A LIOUVILLE THEOREM FOR $F$-HARMONIC MAPS WITH FINITE $F$-ENERGY

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Abstract. Let $(M, g)$ be a $m$-dimensional complete Riemannian manifold with a pole, and $(N, h)$ a Riemannian manifold. Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing $C^2$ function such that $F(0) = 0$ and $d_F := \sup(tF'(t)(F(t))^{-1}) < \infty$. We show that if $d_F < m/2$, then every $F$-harmonic map $u : M \to N$ with finite $F$-energy (i.e. a local extremal of $E_F(u) := \int_M F(|du|^2/2)dv_g$ and $E_F(u)$ is finite) is a constant map provided that the radial curvature of $M$ satisfies a pinching condition depending to $d_F$.

1. Introduction and statement of result

Let $(M, g)$ and $(N, h)$ be two Riemannian manifolds and $F$ be a given $C^2$ function $F : \mathbb{R}^+ \to \mathbb{R}^+$. Then, a map $u : M \to N$ of class $C^2$ is said to be $F$-harmonic if for every compact $K$ of $M$, the map $u$ is extremal of $F$-energy:

$$E_F(u) := \int_K F\left(\frac{|du|^2}{2}\right)dv_g.$$ 

In a normal coordinate system, the tension field associated with $E_F(u)$ by the Euler-Lagrange equations is

$$\tau_F(u) := \sum_{i=1}^{m} (\nabla e_i(F\left(\frac{|du|^2}{2}\right)du))e_i = F'\left(\frac{|du|^2}{2}\right)\tau(u) + du.\left\{\text{grad}\left(F\left(\frac{|du|^2}{2}\right)\right)\right\}$$

where $\tau(u)$ is the usual tension field of $u$ defined by

$$\tau(u)_k = \Delta_M u^k + \sum_{\beta, \gamma;i,j}^{n,m} N^k_{\alpha \gamma}(u)g^{ij} \frac{\partial u^\beta}{\partial x_i} \frac{\partial u^\gamma}{\partial x_j}, \quad k = 1, \ldots, n.$$ 

Then, the map $u$ is $F$-harmonic if $\tau_F(u) = 0$. For further properties of $F$-harmonic maps, we refer the reader to \[1,2\]. For the particular case of $F(t) = t$, the Liouville problem for harmonic maps with finite energy have been studied in \[4,6,7,8,9\]. While for $F(t) = \frac{2}{p}t^{p/2}$, with $p \geq 2$, this is the problem of $p$-harmonic maps with finite $p$-energy (corollary 1.2. If $F(t) = \sqrt{t+2t} - 1$ corresponding to the minimal graph (corollary 1.3). In this paper, we study the same problem for $F$-harmonic maps with finite $F$-energy without condition on the curvature for the
Let \((M, g)\) be a \(m\)-dimensional complete Riemannian manifold, \(m > 2\), with a pole \(x_0\), and let \((N, h)\) be a Riemannian manifold. If \(d_F < m/2\), then every \(F\)-harmonic map of \(M\) into \(N\) with finite \(F\)-energy is constant provided that the radial curvature \(K_r\) of \(M\) satisfies one of the following two conditions:

(i) \(-\alpha^2 \leq K_r \leq -\beta^2\) with \(\alpha > 0, \beta > 0\) and \(1 + (m - 1)\beta - 2d_F\alpha > 0\)

(ii) \(-\frac{\alpha}{1 + \tau^2} \leq K_r \leq \frac{\beta}{1 + \tau^2}\) with \(\alpha \geq 0\) and \(\beta \in [0, \frac{1}{2}]\) such that \(2 + (m - 1)(1 + \sqrt{1 - 4\beta}) - 2d_F(1 + \sqrt{1 + 4\alpha}) > 0\).

Furthermore, we have the following corollaries.

**Corollary 1.2.** Let \((M, g)\) and \((N, h)\) be as in the theorem. Then, every \(C^2\) \(p\)-harmonic map of \(M\) into \(N\) with finite \(p\)-energy, for \(p < m\), is constant.

**Corollary 1.3.** Let \((M, g)\) and \((N, h)\) be as in the theorem. Then, for \(m > 2\), every \(C^2\) map \(u\) of \(M\) into \(N\), with finite energy, solution of

\[
\frac{\tau(u)}{\sqrt{1 + |du|^2}} + du \left\{ \text{grad} \left( \frac{1}{\sqrt{1 + |du|^2}} \right) \right\} = 0
\]

is constant.

