $\mathcal{H}$. Nevertheless, we may construct such an invertible operator $\mathcal{N}$, if we restrict the vectors $(\phi(t), \psi(t))$ to a subspace of $\mathcal{H}$ such that $(\phi(t), \psi(t))$ satisfy the conservation property (6.5). Moreover, it is apparent from the results of Section 3, that the estimates for the solution (ϕ, ψ) satisfying (6.1)–(6.4) are uniform with respect to the parameter b. We could then regard the coefficient b in the boundary conditions (6.3) and (6.4) as a parameter and consider the parameter dependent solution of this system, and construct a nonlinear semigroup $\mathcal{S}_b(t) = S(t,b) : \mathbb{Y}(b) \to \mathbb{Y}(b)$, $t > 0$, for each $b \in [b_0, +\infty)$ with $b_0 > 0$. Here the space $\mathbb{Y}(b)$ will depend explicitly on the parameter b (also, see (2.11)). When $b \to +\infty$, the boundary conditions (6.3) and (6.4) become formally a homogeneous Neumann condition for the chemical potential μ and a homogeneous boundary condition of second order for ϕ, respectively. Hence, we formally obtain the limiting system of equations:

$$\partial_t\phi = \Delta\mu \quad \text{in } [0,T]\times\Omega, \tag{6.6}$$
$$\mu = -\Delta\phi + f(\phi) \quad \text{in } [0,T]\times\Omega, \tag{6.7}$$

and

$$\partial_n \mu = 0 \quad \text{on } [0,T] \times \Gamma, \tag{6.8}$$

$$-\alpha \Delta_\Gamma \psi + \partial_n \phi + \beta \psi = 0 \quad \text{on } [0,T] \times \Gamma. \tag{6.9}$$

Such a system was considered and studied in [21, Section 7]. We note that (6.6), (6.7) imply that the mass

$$\int_\Omega \phi(t,x) dx = \int_\Omega \phi_0(x) dx \tag{6.10}$$

is conserved for all time $t > 0$. Notice that (6.5) becomes (6.10) when $b = +\infty$.

The fundamental question that arises from these initial observations is whether we may be able to construct a family of robust exponential attractors $\mathcal{M}_b$ for the semi-flow $\mathcal{S}_b(t)$ associated with this problem (6.1)–(6.4), whenever $b \in [b_0, +\infty)$. Thus, if the uniform and decaying estimates in Theorem 3.5 and its supporting lemmas hold, it is natural to ask whether the sets $\mathcal{M}_b$ tend to the limit set $\mathcal{M}_\infty$ as $b \to +\infty$ in the following sense:

$$\operatorname{dist}_X(\mathcal{M}_b; \mathcal{M}_\infty) \to 0,$$

where dist_X denotes the Hausdorff distance in the topology of a suitable metric space X. We will investigate such a problem in a forthcoming article.

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