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SOLVING p-LAPLACIAN EQUATIONS ON COMPLETE MANIFOLDS

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Abstract. Using a reduced version of the sub and super-solutions method, we prove that the equation $\Delta_p u + ku^{p-1} - Ku^{p^*-1} = 0$ has a positive solution on a complete Riemannian manifold for appropriate functions $k, K : M \to \mathbb{R}$.

1. Introduction

Let (M, g) be an n-dimensional complete and connected Riemannian manifold $(n \geq 3)$ and let $p \in (1, n)$. We are interested in the existence of positive solutions $u \in H^p_{1,\text{loc}}(M)$ (the standard Sobolev space of order p) of the equation

$$\Delta_p u + ku^{p-1} - Ku^{p^*-1} = 0 \tag{1.1}$$

with $p^* = \frac{pn}{n-p}$ and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u .

As usual $u \in H^p_{1,\text{loc}}(M)$ is defined to be a weak solution of (1.1) if

$$\int_M -|\nabla u|^{p-2}\nabla u \nabla v + (ku^{p-1} - Ku^{p^*-1})v = 0 \tag{1.2}$$

for each $v \in C_0^\infty(M)$. A supersolution (respectively a subsolution) $u \in H^p_{1,\text{loc}}(M)$ is defined in the same way by changing $=$ by $\leq$ (respect $\geq$) in equation(1.2) and requiring that the test function $v \in C_0^\infty(M)$ to be non negative. Throughout this paper, we will assume that k and K are smooth real valued functions on M. Following the terminology in [3], this equation is referred to as the generalized scalar curvature type equation, it's an extension of the equation of prescribed scalar curvature. In the case of a compact manifold, the problem was considered in [3]. One of the results obtained in this latter paper is the following theorem

Theorem 1.1. *Let (M, g) be a compact Riemannian manifold with $n \geq 2$ and let $p \in (1, n)$. Let k and K be smooth real functions on M. If we assume that k and K are both positive, then* (1.1) *possesses a positive solution $u \in C^{1,\alpha}(M)$.*

In this paper, we look for positive solutions of (1.1) on complete Riemannian manifolds. To achieve this task, we use a recent result obtained by the authors in [2]. Before quoting this result we recall some definitions. A nonnegative and smooth function K on a complete manifold is said *essentially positive* if there exists an

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exhaustion by compact domains $\{\Omega_i\}_{i\geq 0}$ such that $M = \cup_{i\geq 0}\Omega_i$ and $K|_{\partial\Omega_j} > 0$ for any $j \geq 0$. Moreover, if there is a positive supersolution $u \in H_1^p(\Omega_i) \cap C^0(\Omega_i)$ on each Ω_i of (1.1) the essentially positive function K is said to be *permissible*. With this terminology the following theorem has been established in [2]

Theorem 1.2. *Let (M, g) be a complete non compact Riemannian manifold of dimension $n \geq 3$ and k, K be smooth real valued functions on M. Suppose that K is permissible and $k \leq K$. If there exists a positive subsolution $u_- \in H^p_{1,\mathrm{loc}}(M) \cap L^\infty(M) \cap C^0(M)$ of (1.1) on M, then (1.1) has a positive and maximal weak solution $u \in H_1^p(M)$. Moreover $u \in C^{1,\alpha}(\Omega_i)$ on each compact Ω_i for some $\alpha \in (0, 1)$.*

The Riemannian manifold M will be said of bounded geometry if the Ricci curvature of M is bounded from below and the injectivity radius is strictly positive everywhere.

We formulate our main result as follows:

Theorem 1.3. *Let (M, g) be a complete non compact Riemannian manifold of dimension $n \geq 3$ and k, K be smooth real valued functions on M. Suppose that*

- (a) *the function K is permissible and $K \geq c_o > 0$ where c_o is a real constant, k is bounded and satisfies $k \leq K$, and $\int_{\Omega_i} k = 0$, on each compact domain Ω_i of the exhaustion of M.*
- (b) *M is of bounded geometry.*

Then (1.1) has a weak positive maximal solution $u \in H_1^p(M)$. Moreover $u \in C^{1,\alpha}_{loc}(M)$ for some $\alpha \in (0, 1)$.

