ON SECOND ORDER PERIODIC BOUNDARY-VALUE PROBLEMS WITH UPPER AND LOWER SOLUTIONS IN THE REVERSED ORDER

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Abstract. In this paper, we study the differential equation with the periodic boundary value

\[ u''(t) = f(t, u(t), u'(t)), \quad t \in [0, 2\pi] \]

\[ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \]

The existence of solutions to the periodic boundary problem above with appropriate conditions is proved by using an upper and lower solution method.

1. Introduction and Main Results

In this paper, we study the second-order periodic boundary-value problem

\[ u''(t) = f(t, u(t), u'(t)), \quad t \in [0, 2\pi] \]

\[ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad (1.1) \]

where \( f(t, u, v) \) is a Caratheodory function. A function \( f : [0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R} \) is said to be a Carathodory function if it possess the following three properties:

(i) For all \((u, v) \in \mathbb{R}^2\), the mapping \( t \to f(t, u, v) \) is measurable on \([0, 2\pi]\).

(ii) For almost all \( t \in [0, 2\pi] \), the mapping \((u, v) \to f(t, u, v) \) is continuous on \( \mathbb{R}^2 \).

(iii) For any given \( N > 0 \), there exists \( g_N(t) \), a Lebesgue integrable function defined on \([0, 2\pi]\), such that

\[ |f(t, u, v)| \leq g_N(t) \quad \text{for a.e. } t \in [0, 2\pi], \]

whenever \(|u|, |v| \leq N\).

To develop upper and lower solutions method, we need the concepts of upper and lower solutions. We say that \( \beta \in W^{2,1}[0, 2\pi] \) is an upper solution to (1.1), if it satisfies

\[ \beta''(t) \leq f(t, \beta(t), \beta'(t)), \quad t \in [0, 2\pi] \]

\[ \beta(0) = \beta(2\pi), \quad \beta'(0) \leq \beta'(2\pi). \quad (1.2) \]
Similarly, a function \( \alpha \in W^{2,1}[0, 2\pi] \) is said to be a lower solution to (1.1), if it satisfies

\[
\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)), \quad t \in [0, 2\pi]
\]

\[
\alpha(0) = \alpha(2\pi), \quad \alpha'(0) \geq \alpha'(2\pi).
\]

We call a function \( u \in W^{2,1}[0, 2\pi] \) a solution to (1.1), if it is an upper and a lower solution to (1.1).

Under the classical assumption that \( \alpha(t) \leq \beta(t) \), a number of authors have studied the existence of the methods of lower and upper solutions or the monotone iterative technique [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Only a few have study the presence of a lower solution \( \alpha \) and Wan [7] have studied (1.1) by means of a monotone iterative technique in the presence of a lower solution \( \alpha(t) \) and an upper solution \( \beta(t) \) with \( \beta(t) \leq \alpha(t) \). Recently, Jiang, Fan and Wan [7] have studied (1.1) by means of a monotone iterative technique in the presence of a lower solution \( \alpha(t) \) and an upper solution \( \beta(t) \) with \( \beta(t) \leq \alpha(t) \). To develop a monotone method, the following hypotheses are needed in [7].

(A1) For any given \( \beta, \alpha \in C[0, 2\pi] \) with \( \beta(t) \leq \alpha(t) \) on \([0, 2\pi]\), there exist \( 0 < A \leq B \) such that

\[
A(v_2 - v_1) \leq f(t, u, v_2) - f(t, u, v_1) \leq B(v_2 - v_1)
\]

or

\[
-B(v_2 - v_1) \leq f(t, u, v_2) - f(t, u, v_1) \leq -A(v_2 - v_1)
\]

for a.e. \( t \in [0, 2\pi] \) whenever \( \beta(t) \leq u \leq \alpha(t) \), \( v_1, v_2 \in \mathbb{R} \), and \( v_1 \leq v_2 \).