For \(m = 2\), the statement of the theorem is false in general. In fact, for the case (i), there exist holomorphic maps of the hyperbolic disc with finite energy \([9]\).

While for the case (ii) there exist holomorphic maps of \(\mathbb{C}\) into \(\mathbb{P}^1\) with finite energy \([9]\).

## 2. PROOF OF THEOREM 1.1

Let \(X\) and \(Y\) be two vector fields on \(M\). It is well-known \([3, 6]\), that the stress-energy for harmonic maps is

\[
S_u := \frac{|du|^2}{2} \langle X, Y \rangle_g - \langle du(X), du(Y) \rangle_h
\]

and satisfies

\[
\langle \text{div} S_u \rangle(X) = -\langle \tau(u), du(X) \rangle_h.
\]

Following \([2]\), we define the stress-energy of \(F\)-harmonic maps by

\[
S_{F,u}(X, Y) := F\left( \frac{|du|^2}{2} \right) \langle X, Y \rangle_g - F\left( \frac{|du|^2}{2} \right) \langle du(X), du(Y) \rangle_h.
\]

When \(F(t) := t\) we have \(S_{F,u} := S_u\). Also \(\langle \text{div} S_{F,u} \rangle(X) = -\langle \tau_F(u), du(X) \rangle_h\) thanks to the following lemma.

**Lemma 2.1.** For every vector field \(X\) on \(M\), we have

\[
\langle \text{div} S_{F,u} \rangle(X) = -\langle \tau_F(u), du(X) \rangle_h, \quad (2.1)
\]

\[
\text{div}(F\langle \frac{|du|^2}{2} \rangle X)
\]

\[
= \text{div}(F'(\frac{|du|^2}{2}) \langle du(X), du(e_i) \rangle_h e_i) - \langle \tau_F(u), du(X) \rangle_h + [S_{F,u}, X], \quad (2.2)
\]
where
\[
[S_{F,u}, X](x) = \sum_{i,j=1}^{m} \left( F\left(\frac{|du|^2}{2}\right)\delta_{ij} - F'\left(\frac{|du|^2}{2}\right)(du(e_i), du(e_j)) \right) \langle \nabla_e, X, e_j \rangle_g.
\]

In particular, if \( u \) is \( F \)-harmonic and \( D \subset M \) is a \( C^1 \) boundary domain, then we have
\[
\int_{\partial D} S_{F,u}(X, \nu) d\sigma_g = \int_D [S_{F,u}, X] dV_g
\]
where \( \nu \) is the normal to \( \partial D \).

Proof. Let \( x \in M \). Chose a normal coordinate system such that at \( x \). \( g_{ij}(x) = \delta_{ij} \)
d\( g(x) = 0 \), where \( (e_1, \ldots, e_m) \) being a normal basis, we have \( \nabla_{e_j} e_k = 0 \) for all \( j, k \)
and
\[
(\text{div } S_{F,u})(X) = \sum_{i=1}^{m} \left\{ \nabla_{e_i} S_{F,u}(e_i, X) - S_{F,u}(e_i, \nabla_e X) - S_{F,u}(\nabla_{e_i} e_i, X) \right\}
\]
\[
= \sum_{i=1}^{m} \left\{ \nabla_{e_i} \left( F\left(\frac{|du|^2}{2}\right)\langle e_i, X \rangle - F'\left(\frac{|du|^2}{2}\right)(du(e_i), du(X)) \right) - F\left(\frac{|du|^2}{2}\right)(e_i, \nabla_e X)
+ F\left(\frac{|du|^2}{2}\right)(du(e_i), du(\nabla_e X)) - S_{F,u}(\nabla_{e_i} e_i, X) \right\}
\]
\[
= \sum_{i=1}^{m} \left\{ \nabla_{e_i} \left( F\left(\frac{|du|^2}{2}\right)(du(e_i), du(X)) \right) - F\left(\frac{|du|^2}{2}\right)(e_i, \nabla_e X)
+ F\left(\frac{|du|^2}{2}\right)(du(e_i), du(\nabla_e X)) - S_{F,u}(\nabla_{e_i} e_i, X) \right\}
\]
\[
= \sum_{i=1}^{m} \left\{ \sum_{j=1}^{m} F'\left(\frac{|du|^2}{2}\right)(\nabla_{e_i}(du(e_j)), du(e_j)) \langle e_i, X \rangle
+ F\left(\frac{|du|^2}{2}\right)\nabla_{e_i} \langle e_i, X \rangle - \nabla_{e_i} \left( F'\left(\frac{|du|^2}{2}\right)(du(e_i)), du(X) \right)
- F'\left(\frac{|du|^2}{2}\right)(du(e_i), \nabla_{e_i} (du(X)))
- F\left(\frac{|du|^2}{2}\right)(e_i, \nabla_{e_i} X) + F'\left(\frac{|du|^2}{2}\right)(du(e_i), du(\nabla_{e_i} X))
- S_{F,u}(\nabla_{e_i} e_i, X) \right\}.
\]
Thus
\[
(\text{div } S_{F,u})(X) = \sum_{i,j=1}^{m} \left\{ F'\left(\frac{|du|^2}{2}\right)(\nabla_{e_i}(du(e_j)), du(e_j)) X_i \right\}
+ \sum_{i=1}^{m} \left\{ F\left(\frac{|du|^2}{2}\right)(\nabla_{e_i} e_i, X) + F\left(\frac{|du|^2}{2}\right)(e_i, \nabla_{e_i} X)
- \nabla_{e_i} \left( F'\left(\frac{|du|^2}{2}\right)(du(e_i)), du(X) \right) \right\}.
Then, by straightforward computation, we obtain

