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STABILITY AND BOUNDEDNESS OF SOLUTIONS TO CERTAIN FOURTH-ORDER DIFFERENTIAL EQUATIONS

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Abstract. We give criteria for the asymptotic stability and boundedness of solutions to the nonlinear fourth-order ordinary differential equation

$$x^{(4)} + \varphi(\ddot{x})\dddot{x} + f(x, \dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x})\,,$$

when $p \equiv 0$ and when $p \neq 0$. Our results include and improve some well-known results in the literature.

1. Introduction

Since Lyapunov [17] proposed his famous theory on the stability of motion, numerous methods have been proposed for deriving suitable Lyapunov functions to study the stability and boundedness of solutions of certain second-, third-, fourth-, fifth- and sixth order non-linear differential equations. See, for example, Anderson [1], Barbasin [3], Cartwright [4], Chin [5, 6], Ezeilo [8, 9], Harrow [10, 11], Ku and Puri [12], Ku et al. [13], Ku [14, 15], Krasovskii [16], Leighton [17], Li [18], Marinosson [20], Miyagi and Taniguchi [21], Ponzo [22], Reissig et al. [23], Schwartz and Yan [24], Shi-zong et al. [25], Sinha [26, 27], Skidmore [28], Szegö [29], Tiryaki and Tunç [30, 31], Tunç [31, 32, 33], Zubov [36] and the references quoted therein. In 1989, Chin [6] has tried to apply a new technique (called the intrinsic method) proposed by himself to construct some new Lyapunov functions to study the stability of solutions of three fourth order non-linear differential equations described as follows:

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + f(x) = 0, \tag{1.1}$$

$$x^{(4)} + a_1\dddot{x} + \psi(\dot{x})\ddot{x} + a_3\dot{x} + a_4 x = 0, \tag{1.2}$$

$$x^{(4)} + a_1\dddot{x} + f(x, \dot{x})\ddot{x} + a_3\dot{x} + a_4 x = 0. \tag{1.3}$$

Later, the authors in [31] based on the results in [6] have applied the method used in [6] to construct some new Lyapunov functions to examine the stability and boundedness of the solutions of non-linear differential equation described by

$$x^{(4)} + \varphi(\ddot{x})\dddot{x} + f(x, \dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{1.4}$$

with $p \equiv 0$ and $p \neq 0$, respectively. In 1998, Wu and Xiong [35] proved both that the Lyapunov functions constructed in Chin [6] are the same as those obtained

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by Cartwright [4] and Ku [14]. Chin's results [4] are not true for the equations (1.1), (1.2), (1.3) in the general cases. Further, the local asymptotic stability of the zero solution of the equations (1.1), (1.2) and (1.3) has been investigated in [35]. Therefore, in this paper, we will revise our results obtained in [31] again and extend and improve the results established in [35]. Now, we consider the fourth order non-linear differential equation (1.4) or its equivalent system in the phase variables form

$$\begin{gathered} \dot{x} = y, \dot{y} = z, \dot{z} = w, \\ \dot{w} = -\varphi(z)w - f(x,y)z - g(y) - h(x) + p(t,x,y,z,w) \end{gathered} \tag{1.5}$$

in which the functions φ, f, g, h and p depend only on the arguments displayed and the dots denote differentiation with respect to t. The functions φ, f, g, h and p are assumed to be continuous on their respective domains. The derivatives $\frac{dg}{dy} \equiv g'(y)$ and $\frac{dh}{dx} \equiv h'(x)$ exist and are continuous. Moreover, the existence and the uniqueness of the solutions of the equation (1.4) will be assumed. That is, the functions φ, f, g, h and p so constructed such that the uniqueness theorem is valid. It is worth mentioning that the continuity of the functions φ, f, g, h and p guarantees at the least the existence of a solution of the equation (1.4). Next, the existence and continuity of the derivatives $\frac{dg}{dy} \equiv g'(y)$ and $\frac{dh}{dx} \equiv h'(x)$ in a compact domain ensure that the functions g and h satisfy the locally Lipschitz condition in the closed domain. This guarantees the uniqueness of the solutions. It should also be noted that the domain of attraction of the zero solution $x = 0$ of the equation (1.4) (for $p \equiv 0$) in the following first result is not going to be determined here.