Our paper is organized as follows: In the first section we construct a supersolution of (1.1) on each compact subset of M. In the second section, we show the existence of a positive eigenfunction of the nonlinear operator $L_p u = -\Delta_p u - ku^{p-1}$ on M which we will use next to construct a global subsolution of our equation.

First, we establish the following result.

Lemma 1.4. *Let Ω be a compact domain of M and f be a C^∞ function on Ω. The equation*

$$\begin{gathered} -\Delta_p \phi = f \quad \text{in } \Omega - \partial\Omega \\ \phi = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

admits a solution $\phi \in C^{1,\alpha}(\Omega)$.

Proof. Letting $A = \{\phi \in H^p_{1,0}(\Omega) : \int_\Omega f\phi = 1\}$, we put

$$\mu = \inf_{\phi \in K} \int_\Omega |\nabla\phi|^p.$$

The set A is non empty since it contains the function $\phi = \frac{\operatorname{sgn}(f)|f|^{p-1}}{\int_\Omega |f|^p}$.

Let $(\phi_i)_{i\in\mathbb{N}}$ be a minimizing sequence in A, that is,

$$\lim_{i\to\infty} \int_\Omega |\nabla\phi_i|^p = \mu.$$

Then, if $\lambda_{1,p}$ denotes the first nonvanishing eigenvalue of the p-Laplacian operator, we have

$$\lambda_{1,p} \leq \frac{\int_\Omega |\nabla\phi_i|^p}{\int_\Omega |\phi_i|^p}$$

so

$$\int_\Omega |\phi_i|^p \le \lambda_{1,p}^{-1}\int_\Omega |\nabla\phi_i|^p < \frac{\mu}{\lambda_{1,p}}+1.$$

The sequence $(\phi_i)_{i\in\mathbb{N}}$ is bounded in $H_1^p(\Omega)$, hence by the reflexivity of the space $H_1^p(\Omega)$ and the Rellich-Kondrakov theorem, there exists a subsequence of $(\phi_i)_{i\in\mathbb{N}}$ still denoted (ϕ_i) such that

(a) $(\phi_i)_{i\in\mathbb{N}}$ converges weakly to $\phi\in H_1^p(\Omega)$
(b) $(\phi_i)_{i\in\mathbb{N}}$ converges strongly to $\phi\in L^p(\Omega)$.

From (b) we deduce that $\phi_i \longrightarrow \phi$ in $L^1(\Omega)$ then $\phi\in A$ and from (a) we get

$$\|\phi\|_{H_1^p(\Omega)} \le \lim_{i\to+\infty}\inf \|\phi_i\|_{H_1^p(\Omega)}.$$

Taking into account of (b) again, we obtain

$$\int_\Omega |\nabla\phi|^p \le \liminf_{i\to+\infty}\int_\Omega |\nabla\phi_i|^p = \mu\,.$$

Since $\phi\in A$, we get

$$\int_\Omega |\nabla\phi|^p = \mu = \inf_{\psi\in K}\int_\Omega |\nabla\psi|^p.$$

The Lagrange multiplier theorem allows us to say that ϕ is a weak solution of (1.3). □

The regularity of ϕ follows from the next proposition, with the following notation

$$W^{1,p}(\Omega) = \begin{cases} H_1^p(\Omega) & \text{if } \partial\Omega=\phi \\ H_{1,0}^p(\Omega) & \text{if } \partial\Omega\ne\phi\,. \end{cases}$$

Proposition 1. *Let $h\in C^o(\Omega\times R)$ be such that, for any $(x,r)\in\Omega\times R$, $|h(x,r)|\le C|r|^{p^*-1}+D$.*

If $u\in W^{1,p}(\Omega)$ is a solution of $-\Delta_p u+h(x,u)=0$, then $u\in C^{1,\alpha}(\Omega)$.