(A2) Inequality

\[
f(t, u_2, v) - f(t, u_1, v) \geq -\frac{A^2}{4}(u_2 - u_1)
\]

holds for a.e. \( t \in [0, 2\pi] \), whenever \( \beta(t) \leq u_1 \leq u_2 \leq \alpha(t) \), \( v \in \mathbb{R} \).

The purpose of this paper is to prove the existence of solutions to (1.1) under the assumption that there exist a lower solution \( \alpha(t) \) and an upper solution \( \beta(t) \) of (1.1) with \( \beta(t) \leq \alpha(t) \) and \( f(t, u, v) \) only satisfies one side Lipschitz condition. We use the upper and lower solutions method and prove that the solution \( u(t) \) of (1.1) satisfies \( \beta(t) \leq u(t) \leq \alpha(t) \). Our result extends and complements those in [15, 16, 17].

To develop upper and lower solutions method, we need one of the following hypotheses

(H1) For any given \( \beta, \alpha \in C[0, 2\pi] \) with \( \beta(t) \leq \alpha(t) \) on \([0, 2\pi]\), there exist \( A > 0 \) and \( B > 0 \) such that \( B^2 \geq 4A \) and

\[
f(t, u_2, v_2) - f(t, u_1, v_1) \geq -A(u_2 - u_1) + B(v_2 - v_1)
\]

for a.e. \( t \in [0, 2\pi] \) whenever \( \beta(t) \leq u_1 \leq u_2 \leq \alpha(t) \), \( v_1, v_2 \in \mathbb{R} \), and \( v_1 \leq v_2 \).

(H1') For any given \( \beta, \alpha \in C[0, 2\pi] \) with \( \beta(t) \leq \alpha(t) \) on \([0, 2\pi]\), there exist \( A > 0 \) and \( B > 0 \) such that \( B^2 \geq 4A \) and

\[
f(t, u_2, v_2) - f(t, u_1, v_1) \geq -A(u_2 - u_1) + B(v_1 - v_2)
\]

for a.e. \( t \in [0, 2\pi] \) whenever \( \beta(t) \leq u_1 \leq u_2 \leq \alpha(t) \), \( v_1, v_2 \in \mathbb{R} \), and \( v_1 \geq v_2 \).
We remark that condition (H1’) is equivalent to

(a1) For any given \( \beta, \alpha \in C[0, 2\pi] \) with \( \beta(t) \leq \alpha(t) \) on \( [0, 2\pi] \), there exists \( B > 0 \) such that

\[
f(t, u, v_1) - f(t, u, v_2) \leq -B(v_1 - v_2)
\]

for a.e. \( t \in [0, 2\pi] \) whenever \( \beta(t) \leq u \leq \alpha(t) \), \( v_1, v_2 \in \mathbb{R} \), and \( v_1 \geq v_2 \).

(a2) There exists \( A > 0 \) such that \( B^2 \geq 4A \) and

\[
f(t, u_2, v) - f(t, u_1, v) \geq -A(u_2 - u_1)
\]

holds for a.e. \( t \in [0, 2\pi] \), whenever \( \beta(t) \leq u_1 \leq u_2 \leq \alpha(t) \), \( v \in \mathbb{R} \).

Also we remark that (H1) or (H1’) weaker than (A1)-(A2) in [7].

Let

\[
\beta
\]

be a function satisfying the hypothesis (H1). Then

\[
\text{Theorem 1.1.}
\]

The main results of this paper are stated as follows.

Lemma 2.1. Let

\[
y \in W^{1,1}[0, 2\pi],
\]

and satisfy

\[
y'/(t) + Ly(t) \geq 0 \quad \text{for a.e.} \quad t \in [0, 2\pi],
\]

\[
y(0) \geq y(2\pi),
\]

where \( |L| > 0 \). Then \( Ly(t) \geq 0 \) on \([0, 2\pi]\), i.e., when \( L > 0 \) the minimum of \( y(t) \) is nonnegative; when \( L < 0 \) the maximum of \( y(t) \) is nonpositive.