\[
\begin{align*}
- F'\left(\frac{|du|^2}{2}\right)\langle du(e_i), \nabla_{e_i}(du(X)) \rangle & - F\left(\frac{|du|^2}{2}\right)\langle e_i, \nabla_{e_i}X \rangle \\
+ F'\left(\frac{|du|^2}{2}\right)\langle du(e_i), du(\nabla_{e_i}X) \rangle & - S_{F,u}(\nabla_{e_i}e_i, X) \\
= \sum_{i,j=1}^{m} \left\{ F'\left(\frac{|du|^2}{2}\right)\langle X_i\nabla_{e_i}(du(e_j)), du(e_j) \rangle \right. \\
- \sum_{i=1}^{m} \left\{ F'\left(\frac{|du|^2}{2}\right)\langle du(e_i), \nabla_{e_i}(du(X)) \rangle \\
+ F'\left(\frac{|du|^2}{2}\right)\langle du(e_i), du(\nabla_{e_i}X) \rangle & + F\left(\frac{|du|^2}{2}\right)\langle \nabla_{e_i}e_i, X \rangle - F\left(\frac{|du|^2}{2}\right)\langle e_i, \nabla_{e_i}X \rangle \\
- \langle \nabla_{e_i}(F'\left(\frac{|du|^2}{2}\right)du(e_i)), du(X) \rangle & - S_{F,u}(\nabla_{e_i}e_i, X) \right\}.
\end{align*}
\]

Since \( \nabla_{e_i}e_i = 0 \), with \((\nabla_{e_i}du)(X) = \nabla_{e_i}(du(X)) - du(\nabla_{e_i}X) \) and by symmetry \((\nabla_{e_i}du)(X) = (\nabla_X du)(e_i) \), we have

\[
\text{div}(S_{F,u})(X) = \sum_{j=1}^{m} \left\{ F'\left(\frac{|du|^2}{2}\right)\langle \nabla_X (du(e_j)), du(e_j) \rangle \right. \\
- \sum_{i=1}^{m} \left\{ F'\left(\frac{|du|^2}{2}\right)\langle du(e_i), \nabla_{e_i}(du(X)) \rangle - du(\nabla_{e_i}X) \rangle \\
- \langle \nabla_{e_i}(F'\left(\frac{|du|^2}{2}\right)du(e_i)), du(X) \rangle \right\}.
\]

Finally,

\[
\text{div}(S_{F,u})(X) = -\langle \tau_F(u), du(X) \rangle.
\]

Also

\[
\text{div}(F\left(\frac{|du|^2}{2}\right)X) \begin{aligned}
= & \sum_{i=1}^{m} \langle \nabla_{e_i}(F\left(\frac{|du|^2}{2}\right)X), e_i \rangle \\
= & \sum_{i=1}^{m} \left\{ \langle \nabla_{e_i}(F\left(\frac{|du|^2}{2}\right)X), e_i \rangle + F\left(\frac{|du|^2}{2}\right)\langle \nabla_{e_i}X, e_i \rangle \right\} \\
= & \nabla_X F\left(\frac{|du|^2}{2}\right) + \sum_{i=1}^{m} F\left(\frac{|du|^2}{2}\right)\langle \nabla_{e_i}X, e_i \rangle.
\end{aligned}
\]