The above proposition was proved in ([3]), in the context of compact Riemannian manifolds without boundary. The proof is in its essence based on the Sobolev inequality and since this latter is also valid in $\mathring{H}_1^p(\Omega)$ as in $H_1^p(\Omega)$, it follows that proposition (1) remains true in the case of compact Riemannian manifolds with boundary.

2. Existence of a supersolution

In this section we construct a positive supersolution of (1.1) on each compact domain of M.

Theorem 2.1. *Let Ω be a compact domain of M. If K is a smooth function such that $K\ge c_0>0$ and k is a smooth function with $k\le K$, then there exists a positive supersolution of* (1.1) *in Ω.*

Proof. Letting $u=e^v$ where $v\in H_1^p(\Omega)$ is a function which will be precise later and $q=p^*-1$, then we get for every $\phi\in H_1^p(\Omega)$ with $\phi\ge 0$

$$\int_\Omega \Delta_p u\phi = \int_\Omega e^{(p-1)v}(\Delta_p v+(p-1)|\nabla v|^p)\phi$$

and

$$\int_\Omega (\Delta_p u + ku^{p-1}-Ku^q)\phi = \int_\Omega e^{(p-1)v}(\Delta_p v+(p-1)|\nabla v|^p + k - Ke^{(q-p+1)v})\phi.$$

So it suffices to show the existence of v such that

$$\int_{\Omega} e^{(p-1)v}(\Delta_p v + (p-1)|\nabla v|^p + k - Ke^{(q-p+1)v}).\phi \leq 0 \tag{2.1}$$

Let $b > 0$ be a constant and consider the solution of $\Delta_p h = -b^{1-p}k$ which is guaranteed by Lemma 1.4.

Now putting $v = bh + t$ where t is a real constant to be chosen later. The inequality (2.1) becomes

$$\int_{\Omega} e^{(p-1)(bh+t)}(b^{p-1}\Delta_p h + (p-1)b^p|\nabla h|^p + k - Ke^{(q-p+1)(bh+t)})\phi \leq 0$$

If we choose t such that $e^{(q-p+1)t} = b^{p-1}$, we will find that

$$\begin{aligned}
&\int_{\Omega} e^{(p-1)(bh+t)}((p-1)b|\nabla h|^p - Ke^{(q-p+1)bh})\phi \\
&\leq \int_{\Omega} e^{(p-1)(bh+t)}((p-1)b|\nabla h|^p - Km_o)\phi \leq 0
\end{aligned}$$

where $m_o = \min_{x\in\Omega} e^{(q-p+1)bh(x)}$ and since the function $K \geq c_o > 0$, we choose b small enough so that

$$|\nabla h|^p \leq \frac{c_o m_o}{b(p-1)}$$

we get the desired result. □

3. Existence of a subsolution

The operator $L_p u = -\Delta_p u - ku^{p-1}$ under Dirichlet conditions has a first eigenvalue $\lambda_{1,p}^{\Omega}$ on each open and bounded domain $\Omega \subset M$ which is variationally defined as

$$\lambda_{1,p}^{\Omega} = \inf(\int_{\Omega} |\nabla\phi|^p - k|\phi|^p) \tag{3.1}$$

where the infimum is extended to the set

$$A = \{\phi \in H_{1,0}^p(\Omega) : \int_{\Omega} |\phi|^p = 1\}.$$

Since $|\nabla\phi| = |\nabla|\phi||$, we can assume that $\phi \geq 0$. The corresponding positive eigenfunction is solution of the Dirichlet problem

$$\begin{gathered}
\Delta_p\phi + k\phi^{p-1} = -\lambda_{1,p}^{\Omega}\phi^{p-1} \quad \text{in } \Omega \\
\phi > 0 \quad \text{in } \Omega \\
\phi = 0 \quad \text{on } \partial\Omega
\end{gathered} \tag{3.2}$$

Let $\{\Omega_i\}_{i\geq 0}$ be an exhaustion of M by compact domains with smooth boundary such that $\Omega_i \subset \mathring{\Omega}_{i+1}$

