A CHARACTERIZATION OF BALLS USING THE DOMAIN DERIVATIVE

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Abstract. In this note we give a characterization of balls in $\mathbb{R}^N$ using the domain derivative. As a byproduct we will show that an overdetermined Stekloff eigenvalue problem is solvable if and only if the domain of interest is a ball.

1. Introduction

In this note we give a characterization of balls in $\mathbb{R}^N$ using the domain derivative. As an application we prove that an overdetermined Stekloff eigenvalue problem is solvable if the domain of interest is a ball. This work is motivated by the following result.

**Theorem 1.1.** A domain $D \subset \mathbb{R}^N$ is a ball if and only if there exists a constant $c$ such that the following integral equality is valid

$$\int_D h \, dx = c \int_{\partial D} h \, d\sigma, \quad (1.1)$$

for every harmonic function $h$.

For the proof of the above theorem, the reader is referred to [1, 3].

Our characterization replaces (1.1) by another integral equation which involves the domain derivative of the solution of the Saint-Venant equation in $D$. This result will enable us to show that an overdetermined Stekloff eigenvalue problem is solvable if and only if the domain of the problem is a ball.

2. Main result

To state the main result we need some preparation. Henceforth $D$ is a smooth simply connected bounded domain in $\mathbb{R}^N$. By $u$ we denote the unique solution of the Saint-Venant problem in $D$; i.e.,

$$-\Delta u = 1 \quad \text{in } D$$
$$u = 0 \quad \text{on } \partial D \quad (2.1)$$

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Given a vector field $V \in C^2(\mathbb{R}^N;\mathbb{R}^N)$, we denote by $u'$, the domain derivative of $u$ at $D$ in direction of $V$; the reader is referred to [5] for a thorough treatment of the concept of domain derivatives. Using [5, Theorems 3.1 and 3.2], it follows that

$$\Delta u' = 0 \text{ in } D$$
$$u' = -\frac{\partial u}{\partial \nu} V \cdot \nu \text{ on } \partial D,$$

where $\nu$ stands for the unit outward normal vector on $\partial D$. Now we state our main result.

**Theorem 2.1.** The domain $D$ is a ball if and only if there exists a constant $c$ such that the following integral equation is valid

$$\int_D u' \, dx = c \int_{\partial D} u' \, d\sigma,$$  \tag{2.3}

for every vector field $V \in C^2(\mathbb{R}^N;\mathbb{R}^N)$.

We need the following result.

**Lemma 2.2.** Suppose $f \in C(\partial D)$ and the following equation holds

$$\int_{\partial D} f V \cdot \nu \, d\sigma = 0,$$  \tag{2.4}

for every $V \in C^2(\mathbb{R}^N,\mathbb{R}^N)$. Then $f$ vanishes on $\partial D$.

**Proof.** To derive a contradiction suppose $f(x_0) \neq 0$, for some $x_0 \in \partial D$. Let us assume that in fact $f(x_0) > 0$; the case $f(x_0) < 0$ can be addressed similarly. Since $f$ is continuous, we readily infer existence of an open component of $\partial D$, denoted $\gamma$, where

$$f(x) \geq \frac{1}{k}, \quad \forall x \in \gamma,$$

for some integer $k$. Thanks to smoothness of $\partial D$ we can make the following observation; namely, $\partial D$ is locally star-shaped. This means: For every $\xi \in \partial D$, there exists a ball $B_\xi$ centered at $\xi$, and a point $x_\xi \in D$, such that

$$(x - x_\xi) \cdot \nu(x) > 0, \quad \forall x \in B_\xi \cap \partial D.$$  

Without loss of generality we may assume there exists $x^* \in D$ such that

$$(x - x^*) \cdot \nu(x) > 0, \quad \forall x \in \gamma.$$  

Let us now consider a non-negative test function $\phi \in C_0^\infty(\mathbb{R}^N)$, where the intersection of the support of $\phi$ with $\partial D$ is a proper subset of $\gamma$ and has positive measure.

Now we choose $V = \phi(x)(x - x^*)$ in (2.4); note that $V$ is admissible since it belongs to $C^2(\mathbb{R}^N,\mathbb{R}^N)$. Thus

$$\int_{\gamma} f(x)\phi(x)(x - x^*) \cdot \nu(x) \, d\sigma = 0.$$  \tag{2.5}

However

$$\int_{\gamma} f(x)\phi(x)(x - x^*) \cdot \nu(x) \, d\sigma \geq \frac{1}{k} \int_{\text{support}(\phi) \cap \gamma} \phi(x)(x - x^*) \cdot \nu(x) \, d\sigma > 0,$$