Then, by straightforward computation, we obtain

\[
\nabla_X F\left(\frac{|du|^2}{2}\right) = \sum_{i=1}^{m} \frac{1}{2} F'\left(\frac{|du|^2}{2}\right)\nabla_X \langle du(e_i), du(e_i) \rangle \\
= \sum_{i=1}^{m} F'\left(\frac{|du|^2}{2}\right)\langle \nabla_X (du(e_i)), du(e_i) \rangle \\
= \sum_{i=1}^{m} F'\left(\frac{|du|^2}{2}\right)\langle \nabla_X du(e_i) + du(\nabla_X e_i), du(e_i) \rangle
\]
EJDE-2006/15
A LIOUVILLE THEOREM FOR $F$-HARMONIC MAPS

Thus

$$
\nabla_X F \left( \frac{|du|^2}{2} \right) = \sum_{i=1}^{m} \left\{ \nabla_{e_i} \langle du(X), F' \left( \frac{|du|^2}{2} \right) du(e_i) \rangle 
- \langle du(X), \nabla_{e_i} \left( F' \left( \frac{|du|^2}{2} \right) du(e_i) \right) \rangle 
- F' \left( \frac{|du|^2}{2} \right) \langle du(\nabla_{e_i} X), du(e_i) \rangle \right\}
$$

Thus

$$
\text{div} \left( \frac{|du|^2}{2} \right) X = \sum_{i=1}^{m} \left\{ \text{div} \left( F' \left( \frac{|du|^2}{2} \right) \langle du(X), du(e_i) \rangle e_i \right) 
- \langle du(X), \tau_{F}\rangle(u) \right\} - \sum_{i=1}^{m} F' \left( \frac{|du|^2}{2} \right) \langle du(\nabla_{e_i} X), du(e_i) \rangle
$$

with

$$
[S_{F,u}, X] = \sum_{i,j=1}^{m} \left( F \left( \frac{|du|^2}{2} \right) \delta_{ij} - F' \left( \frac{|du|^2}{2} \right) \langle du(e_i), du(e_j) \rangle \right) \langle \nabla_{e_i} X, e_j \rangle \delta
$$

because $\nabla_{e_i} X = \langle \nabla_{e_i} X, e_j \rangle e_j$. If $D \subset M$ is a $C^1$ boundary domain, we get by the use of Stokes formula

$$
\int_D (\text{div} \, S_{F,u})(X) + \int_D [S_{F,u}, X] = \int_D \text{div} \left( F' \left( \frac{|du|^2}{2} \right) \langle du(X), du(e_i) \rangle e_i \right) - \int_D \sum_{i=1}^{m} \text{div} \left( F' \left( \frac{|du|^2}{2} \right) \langle du(X), du(e_i) \rangle e_i \right) < du(X), du(e_i) > e_i
$$
Thus, if \( u \) is \( F \)-harmonic:

\[
\int_{\partial D} \left( F\left(\frac{|du|^2}{2}\right) \langle X, \nu \rangle - F'(\frac{|du|^2}{2}) \langle du(X), du(\nu) \rangle \right) = \int_D [S_{F,u}, X].
\]

This completes the proof. \( \square \)

**Lemma 2.2.** Let \( u : M \to N \) be a \( F \)-harmonic with finite \( F \)-energy and \( X \) a vector field on \( M \) such that \(|X| \leq \phi(r)\) for \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying

\[
\int_1^{+\infty} \frac{dt}{\phi(t)} = +\infty.
\]

Then there exists an increasing strictly sequence \( (R_n) \) such that

\[
\lim_{n \to \infty} \int_{B(x_0, R_n)} [S_{F,u}, X] dV_g = 0.
\]

**Proof.** Since \( tF'(t) \leq d_F F(t) \) we have

\[
\left| \int_{B(x_0, R)} [S_{F,u}, X] \right| 
\leq \left| \int_{\partial B(x_0, R)} F\left(\frac{|du|^2}{2}\right) \langle X, \nu \rangle \right| + \left| \int_{\partial B(x_0, R)} F'(\frac{|du|^2}{2}) \langle du(X), du(\nu) \rangle \right|
\leq \int_{\partial B(x_0, R)} F\left(\frac{|du|^2}{2}\right) |X| + \int_{\partial B(x_0, R)} F'(\frac{|du|^2}{2}) |X| dV_g,
\]

