MULTIPlicity OF SOLUTIONS FOR A QUASILINEAR PROBLEM WITH SUPercritical GROWTH

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Abstract. The multiplicity and concentration of positive solutions are established for the equation

$$-\epsilon^p \Delta_p u + V(z) |u|^{p-2} u = |u|^{q-2} u + \lambda |u|^{s-2} u \quad \text{in } \mathbb{R}^N,$$

where $1 < p < N$, $\epsilon > 0$, $p < q < p^* \leq s$, $p^* = \frac{Np}{N-p}$, $\lambda \geq 0$ and $V$ is a positive continuous function.

1. Introduction

This article concerns the multiplicity and concentration of positive solutions for the problem

$$-\epsilon^p \Delta_p u + V(z) |u|^{p-2} u = |u|^{q-2} u + \lambda |u|^{s-2} u \quad \text{in } \mathbb{R}^N \quad (1.1)$$

where $u \in W^{1,p}(\mathbb{R}^N)$ with $1 < p < N$

$$u(z) > 0, \quad \text{for } z \in \mathbb{R}^N,$$

$\epsilon > 0$, $p < q < p^* \leq s$, $p^* = \frac{Np}{N-p}$, $\lambda \geq 0$ and $\Delta_p u$ is the $p$-Laplacian operator; that is,

$$\Delta_p u = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right).$$

We assume that $V$ is a continuous function satisfying

$$V(x) \geq V_0 = \inf_{x \in \mathbb{R}^N} V(x) > 0 \quad \text{for } x \in \mathbb{R}^N; \quad (1.2)$$

Also assume that there exists an open and bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$V_0 < \min_{\partial \Omega} V. \quad (1.3)$$

In recent years, much attention has been paid to the existence and multiplicity of solutions for both subcritical and critical cases and to the concentration behavior of solutions for problem

$$-\epsilon^2 \Delta u + V(z) u = f(u) \quad \text{in } \mathbb{R}^N, \quad (1.4)$$
when $\epsilon$ is small. Interesting results may be found, for example, in [3, 5, 6, 8, 10, 14, 17] and their references.

Cingolani & Lazzo [9], using Lusternik-Schnirelman category and involving the sets

$$M = \{ x \in \Omega : V(x) = V_0 \},$$

$$M_\delta = \{ x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta \}, \quad \delta > 0,$$

showed a result of multiplicity of positive solutions for (1.4), where $\Omega = \mathbb{R}^N$, $f(u) = |u|^{q-2}u$ with $q \in (2, 2^*)$, and

$$V_\infty = \liminf_{|x| \to \infty} V(x) > V_0 = \inf_{\mathbb{R}^N} V(x) > 0. \quad (1.5)$$

Recall that for a closed subset $Y$ of a topological space $X$, the Lusternik-Schnirelman category, denoted by $\text{cat}_X Y$, is the least number of closed and contractible sets in $X$ which cover $Y$.

Alves & Souto [4] showed an existence and concentration result for (1.4) with $f(u) = u^{q-1} + u^{2^*-1}$ assuming that condition (1.5) holds.

Alves & Figueiredo [1] (see also [12]) proved a multiplicity result for

$$-\epsilon p \Delta_p u + V(z)|u|^{p-2}u = f(u) \quad \text{in} \quad \mathbb{R}^N \quad (1.6)$$

using again Lusternik-Schnirelman category and assuming that condition (1.5) holds, $2 \leq p < N$ and $f$ belongs to a large class which includes the model $f(u) = |u|^{q-2}u$ with $q \in (p, p^*)$. Moreover, the authors showed that each solution of $(P_\ast)$ has a phenomenon of concentration near a point of minimum of the potential $V$.

The case with critical growth was proved in [13].

del Pino & Felmer [11] proved that if the conditions (1.2) and (1.3) hold, problem (1.4) has a positive solution for small $\epsilon$, which has a phenomenon of concentration near of one minimum point of potential $V$.

Alves & Figueiredo [2], using the penalization method and Lusternik-Schnirelman category theory, showed again a multiplicity and concentration result for (1.6), using now the conditions (1.2) and (1.3) with $1 < p < N$.