2. Main results

Before stating the major theorems, we introduce the following notation:
Set

$$\varphi_1(z) = \begin{cases} \frac{1}{z}\int_0^z \varphi(\tau)d\tau, & z \neq 0 \\ \varphi(0), & z = 0, \end{cases}$$

$$g_1(y) = \begin{cases} \frac{g(y)}{y}, & y \neq 0 \\ g'(0), & y = 0. \end{cases}$$

In the case $p \equiv 0$, we have the following statement.

Theorem 2.1. *In addition to the basic assumptions on φ, f, g and h, suppose that there are positive constants $a, b, c, d, \delta, \varepsilon$ and η such that the following conditions are satisfied:*

- (i) $h(0) = g(0) = 0$
- (ii) $abc - cg'(y) - ad\varphi(z) \geq \delta > 0$ *for all y and z*
- (iii) $0 \leq d - h'(x) \leq \frac{\sqrt{\delta\varepsilon a}}{4}$ *for all x and $h(x)\operatorname{sgn} x \to +\infty$ as $|x| \to \infty$*
- (iv) $0 \leq g_1(y) - c < \frac{\delta}{8c}\sqrt{\frac{d}{2ac}}$ *and $g'(y) \geq c$ for all y*
- (v) $0 \leq f(x,y) - b \leq \eta$ *for all x and y where*

$$\eta \leq \min\Big[\frac{c}{8d}\sqrt{\frac{\delta\varepsilon}{a}}, \frac{a}{8}\sqrt{\frac{\delta\varepsilon}{c}}\Big], \quad \varepsilon \leq \frac{\delta}{2acD}, \quad D = ab + \frac{bc}{d}$$

- (vi) $\varphi(z) \geq a, \varphi_1(z) - \varphi(z) < \frac{\delta}{2a^2c}$ *for all z.*

Then the zero solution of the system (1.5) *is asymptotically stable.*

Remark 2.2. Assumptions (ii), (iv) and (vi) imply

$$\varphi(z) < \frac{bc}{d}, \quad g'(y) < ab.$$

Remark 2.3. When $\varphi(\ddot{x}) = a, f(x,\dot{x}) = b, g(\dot{x}) = c\dot{x}, h(x) = dx$, equation (1.4) reduces to a linear constant coefficient differential equation and conditions (i)-(vi) of Theorem 2.1 reduce to the corresponding Routh-Hurwitz criterion.

Remark 2.4. Theorem 2.1 revises the first theorem in [31] and includes and improves the results of Ezeilo [8, 9], Harrow [10], and Wu and Xiong [35] except the restriction on $f(x,y)$, that is, $0 \le f(x,y) - b \le \eta$.

For the proof of Theorem 2.1 our main tool is the continuous differentiable function $V = V(x,y,z,w)$ defined by

$$\begin{aligned} 2V =& 2\beta\int_0^x h(\xi)d\xi + \beta by^2 - \alpha dy^2 + 2\int_0^y g(\rho)d\rho + \alpha bz^2 + 2\int_0^z \varphi(\tau)\tau d\tau \\ &- \beta z^2 + \alpha w^2 + 2h(x)y + 2\alpha h(x)z + 2\alpha g(y)z \\ &+ 2\beta y\int_0^z \varphi(\tau)d\tau + 2\beta yw + 2zw, \end{aligned} \tag{2.1}$$

where

$$\alpha = \varepsilon + \frac{1}{a}, \quad \beta = \varepsilon + \frac{d}{c}. \tag{2.2}$$

The following lemmas are used for proving that the function $V(x,y,z,w)$ is a Lyapunov function of the system (1.5).