By the Co-area formula and \(|X| \leq \phi(r(x))\),

\[
\int_0^{\infty} \frac{1}{\phi(t)} \left( \int_{\partial B(x_0, t)} F\left(\frac{|du|^2}{2}\right) |X| \right) dt = \int_M \frac{|X| |
\leq \int_M \frac{\phi(r) |X|}{\phi(r)} F\left(\frac{|du|^2}{2}\right) |X|< \infty.
\]

Since \( \int_1^{+\infty} \frac{dt}{\phi(t)} = +\infty \), there exists an increasing strictly sequence \( (R_n) \) such that

\[
\lim_{n \to \infty} \int_{\partial B(x_0, R_n)} F\left(\frac{|du|^2}{2}\right) |X| = 0.
\]

Hence

\[
\lim_{n \to \infty} \int_{B(x_0, R_n)} [S_{F,u}, X] dV_g = 0.
\]

This completes the proof of Lemma 2.2. \( \square \)

For the theorem, it suffices to choose \( X \) satisfying Lemma 2.2 and the condition

\[ [S_{F,u}, X] \geq cF(|du|^2/2) \]

where \( c > 0 \) is a constant. For that we take \( X = r \nabla r \) and using the comparison theorem of the Hessian [5].

**Theorem 2.3** (Comparison theorem). Let \((M, g)\) be a complete Riemannian manifold with a pole \( x_0 \) and \( k_1, k_2 \) be two continuous functions on \( \mathbb{R}^+ \) such that
where \( k_2(r) \leq K_r \leq k_1(r) \), where \( K_r \) is the radial curvature of \( M \), i.e., the sectional curvature of the tangent planes containing the radial vector \( \nabla r \). Also, let \( J_i, (i = 1, 2) \) be the solution of classical Jacobi equation

\[ J_i'' + k_i J_i = 0; \quad J_i(0) = 0 \quad \text{and} \quad J_i'(0) = 1. \]

Then, if \( J_1 > 0 \) on \( \mathbb{R}^+ \), we have on \( M \setminus \{ x_0 \} \)

\[ \frac{J_1'(r)}{J_1(r)}(g - dr \otimes dr) \leq \text{Hess}(r) \leq \frac{J_2'(r)}{J_2(r)}(g - dr \otimes dr). \]

Case (i) of Theorem 2.3 With \( k_1(r) = -\beta^2 \) and \( k_2(r) = -\alpha^2 \), we have

\[ \beta \coth(\beta r)(g - dr \otimes dr) \leq \text{Hess}(r) \leq \alpha \coth(\alpha r)(g - dr \otimes dr). \]

Case (ii) of Theorem 2.3 With \( k_1(r) = \frac{\beta}{r} \) and \( k_2(r) = -\frac{\alpha}{r^2} \), and the fact that on \( M \setminus \{ x_0 \} \),

\[ -\frac{\alpha}{r^2} \leq -\frac{\alpha}{1+r^2} \leq K_r \leq \frac{\beta}{1+r^2} \leq \frac{\beta}{r^2} \]

we have

\[ \left(1 + \sqrt{1 - 4\beta^2}\right)(g - dr \otimes dr) \leq \text{Hess}(r) \leq \left(1 + \sqrt{1 - 4\alpha^2}\right)(g - dr \otimes dr). \]

Lemma 2.4. Under hypothesis of Theorem 2.3, in case (1), we have

\[ [S_{F,u}, X] \geq (1 + (m - 1)\beta - 2d_F\alpha) F\left(\frac{|du|^2}{2}\right) \]

and in case (ii),

\[ [S_{F,u}, X] \geq \frac{1}{2}(2 + (m - 1)(1 + \sqrt{1 - 4\beta^2}) - 2d_F(1 + \sqrt{1 + 4\alpha^2}) F\left(\frac{|du|^2}{2}\right). \]