Lemma 2.2. Suppose that there exists a lower solution \( \alpha(t) \) and an upper solution \( \beta(t) \) of (1.1) such that \( \beta(t) \leq \alpha(t) \) on \([0, 2\pi]\), and \( f(t, u, v) \) is a Caratheodory function satisfying the hypothesis (H1). Then \( A(t) \leq B(t) \) on \([0, 2\pi]\).
Proof. It follows from (1.2) and (1.3) that
\[ A'(t) + MA(t) \geq f(t, \alpha(t), A(t) - m\alpha(t)) + (m + M)A(t) - m^2\alpha(t), \quad t \in [0, 2\pi] \]
\[ A(0) \geq A(2\pi), \]
and
\[ B'(t) + MB(t) \leq f(t, \beta(t), B(t) - m\beta(t)) + (m + M)B(t) - m^2\beta(t), \quad t \in [0, 2\pi] \]
\[ B(0) \leq B(2\pi). \]

Let \( y(t) = A(t) - B(t) \), then \( y(0) \geq y(2\pi) \). Assume that \( y(t) > 0 \) for some \( t \in [0, 2\pi] \). Indeed, if \( y(t) > 0 \) on \([0, 2\pi]\), then by (H1) we have
\[
y'(t) + My(t) \geq f(t, \alpha(t), A(t) - m\alpha(t)) - f(t, \beta(t), B(t) - m\beta(t)) + (m + M)y(t) - m^2(\alpha(t) - \beta(t))
\geq -(A + Bm + m^2)(\alpha(t) - \beta(t)) + (B + m + M)y(t) = 0, \quad t \in [0, 2\pi],
\]
then by Lemma 2.1 we have \( y(t) \leq 0 \) on \([0, 2\pi]\), which is a contradiction.

If \( y(0) \leq 0 \) (then \( y(2\pi) \leq y(0) \leq 0 \)), and hence there exists a \( s \in (0, 2\pi) \) with \( y(s) > 0 \) such that \( y(t) > 0 \) in \((a, b)\) with \( y(a) = y(b) = 0 \). By (1.2) and (1.3), we have
\[
y'(t) + My(t) \geq 0, \quad t \in [a, b], \quad y(a) = y(b) = 0.
\]
This leads to \( y'(t) \geq -My(t) > 0 \) on \([a, b]\), which is again a contradiction.

If \( y(0) > 0 \), then there exists a \( a \in (0, 2\pi) \) such that \( y(t) > 0 \) on \([a, b]\) with \( y(a) = 0 \). So we have \( y'(t) + My(t) \geq 0 \) on \([0, a]\), hence \( y'(t) > 0 \) in \([0, a]\), which implies that \( y(0) < y(a) = 0 \), this is also a contradiction. The proof of Lemma 2.2 is completed.

Similarly, we have the following result.

Lemma 2.3. Suppose that there exists a lower solution \( \alpha(t) \) and an upper solution \( \beta(t) \) of (1.1) such that \( \beta(t) \leq \alpha(t) \) on \([0, 2\pi]\), and \( f(t, u, v) \) is a Carathéodory function satisfying the hypothesis (H1'). Then \( B_0(t) \leq A_0(t) \) on \([0, 2\pi]\).