In this work, motivated by [2] and by some ideas developed [16, 15] and [7], we prove the multiplicity and concentration of positive solutions to (1.1) using Lusternik-Schnirelman category. For $\lambda = 0$ and $p = 2$, we have the result obtained in [9]. Hence the results of this paper complete those [9] in three senses: because we deal with $1 < p < N$ instead of $p = 2$, because we do restrict the behavior of $V$ at infinity, and because we have $f(u) = |u|^{q-2}u + \lambda|u|^{s-2}u$ with $s \geq p^*$. Moreover, in the present paper, we continue the study of [2] and [13], because we consider supercritical nonlinearities. To our knowledge there is no results on existence of solutions to problem $(P_\lambda)$ via the penalization method, and multiplicity results with supercritical growth via the Lusternik-Schnirelman category theory.

Our main result for problem (1.1) is the following.

**Theorem 1.1.** Suppose that the function $V$ satisfies (1.2)-(1.3). Then, for any $\delta > 0$, there exists $\tau = \tau(\delta) > 0$ and $\lambda_0 > 0$ such that (1.1) has at least $\text{cat}_{M_\delta} M$ positive solutions for all $\epsilon \in (0, \tau)$ and for all $\lambda \in [0, \lambda_0]$. Moreover, if $u_\epsilon$ is a positive solution of (1.1) and $\eta_\epsilon \in \mathbb{R}^N$ a global maximum point of $u_\epsilon$, then

$$\lim_{\epsilon \to 0} V(\eta_\epsilon) = V_0.$$
To solve problem (1.1), we first consider a truncated problem which involves only a subcritical Sobolev exponent. We show that any positive solution of truncated problem is a positive solution of (1.1).

Hereafter, we will work with the following problem equivalent to (1.1), which is obtained under change of variable $z = \varepsilon x$

$$
-\Delta_p u + V(\varepsilon x)|u|^{p-2}u = |u|^{q-2}u + \lambda|u|^{s-2}u \quad \text{in } \mathbb{R}^N
$$

$$
u \in W^{1,p}(\mathbb{R}^N) \quad \text{with } 1 < p < N
$$

$$
u(x) > 0, \quad \forall x \in \mathbb{R}^N.
$$

\section{Truncated Problem}

First of all, we have to note that because $f$ has supercritical growth we cannot use directly variational techniques because of the lack of compactness of the Sobolev immersions.

So we construct a suitable truncation of $f$ in order to use variational methods or more precisely, the Mountain Pass Theorem. This truncation was used in [16] (see also [7] and [12]).

Let $K > 0$, be a constant to be determined later, and $\hat{f}_K : \mathbb{R} \to \mathbb{R}$ given by

$$
\hat{f}_K(t) = \begin{cases}
0 & \text{if } t < 0 \\
 t^{q-1} + \lambda t^{s-1} & \text{if } 0 \leq t < K \\
 (1 + \lambda K^{s-q})t^{q-1} & \text{if } t \geq K.
\end{cases}
$$

Consider $\alpha, \gamma \in \mathbb{R}$ such that $\alpha < 1 < \gamma$ and $\eta \in C^1([\alpha K, \gamma K])$ with $\alpha$ and $\gamma$ independent of $K$ and $\eta$ satisfying

$$
\eta(t) \leq \hat{f}_K(t) \quad \text{for all } t \in [\alpha K, \gamma K],
$$

$$
\eta(\alpha K) = \hat{f}_K(\alpha K), \quad \eta(\gamma K) = \hat{f}_K(\gamma K),
$$

$$
\eta'(\alpha K) = \hat{f}_K'(\alpha K), \quad \eta'(\gamma K) = \hat{f}_K'(\gamma K),
$$

$$
t \mapsto \frac{\eta(t)}{t^{q-1}} \text{ is increasing for all } t \in [\alpha K, \gamma K].
$$