Lemma 2.5. *Suppose that all the conditions of Theorem 2.1 hold. Then there are positive constants $D_i \equiv D_i(a,b,c,d,\varepsilon,\delta)$, $(i = 1, 2, 3, 4)$, such that for all x,y,z,w,*

$$V \ge D_1\int_0^x h(\xi)d\xi + D_2y^2 + D_3z^2 + D_4w^2\,.$$

Proof. We observe that the function $2V$ in (2.1) can be rewrittten as

$$\begin{aligned} 2V =& \frac{1}{c}[h(x) + cy + \alpha cz]^2 + \frac{1}{\varphi_1(z)}[w + \varphi_1(z)z + \beta\varphi_1(z)y]^2 + [\alpha - \frac{1}{\varphi_1(z)}]w^2 \\ &+ [\alpha b - \beta - \alpha^2 c]z^2 + [\beta b - \alpha d - \beta^2\varphi_1(z)]y^2 + 2\int_0^y g(\rho)d\rho - cy^2 \\ &+ 2\alpha[g_1(y) - c]yz + 2\beta\int_0^x h(\xi)d\xi - (\frac{1}{c})h^2(x) + [2\int_0^z \varphi(\tau)\tau d\tau - \varphi_1(z)z^2]. \end{aligned}$$

In light of the hypothesis of the theorem, the use of (2.2) and the mean value theorem (both for the derivative and integral), it can be easily obtained that

$$2V \ge \varepsilon\int_0^x h(\xi)d\xi + (\frac{\delta d}{2ac^2})y^2 + (\frac{\delta}{4a^2c})z^2 + \varepsilon w^2 + 2\alpha[g_1(y) - c]yz.$$

The remaining of this proof follows the strategy indicated in [31], and hence it is omitted. This completes the proof. $\square$

Lemma 2.6. *Assume that all the conditions of Theorem 2.1 hold, Then there exist positive constants* $D_i \equiv D_i(a, b, c, \varepsilon, \delta)$, $(i = 5, 6, 7)$, *such that if* $(x(t), y(t), z(t), w(t))$ *is a solution of the system* (1.5), *then*

$$\dot{V} \equiv \frac{d}{dt}V(x, y, z, w) \leq -(D_5 y^2 + D_6 z^2 + D_7 w^2). \tag{2.3}$$

Proof. Along any solution (x, y, z, w) of system (1.5), it follows from (2.1) and (1.5) that

$$\begin{aligned} \dot{V} = &- [f(x, y) - \alpha g'(y) - \beta\varphi_1(z)]z^2 - [\alpha\varphi(z) - 1]w^2 - [\beta g_1(y) - h'(x)]y^2 \\ &- \beta[f(x, y) - b]yz - \alpha[f(x, y) - b]zw - \alpha[d - h'(x)]yz. \end{aligned} \tag{2.4}$$

It is clear from (ii)-(vi) and (2.2) that

$$\dot{V} \leq -(\frac{\varepsilon c}{2})y^2 - (\frac{\delta}{8ac})z^2 - (\frac{3\varepsilon a}{4})w^2 - W_6 - W_7 - W_8, \tag{2.5}$$

where

$$W_6 = (\frac{\varepsilon c}{4})y^2 + \beta[f(x, y) - b]yz + (\frac{\delta}{16ac})z^2, \tag{2.6}$$

$$W_7 = (\frac{\varepsilon a}{4})w^2 + \alpha[f(x, y) - b]zw + (\frac{\delta}{16ac})z^2, \tag{2.7}$$

$$W_8 = (\frac{\varepsilon c}{4})y^2 + \alpha[d - h'(x)]yz + (\frac{\delta}{4ac})z^2. \tag{2.8}$$

It should be noted that all six coefficients in the expressions (2.6)-(2.8) are non-negative. By using the conditions (ii), (iii),(v), and the inequalities

$$\begin{gathered} \beta^2[f(x, y) - b]^2 < \frac{4d^2}{c^2}[f(x, y) - b]^2 < \frac{\delta\varepsilon}{16a}, \\ \alpha^2[f(x, y) - b]^2 < \frac{4}{a^2}[f(x, y) - b]^2 < \frac{\delta\varepsilon}{16c}, \\ \alpha^2[d - h'(x)]^2 < \frac{4}{a^2}[d - h'(x)]^2 < \frac{\delta\varepsilon}{4a} \end{gathered}$$

respectively, it follows that

$$W_6 \geq (\frac{\varepsilon c}{4})y^2 - (\frac{\sqrt{\delta\varepsilon}}{4\sqrt{a}})|yz| + (\frac{\delta}{16ac})z^2 \geq [\frac{\sqrt{\varepsilon c}}{2}|y| - \frac{1}{4}\sqrt{\frac{\delta}{ac}}|z|]^2 \geq 0, \tag{2.9}$$