Proof. It follows from (1.2) and (1.3) that for \( t \in [0, 2\pi] \),
\[
A'_0(t) + M_0A_0(t) \geq f(t, \alpha(t), A_0(t) - m_0\alpha(t)) + (m_0 + M_0)A_0(t) - m_0^2\alpha(t),
A_0(0) \geq A_0(2\pi)
\]
and for \( t \in [0, 2\pi] \)
\[
B'_0(t) + MB(t) \leq f(t, \beta(t), B_0(t) - m_0\beta(t)) + (m_0 + M_0)B_0(t) - m_0^2\beta(t),
B_0(0) \leq B_0(2\pi).
\]
Let \( y(t) = A_0(t) - B_0(t) \), then \( y(0) \geq y(2\pi) \). Assume that \( y(t) < 0 \) for some \( t \in [0, 2\pi] \). Indeed, if \( y(t) < 0 \) on \([0, 2\pi]\), then by (H1') we have
\[
y'(t) + M_0y(t) \geq f(t, \alpha(t), A_0(t) - m_0\alpha(t)) - f(t, \beta(t), B_0(t) - m_0\beta(t))
+ (m_0 + M_0)y(t) - m_0^2(\alpha(t) - \beta(t))
\geq -(A - Bm_0 + m_0^2)(\alpha(t) - \beta(t)) + (-B + m_0 + M_0)y(t)
= 0, \quad t \in [0, 2\pi],
\]
then by Lemma 2.1, we have $y(t) \geq 0$ on $[0, 2\pi]$, which is a contradiction.

If $y(2\pi) \geq 0$ (then $y(0) \geq y(2\pi) \geq 0$), and hence there exists $s \in (0, 2\pi)$ with $y(s) < 0$, then there would be $0 < a < s < b \leq 2\pi$ such that $y(t) < 0$ in $(a, b)$ with $y(a) = y(b) = 0$. By (1.2) and (1.3), we have

$$y'(t) + M_0y(t) \geq 0, \quad t \in [a, b], \quad y(a) = y(b) = 0.$$ 

This leads to $y'(t) \geq -M_0y(t) > 0$ on $[a, b]$, which is again a contradiction.

If $y(2\pi) < 0$, then there exists $a \in (0, 2\pi)$ such that $y(t) < 0$ on $(a, 2\pi]$ with $y(a) = 0$. So we have $y'(t) + M_0y(t) \geq 0$ on $(a, 2\pi]$, hence $y'(t) > 0$ in $(a, 2\pi]$, which implies that $y(2\pi) > y(a) = 0$, this is also a contradiction. The proof of Lemma 2.3 is complete. □

In the following arguments, we only give the proof of Theorem 1.1 since the proof of Theorem 1.2 can be treated in a similar way.

Let

$$p(t, x) = \begin{cases} A(t), & x < A(t), \\ x, & A(t) \leq x \leq B(t), \\ B(t), & x > B(t). \end{cases}$$

It is interesting to give an introduction to Lemma 2.4 and a reference where it can be found.

**Lemma 2.4.** If $m > 0$, then for any $q(t) \in L^1[0, 2\pi]$, the problem

$$u'(t) + mu(t) = q(t), \quad \text{for a.e. } t \in [0, 2\pi]$$

$$u(0) = u(2\pi),$$

has a unique solution $u \in W^{1,1}[0, 2\pi]$, and

$$u(t) = L^{-1}q(t) = \int_0^{2\pi} G_m(t, s)q(s)ds,$$

where

$$G_m(t, s) := \begin{cases} \frac{e^{m(2\pi + s - t)}}{e^{m\pi} - 1}, & 0 \leq s \leq t \leq 2\pi, \\ \frac{e^{m(s - t)}}{e^{m\pi} - 1}, & 0 \leq t \leq s \leq 2\pi. \end{cases}$$

By Lemma 2.1, we have

$$\alpha(t) = L^{-1}A(t), \quad \beta(t) = L^{-1}B(t), \quad \beta(t) \leq L^{-1}p(t, x) \leq \alpha(t).$$

Now we consider the modified problem

$$x'(t) + Mx(t) = f(t, L^{-1}p(t, x(t)), (I - mL^{-1})p(t, x(t)))$$

$$+(m + M)p(t, x(t)) - m^2L^{-1}p(t, x(t)), \quad x(0) = x(2\pi).$$

(2.1)

For each $x \in C[0, 2\pi]$, we define the mapping $\Phi : C[0, 2\pi] \rightarrow C[0, 2\pi]$,

$$\Phi x(t) = \int_0^{2\pi} G_M(t, s)(Fx)(s)ds,$$

(2.2)

where

$$(Fx)(t) := f(t, L^{-1}p(t, x(t)), (I - mL^{-1})p(t, x(t)))$$

$$+(m + M)p(t, x(t)) - m^2L^{-1}p(t, x(t)).$$
Thus \((\Phi x)(t)\) is also bounded.