Now using the functions $\eta$ and $\hat{f}_K$, we define

$$
f_K(t) = \begin{cases}
\eta(t) & \text{if } t \in [\alpha K, \gamma K], \\
\hat{f}_K(t) & \text{if } t \notin [\alpha K, \gamma K]
\end{cases}
$$

and the truncated problem

$$
-\Delta_p u + V(\varepsilon x)|u|^{p-2}u = f_K(u)
$$

$$
u \in W^{1,p}(\mathbb{R}), \quad u > 0 \quad \text{in } \mathbb{R}^N.
$$

It is easy to check that $f_K \in C^1(\mathbb{R})$, and that

$$
f_K(t) = 0, \quad \text{for all } t < 0,
$$

$$
f_K(t) \leq (1 + \lambda K^{s-q})t^{q-1} \quad \text{for all } t \geq 0,
$$

$$
F_K(t) = \int_0^t f_K(\xi)d\xi
$$
there exists \( \theta \in \mathbb{R} \) such that \( p < \theta \) and
\[
0 < \theta F_K(t) \leq f_K(t)t \quad \text{for all } t > 0,
\]
the function
\[
t \mapsto \frac{f_K(t)}{t^{p-1}} \quad \text{is increasing for all } t > 0,
\]
\[
f_K'(t)t^2 - (p-1)f_K(t)t \geq (q-p)t^q.
\]

**Remark 2.1.** Note that if \( u_{\epsilon,\lambda} \) is a positive solution of (2.1) such that there exists \( K_0 > 0 \), where for each \( K \geq K_0 \), there exists \( \lambda_0(K) > 0 \) such that \( |u_{\epsilon,\lambda}|_{L^\infty(\mathbb{R}^N)} \leq \alpha K \) for all \( \epsilon \in (0, \bar{\epsilon}) \) and for all \( \lambda \in [0, \lambda_0] \), then \( u_{\epsilon,\lambda} \) is a positive solution of (1.7).

### 3. Multiplicity and Concentration of positive solutions for Truncated Problem

The result below is related to the multiplicity and concentration of solutions for (2.1) and its proof can be found in [2, Theorem 1.1] or [12].

**Theorem 3.1.** Suppose that \( V \) verify \( (1.2), (1.3) \). Then, for any \( \delta > 0 \), there exists \( \epsilon_0 = \epsilon_0(\delta, \lambda, K) > 0 \) such that \( (T_\lambda) \) has at least \( c_{M_\delta} \) positive solutions for all \( \epsilon \in (0, \epsilon_0) \) and for each \( \lambda > 0 \). Moreover, if \( u_{\epsilon,\lambda} \) is a positive solution of (2.1) and \( \eta_\epsilon \in \mathbb{R}^N \) a global maximum point of \( u_{\epsilon,\lambda} \), then
\[
\lim_{\epsilon \to 0} V(\eta_\epsilon) = V_0.
\]

### 4. Multiplicity of positive solutions for (1.7)

We recall that the weak solutions of (2.1) are the critical points of the functional
\[
I_{\epsilon,\lambda}(u) = \int_{\mathbb{R}^N} |\nabla u|^p + \frac{1}{p} \int_{\mathbb{R}^N} V(\epsilon x)|u|^p - \int_{\mathbb{R}^N} F_K(u),
\]
which is well defined for \( u \in W_\epsilon \), where
\[
W_\epsilon = \{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)|u|^p < \infty \}
\]
endowed with the norm
\[
\|u\|_\epsilon^p = \int_{\mathbb{R}^N} |\nabla u|^p + \int_{\mathbb{R}^N} V(\epsilon x)|u|^p.
\]

Let us also denote by \( E_{V_0,\lambda} \) the energy functional associated to the problem
\[
-\Delta_p u + V_0|u|^{p-2}u = f_K(u) \\
\text{for all } u \in W^{1,p}(\mathbb{R}), \quad u > 0 \text{ in } \mathbb{R}^N,
\]
that is,
\[
E_{V_0,\lambda}(u) = \int_{\mathbb{R}^N} |\nabla u|^p + \frac{1}{p} \int_{\mathbb{R}^N} V_0|u|^p - \int_{\mathbb{R}^N} F_K(u),
\]
Here we will establish a preliminary estimative for \( \|u_{\epsilon,\lambda}\|_\epsilon \).