Proof. First note that

\[ [S_{F,u}, X] = \sum_{i,j=1}^{m} \left(F\left(\frac{|du|^2}{2}\right)\delta_{ij} - F'(\frac{|du|^2}{2})(du(e_i), du(e_j))h \right) < \nabla e_i, X, e_j, g, \]

where \( (e_1, \ldots, e_{m-1}, \frac{\partial}{\partial r}) \) with \( e_m = \frac{\partial}{\partial r} \), being a normal basis on \( B(x_0, R) \). Then, since \( X = r \frac{\partial}{\partial r} \), it follows that \( \nabla \frac{\partial}{\partial r} X = \frac{\partial}{\partial r} \) and so we get

\[ \langle \nabla_{\frac{\partial}{\partial r}} X, \frac{\partial}{\partial r} \rangle_g = 1, \]

\[ \langle \nabla e_i, X, e_i \rangle_g = r \text{Hess}(r)(e_i, e_i), \quad \text{for} \quad i = 1, \ldots, m - 1, \]

\[ \nabla e_i, X = \sum_{j=1}^{m-1} r \text{Hess}(r)(e_i, e_j), \quad \text{for} \quad i = 1, \ldots, m - 1. \]

Therefore,

\[ [S_{F,u}, X] = F\left(\frac{|du|^2}{2}\right)(1 + \sum_{i=1}^{m-1} r \text{Hess}(r)(e_i, e_i)) \]

\[ - \sum_{i,j=1}^{m-1} F'(\frac{|du|^2}{2})(du(e_i), du(e_j))h \langle \nabla e_i, X, e_j \rangle_g \]

\[ - F'(\frac{|du|^2}{2})(du(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r}))h \langle \nabla \frac{\partial}{\partial r}, X, \frac{\partial}{\partial r} \rangle_g \]
\[- \sum_{j=1}^{m-1} F'(\frac{|du|^2}{2}) \langle du(\partial_r), du(e_j) \rangle_h \langle \nabla_\partial X, e_j \rangle_g \]
\[- \sum_{i=1}^{m-1} F'(\frac{|du|^2}{2}) \langle du(e_i), du(\partial_r) \rangle_h \langle \nabla_\partial e_i, \partial_r \rangle_g \]
\[= F(\frac{|du|^2}{2})(1 + \sum_{i=1}^{m-1} r \text{Hess}(r)(e_i, e_i)) \]
\[- \sum_{i,j=1}^{m-1} F'(\frac{|du|^2}{2}) \langle du(e_i), du(e_j) \rangle_r \text{Hess}(r)(e_i, e_j) \]
\[- F'(\frac{|du|^2}{2}) \langle du(\partial_r), du(\partial_r) \rangle \]

For the case (i), we have
\[\langle S_{F,u}, X \rangle \geq F(\frac{|du|^2}{2}) + (m-1)(\beta r) \text{coth}(\beta r) F(\frac{|du|^2}{2}) \]
\[- F'(\frac{|du|^2}{2})|du|^2(\alpha r) \text{coth}(\alpha r) \]
\[+ F'(\frac{|du|^2}{2})(\alpha r) \text{coth}(\alpha r) - 1) \langle du(\partial_r), du(\partial_r) \rangle \]
\[\geq F(\frac{|du|^2}{2}) + F(\frac{|du|^2}{2})((m-1)(\beta r) \text{coth}(\beta r) - 2d_F(\alpha r) \text{coth}(\alpha r)) \]
\[\geq F(\frac{|du|^2}{2}) + F(\frac{|du|^2}{2}) r \text{coth}(\beta r)((m-1)\beta - 2d_F(\alpha r) \text{coth}(\alpha r)). \]

Since the function $\text{coth}(x)$ is decreasing and, $x \text{coth}(x)$ is bounded below by a positive constant in $\mathbb{R}^+$, we have
\[\langle S_{F,u}, X \rangle \geq (1 + (m-1)\beta - 2d_F(\alpha r))F(\frac{|du|^2}{2}) \]

For the case (ii), we have
\[\langle S_{F,u}, X \rangle \geq F(\frac{|du|^2}{2}) + (m-1)a F(\frac{|du|^2}{2}) - b F'(\frac{|du|^2}{2})|du|^2 \]
\[+ (b-1) F'(\frac{|du|^2}{2}) \langle du(\partial_r), du(\partial_r) \rangle \]
\[\geq (1 + (m-1)a - 2d_F(\alpha r))F(\frac{|du|^2}{2}), \]

where we have set
\[a = \frac{1 + \sqrt{1 - 4\beta}}{2} \text{ and } b = \frac{1 + \sqrt{1 + 4\alpha}}{2} \geq 1. \]

\[\square \]

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