Lemma 3.1. *If k is bounded function, then the sequence $\lambda_{1,p}^{\Omega_i}$ defined by* (3.1) *converges.*

Proof. By definition, $\lambda_{1,p}^{\Omega_i}$ is a decreasing sequence. Let $\lambda_{1,p}$ its limit, since the function k is bounded, there exists a constant $c > 0$ such that $-k + c \geq 1$, then

$$\begin{aligned}\int_\Omega |\nabla\phi|^p + (c-k)\phi^p &\geq \int_\Omega |\nabla\phi|^p + \phi^p \\ &\geq 2^{1-p}((\int_\Omega |\nabla\phi|^p)^{1/p} + (\int_\Omega \phi^p)^{1/p})^p \\ &= 2^{1-p}\|\phi\|^p_{H_1^p(\Omega)}\end{aligned}$$

so the operator $L_p u = -\Delta_p u + (c-k)u^{p-1}$ is coercive and we have, for ϕ_i any eigenfunction corresponding to $\lambda_{1,p}^{\Omega_i}$,

$$\begin{aligned}\lambda_{1,p}^{\Omega_i} &= \int_{\Omega_i} |\nabla\phi_i|^p - k\phi_i^p \\ &\geq -c + 2^{1-p}\|\phi_i\|^p_{H_1^p(\Omega)} \\ &\geq -c + 2^{1-p} \geq -c + 2^{1-n}\,.\end{aligned}$$

Then $\lambda_{1,p} > -\infty$. ∎

Lemma 3.2. *If k is bounded, then the eigenfunction problem*

$$\begin{gathered}\Delta_p\phi + k\phi^{p-1} = -\lambda_{1,p}\phi^{p-1} \quad \text{in } M \\ \phi > 0 \quad \text{in } M\end{gathered} \tag{3.3}$$

has a positive solution $\phi \in C^{1,\alpha}_{loc}(M)$.

Proof. Letting $(\Omega_i)_{i\geq 1}$ be an exhaustive covering of the complete manifold M by compact subsets and (ϕ_i) be the sequence of the first nonvanishing eigenfonctions (positive) of the operator $L_p u = -\Delta_p u - ku^{p-1}$ on each Ω_i. Multiplying (3.3) by ϕ_i and integrating over Ω_i, we get

$$\int_{\Omega_i} |\nabla\phi_i|^p - k\phi_i^p = \lambda_{1,p}^{\Omega_i}\int_{\Omega_i}\phi_i^p = \lambda_{1,p}^{\Omega_i} \leq \lambda_{1,p}^{\Omega_1}$$

so that

$$\int_{\Omega_i} |\nabla\phi_i|^p \leq \max_{x\in M}|k| + \lambda_{1,p}^{\Omega_1} < \infty.$$

On the other hand,

$$\begin{aligned}\Big(\Big(\int_{\Omega_i}|\nabla\phi_i|^p\Big)^{1/p} + \Big(\int_{\Omega_i}\phi_i^p\Big)^{1/p}\Big)^p &\leq 2^{p-1}(\int_{\Omega_i}|\nabla\phi_i|^p + \phi_i^p) \\ &\leq 2^{p-1}\Big(1 + \max_{x\in M}|k| + \lambda_{1,p}^{\Omega_1}\Big) < \infty\end{aligned} \tag{3.4}$$

and by the reflexivity of the space $H_1^p(M)$, we deduce that

$$\phi_i \to \phi \text{ weakly in } H_1^p(M)$$

and

$$\|\phi\|^p_{H_1^p(M)} \leq \liminf \|\phi_i\|^p_{H_1^p(M)}. \tag{3.5}$$

Now since $\int_M \phi_i^p = 1$, for every $\varepsilon > 0$ there exists a compact domain $K_i \subset M$ such that $\int_{M\setminus K_i}\phi_i^p < \frac{\varepsilon}{2^i}$, let $K = \cap_{i=1}^\infty K_i$ and

$$\int_{M\setminus K}\phi_i^p = \int_{\cup_{i=1}^\infty (M\setminus K_i)}\phi_i^p \leq \sum_{i=1}^\infty \int_{M\setminus K_i}\phi_i^p < \epsilon.$$