**Lemma 4.1.** For any solution \( u_{\epsilon,\lambda} \) of (2.1), there exists \( \bar{C} > 0 \), such that
\[
\|u_{\epsilon,\lambda}\|_\epsilon \leq \bar{C},
\]
for \( \epsilon > 0 \) sufficiently small and uniformly in \( \lambda \).
Proof. By [2] Theorem 1.1 (see [12] too), we have that all solutions $u_{\epsilon,\lambda}$ from (2.1) verify the inequality

$$I_{c,\lambda}(u_{\epsilon,\lambda}) \leq cv_{0,\lambda} + h_{\lambda}(\epsilon),$$

where $cv_{0,\lambda}$ is the level Mountain Pass related of functional $E_{V_{0,\lambda}}$ and $h_{\lambda}(\epsilon) \to 0$ as $\epsilon \to 0$ for each $\lambda \geq 0$. In this case, we may suppose that

$$I_{c,\lambda}(u_{\epsilon,\lambda}) \leq cv_{0,\lambda} + 1,$$

for all $\epsilon \in (0, \bar{\epsilon}(K, \lambda))$. Since $cv_{0,\lambda} \leq c_{V_{0,0}}$, we have

$$I_{c,\lambda}(u_{\epsilon,\lambda}) \leq cv_{0,0} + 1,$$  

for all $\epsilon \in (0, \bar{\epsilon}(K, \lambda))$ and for all $\lambda \geq 0$. Moreover,

$$I_{c,\lambda}(u_{\epsilon,\lambda}) = I_{c,\lambda}(u_{\epsilon,\lambda}) - \frac{1}{\theta}I_{c,\lambda}'(u_{\epsilon,\lambda})u_{\epsilon,\lambda}$$

$$= \left(1 - \frac{1}{\theta}\right)\|u_{\epsilon,\lambda}\|_\varepsilon^p + \int_{\mathbb{R}^N} \frac{1}{\theta}f_K(u_{\epsilon,\lambda})u_{\epsilon,\lambda} - F_K(u_{\epsilon,\lambda})\].$$

By [2,2],

$$I_{c,\lambda}(u_{\epsilon,\lambda}) \geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_{\epsilon,\lambda}\|_\varepsilon^p$$

Therefore, by (4.2), $\|u_{\epsilon,\lambda}\|_\varepsilon \leq \bar{C}$, for $\epsilon \in (0, \bar{\epsilon}(K, \lambda))$ and for all $\lambda \geq 0$, where

$$\bar{C} = \left[c_{V_{0,0}} + 1\left(\frac{\theta p}{\theta - p}\right)\right]^{1/p}. $$

Now, we use the Moser iteration technique [15] (see also [7]) to prove that each solution found of (2.1) is a solution of (1.7).

Proof of Theorem 1.1. We use the notation $u_{\epsilon,\lambda} := u$. For each $L > 0$, we define

$$u_L = \begin{cases} 
  u & \text{if } u \leq L, \\
  L & \text{if } u \geq L,
\end{cases}$$

$$z_L = u_L^{p(\beta - 1)} u \quad \text{and} \quad w_L = uu_L^{\beta - 1}$$

with $\beta > 1$ to be determined later. Taking $z_L$ as a test function, we obtain

$$\int_{\mathbb{R}^N} u_L^{p(\beta - 1)}|\nabla u|^p = -p(\beta - 1) \int_{\mathbb{R}^N} u_L^{p(\beta - 1) - 1} |\nabla u|^{p - 2} \nabla u \nabla u_L$$

$$+ \int_{\mathbb{R}^N} f_K(u)u_L^{p(\beta - 1)} - \int_{\mathbb{R}^N} V(\epsilon x)|u|^{p}u_L^{p(\beta - 1)},$$