$$W_7 \geq (\frac{\varepsilon a}{4})w^2 - (\frac{\sqrt{\delta\varepsilon}}{4\sqrt{c}})|zw| + (\frac{\delta}{16ac})z^2 = [\frac{\sqrt{\varepsilon a}}{2}|w| - \frac{1}{4}\sqrt{\frac{\delta}{ac}}|z|]^2 \geq 0, \tag{2.10}$$

$$W_6 \geq (\frac{\varepsilon c}{4})y^2 - \frac{1}{2}\sqrt{\frac{\delta\varepsilon}{a}}|yz| + (\frac{\delta}{4ac})z^2 = [\frac{\sqrt{\varepsilon c}}{2}|y| - \frac{1}{2}\sqrt{\frac{\delta}{ac}}|z|]^2 \geq 0. \tag{2.11}$$

By collecting the estimates (2.9)-(2.11) into (2.5) we obtain

$$\dot{V} \leq -(\frac{\varepsilon c}{2})y^2 - (\frac{\delta}{8ac})z^2 - (\frac{3\varepsilon a}{4})w^2$$

which proves the lemma. $\square$

Proof of Theorem 2.1. From Lemma 2.5, Lemma 2.6 and condition (iii) of Theorem 2.1, we see that

$$\begin{aligned} V(x,y,z,w) = 0 \quad &\text{if and if only if} \quad x^2+y^2+z^2+w^2=0,\\ V(x,y,z,w) > 0 \quad &\text{if and if only if} \quad x^2+y^2+z^2+w^2>0,\\ V(x,y,z,w) \to \infty \quad &\text{if and if only if} \quad x^2+y^2+z^2+w^2\to\infty. \end{aligned}$$

Let γ denote a trajectory $(x(t), y(t), z(t), w(t))$ of system (1.5) with $p(t, x, y, z, w) \equiv 0$ such that $t = 0$, $x = x_0$, $y = y_0$, $z = z_0$, $w = w_0$, where (x_0, y_0, z_0, w_0) is an arbitrary point in x, y, z, w-space from which motions may originate. Then by Lemma 2.6 for $t \geq 0$,

$$V(x,y,z,w) = V(x(t),y(t),z(t),w(t)) = V(t) \leq V(0).$$

Moreover, $V(t)$ is nonnegative and non-increasing and therefore tends to a nonnegative limit, $V(\infty)$ say, as $t \to \infty$. Suppose $V(\infty) > 0$. Consider the set

$$S\left\{(x,y,z,w) \mid V(x,y,z,w) \leq V(x_0,y_0,z_0,w_0)\right\}.$$

Because of the properties of the function V we know that S is bounded, and therefore the set $\gamma \subset S$ is also bounded. Further, the nonempty set of all limit points of γ consists of whole trajectories of the system

$$\begin{gathered} \dot{x} = y, \dot{y} = z, \dot{z} = w,\\ \dot{w} = -\varphi(z)w - f(x,y)z - g(y) - h(x) \end{gathered}$$

lying on the surface $V(x, y, z, w) = V(\infty)$. Thus if P is a limit point of γ, then there exists a half-trajectory, say γ_P of the above system, issuing from P and lying on the surface $V(x, y, z, w) = V(\infty)$. Since for every point (x, y, z, w) on γ_P we have $V(x, y, z, w) \geq V(\infty)$, this implies that $\dot{V} = 0$ on γ_P. Also, in view of the inequality obtained in Lemma 2.6, that is

$$\dot{V} \leq -(\frac{\varepsilon c}{2})y^2 - (\frac{\delta}{8ac})z^2 - (\frac{3\varepsilon a}{4})w^2,$$

$\dot{V} = 0$ implies $y = z = w = 0$; and by the above system and conditions (i) and (iii) of Theorem 2.1, this means that $x = 0$. Thus, the point $(0, 0, 0, 0)$ lies on the surface $V(x, y, z, w) = V(\infty)$ and hence $V(\infty) = 0$. This completes the proof of Theorem 2.1. □