which contradicts (2.5). Thus $f$ must vanish on $\partial D$, as desired. □
Proof of Theorem 2.1. Assume that (2.3) is satisfied. Let us fix $V \in C^2(\mathbb{R}^N; \mathbb{R}^N)$. We claim
\[
\int_D u' \, dx = \int_{\partial D} \left( \frac{\partial u}{\partial \nu} \right)^2 V \cdot \nu \, d\sigma. \tag{2.6}
\]
To prove (2.6) we observe that from the differential equation in (2.1) we have $\int_D u' \, dx = -\int_D u' \Delta u \, dx$. Since $u'$ is harmonic in $D$ it then follows that
\[
\int_D u' \, dx = \int_D (u \Delta u' - u' \Delta u) \, dx.
\]
Now an application of the Green identity to the right hand side of the above equation yields
\[
\int_D u' \, dx = \int_{\partial D} \left( u \frac{\partial u'}{\partial \nu} - u' \frac{\partial u}{\partial \nu} \right) \, d\sigma.
\]
Since $u$ vanishes on $\partial D$, the above equation implies
\[
\int_D u' \, dx = -\int_{\partial D} u' \frac{\partial u}{\partial \nu} \, d\sigma. \tag{2.7}
\]
From (2.7) and the boundary condition in (2.2) we derive (2.6). From the hypothesis and (2.6) we obtain $c \int_{\partial D} u' \, d\sigma = \int_{\partial D} \left( \frac{\partial u}{\partial \nu} \right)^2 V \cdot \nu \, d\sigma$. So again using the boundary condition in (2.2) we derive
\[
-c \int_{\partial D} \frac{\partial u}{\partial \nu} V \cdot \nu \, d\sigma = \int_{\partial D} \left( \frac{\partial u}{\partial \nu} \right)^2 V \cdot \nu \, d\sigma.
\]
So
\[
\int_{\partial D} \left( \left( \frac{\partial u}{\partial \nu} \right)^2 + c \frac{\partial u}{\partial \nu} \right) V \cdot \nu \, d\sigma = 0.
\]
Since $V \in C^2(\mathbb{R}^N; \mathbb{R}^N)$ is arbitrary Lemma 2.2 applied to the above equation, guarantees that
\[
\frac{\partial u}{\partial \nu} \left( \frac{\partial u}{\partial \nu} + c \right) = 0 \quad \text{on } \partial D.
\]
By the Hopf boundary point lemma applied to (2.1) we infer that $\frac{\partial u}{\partial \nu}$ is negative on $\partial D$. So the last equation implies $\frac{\partial u}{\partial \nu} = -c$ on $\partial D$. This result added to (2.1) yields the following overdetermined boundary value problem
\[
\begin{align*}
-\Delta u &= 1 \quad \text{in } D \\
u &= 0 \quad \text{on } \partial D \\
\frac{\partial u}{\partial \nu} &= -c \quad \text{on } \partial D
\end{align*}
\tag{2.8}
\]
It is classical, see [1, 6], that (2.8) is solvable if and only if $D$ is a ball.

Conversely, let us assume that $D$ is a ball. Without loss of generality we may assume that $D$ is the ball with radius $R$ centered at the origin. Note that in this case the solution of (2.1) is
\[
u(x) = \frac{1}{2N} (R^2 - |x|^2).
\]
Therefore $\frac{\partial u}{\partial \nu}$ will be equal to $-R/N$ on $\partial D$. So if we apply (2.7) we find that
\[
\int_D u' \, dx = -\frac{R}{N} \int_{\partial D} u' \, d\sigma,
\]
which coincides with the integral equation (2.3), with $c = -R/N$. This completes the proof. □
Note that \( c = -\frac{R}{N} \), as in the above argument, could also be written as \( c = -\frac{\omega_N R^N}{S_N} = -\frac{V(D)}{S(D)} \), where \( \omega_N \) stands for the volume of the unit \( N \)-dimensional ball, and \( V(D), S(D) \) denote the volume and the surface area of \( D \), respectively.

In the remaining of this section we focus on the Stekloff eigenvalue problem; i.e.,

\[
\Delta w = 0 \quad \text{in } D.
\]
\[
\frac{\partial w}{\partial \nu} = pw \quad \text{on } \partial D
\]

In (2.9), \( p \) denotes the eigenvalue. It is well known that there are infinitely many eigenvalues \( 0 = p_1 < p_2 \leq p_3 \leq \ldots \) for which (2.9) has non trivial solutions. These solutions are the corresponding eigenfunctions denoted by \( w_1, w_2, \ldots \), where \( w_1 \) is clearly constant. We now prove the following result.

**Theorem 2.3.** The overdetermined boundary-value problem

\[
\Delta w = 0 \quad \text{in } D
\]
\[
\frac{\partial w}{\partial \nu} = pw \quad \text{on } \partial D
\]
\[
\int_D w_k \, dx = 0 \quad \forall k \geq 2
\]

is solvable if and only if \( D \) is a ball.

*Proof.* Let us assume \( D \) is a ball. Let \( w_k \) be an eigenfunction corresponding to \( p_k, k = 2, 3, \ldots \). Since \( w_k \) is harmonic it follows from the mean value property that

\[
\int_D w_k \, dx = d \int_{\partial D} w_k \, d\sigma,
\]

for some constant \( d \). Thus using the boundary condition in (2.9) in conjunction with the Divergence Theorem we infer

\[
\int_D w_k \, dx = \frac{d}{p_k} \int_D \Delta w_k \, dx.
\]

Since \( w_k \) is harmonic in \( D \) we obtain \( \int_D w_k \, dx = 0 \), as desired.

To prove the converse we proceed along the same lines as in [2, Theorem 2] to prove the converse. To this end, let \( u \) be the solution of the Saint-Venant problem in \( D, V \in C^2(\mathbb{R}^N; \mathbb{R}^N) \), and \( u' \) the domain derivative of \( u \) in direction of \( V \). Since \( D \) is smooth it follows from (2.2) that \( u' \in C^2(\overline{D}) \). Hence \( u' \) can be represented in terms of the eigenfunctions \( w_k \) as follows

\[
u'(x) = \sum_{i=1}^{\infty} \gamma_i \, w_i(x),
\]

where

\[
\gamma_i = \int_{\partial D} w_i u' \, d\sigma.
\]

Integrating the equation before the last, over \( D \), and taking into account that \( \int_D w_i \, dx = 0 \), for \( i = 2, 3, \ldots \) yields

\[
\int_D u' \, dx = \gamma_1 \int_D w_1 \, dx = k \int_{\partial D} u' \, d\sigma,
\]

where \( k \) is a constant independent of the vector field \( V \). Since \( V \) is arbitrary we can apply Theorem 2.1 to conclude that \( D \) must be a ball, as desired. \( \square \)
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