From (3.4) we obtain by Rellich-Kondrakov theorem that

$$\phi_i \to \phi \text{ strongly in } L^p(K).$$

We claim that

$$\int_M \phi^p = 1; \tag{3.6}$$

since, if it is not the case we have by (3.5)

$$1 - \int_M \phi^p > 0,$$

consequently

$$1 = \lim_{i\to\infty} \int_M \phi_i^p \le \varepsilon + \lim_{i\to\infty} \int_K \phi_i^p = \varepsilon + \int_K \phi^p$$

and hence $\varepsilon \ge 1 - \int_M \phi^p$. A contradiction with the fact that ε is arbitrary fixed.

Now from (3.5) and (3.6) we get

$$\int_M |\nabla\phi|^p \le \liminf \int_M |\nabla\phi_i|^p$$

hence

$$\int_M |\nabla\phi|^p - k\phi^p \le \liminf(\int_M |\nabla\phi_i|^p - k\phi_i^p)$$

which by lemma3.1 goes to $\lambda_{1,p}$, and since $\int_M \phi^p = 1$, we obtain

$$\int_M |\nabla\phi|^p - k\phi^p = \lambda_{1,p}.$$

So ϕ is a weak solution of the equation

$$\Delta_p\phi + k\phi^{p-1} = -\lambda_{1,p}\phi^{p-1}$$

From proposition 1, we deduce that $\phi \in C^{1,\alpha}_{loc}(M)$.

It remains to show that ϕ is positive, which is deduced from the next proposition.

Proposition 2 (Druet [3])**.** *Let (Ω, g) be a compact Riemannian n-manifold $n \ge 2$, $1 < p < n$. Let $u \in C^1(\Omega)$ be such that $-\Delta_p u + h(x,u) \ge 0$ on Ω, h fulfilling the conditions*

$$h(x,r) < h(x,s), \quad x \in \Omega,\ 0 \le r < s$$
$$|h(x,u)| \le C(K + |r|^{p-2})|r|, \quad (x,r) \in M \times R,\ C > 0.$$

If $u \ge 0$ on Ω and u does not vanish identically, then $u > 0$ on Ω.

□

If λ is an eigenvalue of the operator

$$L_p u = -\Delta_p\phi - k|\phi|^{p-2}\phi,$$

so is $\lambda + c$ for the operator

$$L_c u = -\Delta_p\phi - (k-c)|\phi|^{p-2}\phi$$

where c is a constant and since k is bounded function we choose c such that $c-k > 0$, and then we get

$$-\Delta_p\phi + h(x,\phi) \ge 0$$

where

$$h(x,\phi) = (c - k(x))\phi^{p-1}.$$

Obviously the function h satisfies the assumptions of proposition 2 and we have $\phi > 0$.

Now we establish the following lemma which will be used later.

Lemma 3.3. *Let M be a Riemannian manifold of bounded geometry. Suppose that $a(x)$ is a bounded smooth function on M and $u \in H_1^p(M)$ be a weak solution of the equation*

$$\Delta_p u + a(x)u^{p-1} = 0 \tag{3.7}$$

then $u \in L^\infty(M)$.

Proof. We are going to use Moser's iteration scheme. Let $k \geq 1$ be any real and $t = k + p - 1$. Multiplying (3.7) by u^k $(k > 1)$ and integrating over M, we get

$$-k \int_M |\nabla u|^p u^{k-1} + \int_M a(x)u^{p+k-1} = 0. \tag{3.8}$$

Using Sobolev's inequality, we get for any fixed $\varepsilon > 0$

$$\begin{aligned} \|u^{\frac{t}{p}}\|_{p^*}^p &= \|u\|_{t\frac{p^*}{p}}^t \\ &\leq (K(n,p)^p + \varepsilon)\|\nabla u^{\frac{t}{p}}\|_p^p + B\|u\|_t^t \\ &= (K(n,p)^p + \varepsilon)(\frac{t}{p})^p \|u^{\frac{t}{p}-1}\nabla u\|_p^p + B\|u\|_t^t \end{aligned} \tag{3.9}$$