By [2],

$$\int_{\mathbb{R}^N} u_L^{p(\beta - 1)}|\nabla u|^p \leq C_{\lambda,K} \int_{\mathbb{R}^N} u_L^{p(\beta - 1)},$$

where $C_{\lambda,K} = (1 + \lambda |K|^s - q)$. From Sobolev imbedding, Hölder inequalities and (4.3),

$$|w_L|^p_{p^*} \leq C_1\beta^p C_{\lambda,K} \left(\int_{\mathbb{R}^N} u_L^{p^*}\right)^{(q-p)/p^*} \left(\int_{\mathbb{R}^N} u_L^{pp^*/p^*} - (q-p)\right)^{[p^*-(q-p)]/p^*},$$

where $p < \frac{pp^*}{p^* - (q-p)} < p^*$. Recalling that $\|u_{\epsilon,\lambda}\|_\varepsilon \leq \bar{C}$, we have

$$|w_L|^p_{p^*} \leq C_2\beta^p C_{\lambda,K} \bar{C}^{(q-p)/p^*} |w_L|^p_{p^*},$$

where $C_2$ is a constant depending only on $\lambda$.
where $\alpha^* = \frac{p^*}{p^* - (q - p)}$. Note that if $u^\beta \in L^{\alpha^*}(\mathbb{R}^N)$, using the definition of $w_L$ and the fact that $u_L \leq u$, we obtain
\[
\left( \int_{\mathbb{R}^N} |uu^\beta L-1|^{p^*} \right)^{p/p^*} \leq C_{\beta} C_{\lambda, K} \left( \int_{\mathbb{R}^N} u^{\beta \alpha^*} \right)^{p/\alpha^*} < +\infty.
\]
By Fatou’s Lemma on the variable $L$, we get
\[
|u|_{\beta p^*} \leq (C_{4} C_{\lambda, K})^{1/\beta} \beta^{1/\beta} |u|_{\beta \alpha^*}.
\]
(4.4)
The assertion is obtained by iteration of estimative (4.4). Namely, let $\chi = \frac{p^*}{\alpha^*}$; i.e., $p^* = \chi \alpha^*$. Then
\[
|u|_{\chi (m+1) \alpha^*} \leq C_5 (C_{4} C_{\lambda, K}) \sum_{i=1}^{m} \frac{\chi^i}{\beta} \chi \sum_{i=1}^{m} i \chi^{-i} \tilde{C}.
\]
Passing to the limit as $m \to \infty$, we have
\[
|u|_{L^\infty(\mathbb{R}^N)} \leq C_5 (C_{4} C_{\lambda, K}) \alpha^2 \tilde{C},
\]
where $\sigma_1 = \sum_{i=1}^{\infty} \frac{\chi^i}{\beta}$ and $\sigma_2 = \sum_{i=1}^{\infty} i \chi^{-i}$. To choose $\lambda_0$, we consider the inequality
\[
\left[ C_{4} (1 + \lambda K^{s-q}) \right]^{\sigma_1} \chi \sigma_2 \tilde{C} \leq C_5 \leq \alpha K.
\]
We conclude that
\[
(1 + \lambda K^{s-q})^{\sigma_1} \leq \frac{\alpha K C_6}{C_{4}^{\sigma_1} \chi \sigma_2 \tilde{C}}.
\]
We choose $\lambda_0$ verifying the inequality
\[
\lambda_0 \leq \left[ \frac{(\alpha K C_6)^{1/\sigma_1}}{C_{4}^{\sigma_1} \chi \sigma_2 \tilde{C}^{1/\sigma_1}} - 1 \right] \frac{1}{K^{s-q}}
\]
and fixing $K$ such that
\[
\left[ \frac{(\alpha K C_6)^{1/\sigma_1}}{C_{4}^{\sigma_1} \chi \sigma_2 \tilde{C}^{1/\sigma_1}} - 1 \right] > 0,
\]
we have $|u_{\lambda,\epsilon}|_{L^\infty(\mathbb{R}^N)} \leq \alpha K$ for all $\epsilon \in (0, \tilde{\epsilon}(K, \lambda))$ and all $\lambda \in [0, \lambda_0]$. The result follows from Remark [2.1].

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References


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