We can easily prove that \(\Phi : C[0, 2\pi] \to C[0, 2\pi]\) is completely continuous. Then Leray-Schauder fixed point Theorem assures that \(\Phi\) has a fixed point \(x \in C[0, 2\pi]\) and

\[
x(t) = \int_0^{2\pi} G_M(t, s)(Fx)(s)ds,
\]

thus the modified problem (2.1) has one solution \(x \in W^{1,1}[0, 2\pi]\).

**Lemma 2.5.** Suppose that (H1) holds. Assume that \(\alpha(t)\) and \(\beta(t)\) are lower and upper solutions to (1.1) and \(\beta(t) \leq \alpha(t)\) on \([0, 2\pi]\). Let \(x \in W^{1,1}[0, 2\pi]\) be a solution to (2.1), then \(A(t) \leq x(t) \leq B(t)\) on \([0, 2\pi]\).

**Remark 2.6.**Lemma 2.5 implies \(u(t) = L^{-1}x(t) = \int_0^{2\pi} G_m(t, s)x(s)ds\) is a solution to (1.1), since \(u'(t) + mu(t) = x(t)\), \(u(0) = u(2\pi)\) and \(A(t) \leq x(t) \leq B(t)\).

**Proof of Lemma 2.7.** Since \(\alpha(t) = L^{-1}A(t), \beta(t) = L^{-1}B(t)\),

\[
B'(t) + MB(t) \leq f(t, L^{-1}B(t), (I - mL^{-1})B(t)) - m^2L^{-1}B(t) + (m + M)B(t),
\]

\[
B(0) \leq B(2\pi)
\]

and

\[
A'(t) + MA(t) \geq f(t, L^{-1}A(t), (I - mL^{-1})A(t)) - m^2L^{-1}A(t) + (m + M)A(t),
\]

\[
A(0) \geq A(2\pi).
\]

We shall prove only that \(x(t) \leq B(t)\) on \([0, 2\pi]\), because \(A(t) \leq x(t)\) can be proved by a similar manner. Let \(y(t) = x(t) - B(t)\), then

\[
y(0) \geq y(2\pi).
\]

Assume that \(y(t) > 0\) for some \(t \in [0, 2\pi]\). Indeed, if \(y(t) > 0\) on \([0, 2\pi]\), we have

\[
x'(t) + Mx(t) = f(t, L^{-1}B(t), (I - mL^{-1})B(t)) - m^2L^{-1}B(t) + (m + M)B(t)
\geq B'(t) + MB(t),
\]

i.e., \(y'(t) + My(t) \geq 0\) on \([0, 2\pi]\). Lemma 2.1 implies \(y(t) \leq 0\) on \([0, 2\pi]\), which is a contradiction. Therefore, there would be a point \(s \in [0, 2\pi]\) with \(y(s) \leq 0\).

If \(y(0) \leq 0\) (then \(y(2\pi) \leq y(0) \leq 0\)), and hence there exist \(0 \leq a < s < b \leq 2\pi\) such that \(y(t) > 0\) in \((a, b)\) with \(y(a) = y(b) = 0\). Then \(p(t, x(t)) = B(t)\) on \([a, b]\) and

\[
y'(t) + My(t)
\geq f(t, L^{-1}p(t, x(t)), B(t) - mL^{-1}p(t, x(t))) + (m + M)B(t) - m^2L^{-1}p(t, x(t))
- [f(t, L^{-1}B(t), B(t) - mL^{-1}B(t)) + (m + M)B(t) - m^2L^{-1}B(t)]
\geq (-A - Bm - m^2)(L^{-1}p(t, x(t)) - L^{-1}B(t))
= 0, \quad t \in (a, b).
\]