where $K(n,p)$ is the best constant in the Sobolev's embedding $H_1^p(R^n) \subset L^{p*}(R^n)$ (see Aubin [1] or Talenti [4]) and B a positive constant depending on ϵ; since

$$\|u^{\frac{t}{p}-1}\nabla u\|_p^p = \int u^{t-p}|\nabla u|^p$$

and taking account of (3.8) we get

$$\int u^k \Delta_p u = -k \int u^{k-1}|\nabla u|^p \leq \|a\|_\infty \|u\|_t^t .$$

Then (3.9) becomes

$$\|u\|_{t\frac{p^*}{p}}^t \leq (K(n,p)^p + \varepsilon)(\frac{t}{p})^p \frac{1}{k}(\|a\|_\infty + B)\|u\|_t^t$$

so that

$$\|u\|_{t\frac{p^*}{p}} \leq \Big((K(n,p)^p + \varepsilon)\big(\frac{t}{p}\big)^p \frac{1}{k}(\|a\|_\infty + B)\Big)^{\frac{1}{t}} \|u\|_t . \tag{3.10}$$

Putting

$$\frac{t}{p} = \beta^i$$

where i is a positive integer and $\beta = \frac{p^*}{p} = \frac{n}{n-p}$, (3.10) becomes

$$\|u\|_{p\beta^{i+1}} \leq ((K(n,p)^p + \varepsilon)\beta^{pi}(\|a\|_\infty + B))^{\frac{1}{p\beta^i}} \|u\|_{p\beta^i} . \tag{3.11}$$

Recurrently, we obtain

$$\|u\|_{p\beta^{i+1}} \leq (K(n,p)^p + \varepsilon)^{\frac{1}{p}(\sum_{j=0}^i \frac{1}{\beta^j})} \beta^{\sum_{j=0}^i \frac{j}{\beta^j}} (\|a\|_\infty + B)^{\frac{1}{p}(\sum_{j=0}^i \frac{1}{\beta^j})} \|u\|_p. \tag{3.12}$$

Now, since

$$\sum_{j=0}^\infty \frac{1}{\beta^j} = \frac{\beta}{\beta - 1} = \frac{n}{p}$$

and

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{j}{\beta^j}&=\sum_{j=1}^{\infty}\frac{j}{(1+\pi)^j}\\&\leq\sum_{j=1}^{\infty}\frac{j}{\sum_{p=0}^{j}C_j^p\pi^p}=\sum_{j=1}^{\infty}\frac{1}{\pi\sum_{p=0}^{j-1}C_j^p\pi^p}\\&=\frac{1}{\pi}\sum_{j=1}^{\infty}\frac{1}{(1+\pi)^{j-1}}=\frac{1}{\pi}\sum_{j=0}^{\infty}\frac{1}{(1+\pi)^j}\\&=\frac{n-p}{p}\sum_{j=0}^{\infty}\frac{1}{\beta^j}=\frac{n(n-p)}{p^2},\end{aligned}$$

it follows by letting $j\to\infty$ in (3.12) that $u\in L^\infty(M)$. □

Theorem 3.4. *Let (M,g) be a complete noncompact Riemannian manifold of dimension $n\geq 3$ with bounded geometry. Suppose that $k\in C^\infty(M)\cap L^\infty(M)$; then there exists a positive subsolution of the equation $\Delta_p u+ku^{p-1}-Ku^{p^*-1}=0$ on M.*

Proof. Since $k\in L^\infty(M)$, there exists a positive constant $c>0$ such that the operator $L_c u=-\Delta_p\phi+(c-k)\phi^{p-1}$ is coercive, so by lemma 3.2 its first non vanishing eigenvalue $\lambda_{1,p}+c>0$. If ϕ denotes the corresponding positive eigenfunction to $\lambda_{1,p}$,by lemma 3.3 we may assume that $\phi<1$.