This leads to \(y'(t) \geq -My(t) > 0\) on \((a, b)\), which is again a contradiction.
If $y(0) > 0$, there exists $a \in (0, 2\pi)$ such that $y(t) > 0$ on $[0, a)$ with $y(a) = 0$. So we have $y'(t) + My(t) \geq 0$, hence $y'(t) > 0$ in $[0, a)$, which implies that $y(0) < y(a) = 0$, this is also a contradiction. The proof is complete. \hfill $\Box$

By Remark 2.6 we have obtained the results of Theorem 1.1

3. Example

In this section, we consider the periodic boundary-value problem

\begin{equation}
\begin{aligned}
u''(t) + ku'(t) &= F(t, u), \quad t \in [0, 2\pi] \\
u(0) &= u(2\pi), u'(0) = u'(2\pi),
\end{aligned}
\end{equation}

where $F(t, u)$ is a Caratheodory function, $k > 0$ or $k < 0$.

We say that $\beta \in W^{2,1}[0, 2\pi]$ is an upper solution to the problem (3.1), if it satisfies

\begin{equation}
\begin{aligned}
\beta''(t) + k\beta'(t) &\leq F(t, \beta(t)), \quad t \in [0, 2\pi] \\
\beta(0) &= \beta(2\pi), \beta'(0) \leq \beta'(2\pi).
\end{aligned}
\end{equation}

Similarly, a function $\alpha \in W^{2,1}[0, 2\pi]$ is said to be a lower solution to (3.1), if it satisfies

\begin{equation}
\begin{aligned}
\alpha''(t) + k\alpha'(t) &\geq F(t, \alpha(t)), \quad t \in [0, 2\pi] \\
\alpha(0) &= \alpha(2\pi), \alpha'(0) \geq \alpha'(2\pi).
\end{aligned}
\end{equation}

To develop the upper and lower solutions method, we also need the following hypothesis:

(H) For any given $\beta, \alpha \in C[0, 2\pi]$ with $\beta(t) \leq \alpha(t)$ on $[0, 2\pi]$, the inequality

$$F(t, u_2) - F(t, u_1) \geq -\frac{k^2}{4}(u_2 - u_1)$$

holds for a.e. $t \in [0, 2\pi]$, whenever $\beta(t) \leq u_1 \leq u_2 \leq \alpha(t)$.

Let $A = k^2/4$, $B = |k|$, then (H1) holds when $k < 0$, and (H1') holds when $k > 0$. Hence the conclusions of Theorem 1.1 hold when $k < 0$, thus $\alpha'(t) + \frac{k}{2}\alpha(t) \leq \beta'(t) + \frac{k}{2}\beta(t)$ and problem (3.1) has one solution $u \in W^{2,1}[0, 2\pi]$ such that

$$\beta(t) \leq u(t) \leq \alpha(t), \alpha'(t) + \frac{k}{2}\alpha(t) \leq u'(t) + \frac{k}{2}u(t) \leq \beta'(t) + \frac{k}{2}\beta(t).$$

The conclusions of Theorem 1.2 hold when $k > 0$, thus $\alpha'(t) + \frac{k}{2}\alpha(t) \geq \beta'(t) + \frac{k}{2}\beta(t)$ and problem (3.1) has one solution $u \in W^{2,1}[0, 2\pi]$ such that

$$\beta(t) \leq u(t) \leq \alpha(t), \beta'(t) + \frac{k}{2}\beta(t) \leq u'(t) + \frac{k}{2}u(t) \leq \alpha'(t) + \frac{k}{2}\alpha(t).$$

In [7, 18], the authors obtained one solution $u \in W^{2,1}[0, 2\pi]$ of (3.1) such that $\beta(t) \leq u(t) \leq \alpha(t)$. Here we have improved the results of [7, 18].

References


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