For $r>0$ we consider

$$u_-=\left(e^{r^2}-\phi^{r^3}\right)^{\frac{1}{r}+1}$$

and by a direct computations we obtain in the sense of distribution

$$\begin{aligned}\nabla u_-&=-r^2(r+1)(e^{r^2}-\phi^{r^3})^{\frac{1}{r}}\phi^{r^3-1}\nabla\phi,\\ \Delta_p u_-&=\Big[r^2(r+1)(e^{r^2}-\phi^{r^3})^{1/r}\phi^{r^3-1}\Big]^{p-1}\\&\quad\times\big[-\Delta_p\phi+(p-1)\big(\frac{1-r^3}{\phi}+\frac{r^2\phi^{r^3-1}}{e^{r^2}-\phi^{r^3}}\big)|\nabla\phi|^p\big].\end{aligned}$$

Hence

$$\begin{aligned}&\Delta_p u_-+ku_-^{p-1}-Ku_-^q\\&=\big[r^2(r+1)(e^{r^2}-\phi^{r^3})^{\frac{1}{r}}\phi^{r^3}\big]^{p-1}\\&\quad\times\Big[-\Delta_p\phi+(p-1)\big(\frac{1-r^3}{\phi}+\frac{r^2\phi^{r^3-1}}{e^{r^2}-\phi^{r^3}}\big)|\nabla\phi|^p+k\big(\frac{e^{r^2}-\phi^{r^3}}{r^2(r+1)\phi^{r^3}}\big)^{p-1}\phi^{p-1}\\&\quad-K\big(\frac{e^{r^2}-\phi^{r^3}}{r^2(r+1)\phi^{r^3}}\big)^{p-1}(e^{r^2}-\phi^{r^3})^{(q-p+1)(1+\frac{1}{r})}\phi^{p-1}\Big]\\&=\big[r^2(r+1)(e^{r^2}-\phi^{r^3})^{\frac{1}{r}}\phi^{r^3-1}\big]^{p-1}\\&\quad\times\Big[\lambda_{1,p}+(p-1)\frac{1}{\phi^p}\big(1-r^3+\frac{r^2\phi^{r^3}}{e^{r^2}-\phi^{r^3}}\big)|\nabla\phi|^p+k\Big(\big(\frac{e^{r^2}-\phi^{r^3}}{r^2(r+1)\phi^{r^3}}\big)^{p-1}+1\Big)\\&\quad-K\big(\frac{e^{r^2}-\phi^{r^3}}{r^2(r+1)\phi^{r^3}}\big)^{p-1}(e^{r^2}-\phi^{r^3})^{(q-p+1)(1+\frac{1}{r})}\Big].\end{aligned}$$

Now since

$$\lim_{r\to 0}(e^{r^2}-\phi^{r^3})^{1+\frac{1}{r}}=0$$

and

$$\lim_{r\to 0}\frac{r^2}{e^{r^2}-\phi^{r^3}}=1\,,$$

we deduce that

$$u_-=(e^{r^2}-\phi^{r^3})^{1+\frac{1}{r}}\in H^p_{1,\mathrm{loc}}(M)$$

is a subsolution of (1.1) and clearly $u_-\in C^o(M)\cap L^\infty(M)$. The main theorem (Theorem 1.3) is a consequence of theorem 2.1 and theorem 3.4. □

References

[1] Aubin.T, problèmes isopérimétriques et espaces de Sobolev. J. Diff. Géom 11 (1976), 573-598.

[2] M. Benalili, Y Maliki, Reduction method for proving the existence of solutions to elliptic equations involving the p-Laplacian. Electr.Journal of Differential Equations 106 (2003) 10pp.

[3] O. Druet, Generalized scalar curvature type equations on compact Riemannian manifolds. Proc. Roy. Soc. Edinburgh Sect. A130 (2000) No 4, 767-788.

[4] E. Hebey, Introduction à l'analyse non linéaire sur les variétés. Ed. Diderot (1997).

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