In the case $p \neq 0$ we have

Theorem 2.7. *Suppose the following conditions are satisfied:*

(i) $g(0) = 0$
(ii) *the conditions (ii)-(vi) of Theorem 2.1 hold*
(iii) $|p(t, x, y, z, w)| \leq (A + |y| + |z| + |w|)q(t)$, *where* $q(t)$ *is a non-negative and continuous function of* t, *and satisfies* $\int_0^t q(s)ds \leq B < \infty$ *for all* $t \geq 0$, A *and* B *are positive constants.*

Then for any given finite constants x_0, y_0, z_0 *and* w_0, *there exists a constant* $K = K(x_0, y_0, z_0, w_0)$, *such that any solution* $(x(t), y(t), z(t), w(t))$ *of the system* (1.5) *determined by*

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0, \quad w(0) = w_0$$

satisfies for all $t \geq 0$,

$$|x(t)| \leq K, |y(t)| \leq K, |z(t)| \leq K, |w(t)| \leq K.$$

Remark 2.8. Theorem 2.7 revises the second theorem in [31], and generalizes the results of Ezeilo [8] and Harrow [11], and improves the results of Wu and Xiong [35] except the restriction on $f(x,y)$, that is, $0 \leq f(x,y) - b \leq \eta$.

Proof of Theorem 2.7. The proof here is based essentially on the method devised by Antosiewicz [2]. Let $(x(t), y(t), z(t), w(t))$ be an arbitrary solution of the system (1.5) satisfying the initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0, \quad w(0) = w_0.$$

Next, consider the function $V(t) = V(x(t), y(t), z(t), w(t))$, where V is defined by (2.1). Because $h(0)$ is not necessarily zero now; we only have the following estimate, in the proof of the theorem,

$$V \geq D_1 \int_0^x h(\xi)d\xi + D_2y^2 + D_3z^2 + D_4w^2 - (\frac{1}{c})h^2(0) \tag{2.12}$$

and since $p \neq 0$, the conclusion of Lemma 2.6 can be revised as follows

$$\dot{V} \leq -(D_5y^2 + D_6z^2 + D_7w^2) + (\alpha w + z + \beta y)p(t, x, y, z, w).$$

Let $D_8 = \max(\alpha, 1, \beta)$. Then, we have

$$\dot{V} \leq -D_8(|y| + |z| + |w|)(A + |y| + |z| + |w|)q(t).$$

Using the inequalities

$$|w| \leq 1 + w^2 \text{ and } |2yz| \leq y^2 + z^2,$$

we obtain

$$\dot{V} \leq D_9[3 + 4(y^2 + z^2 + w^2)]q(t), \tag{2.13}$$

where $D_9 = D_8(A + 1)$. It follows from from the result of Lemma 2.5 that

$$V \geq D_{10}(y^2 + z^2 + w^2) - D_0, \tag{2.14}$$

$D_{10} = \min(D_2, D_3, D_4)$. Now, from (2.13) and (2.14) we have

$$\dot{V} \leq D_{11}q(t) + D_{12}Vq(t) \tag{2.15}$$

where $D_{11} = D_9(3 + \frac{4D_0}{D_{10}})$, $D_{12} = \frac{4D_9}{D_{10}}$. Integrating (2.15) from 0 to t, we obtain

$$V(t) - V(0) \leq D_{11} \int_0^t q(s)ds + D_{12} \int_0^t V(s)q(s)ds.$$

Setting $D_{13} = D_{11}B + V(0)$, and using condition (iii) of Theorem 2.7 we have

$$V(t) \leq D_{13} + D_{12} \int_0^t V(s)q(s)ds.$$

Hence, Gronwall-Bellman inequality yields

$$V(t) \leq D_{13} \exp(D_{12} \int_0^t q(s)ds).$$

This completes the proof of Theorem 2.7. $\square$

Finally, if p is a bounded function, then the constant K above can be fixed independent of x_0, y_0, z_0 and w_0, as will be seen from our next result.

Theorem 2.9. *Suppose that $g(0) = 0$ and conditions (ii)-(vi) of Theorem 2.1 hold, and that $p(t, x, y, z, w)$ satisfies*

$$|p(t, x, y, z, w)| \leq \Delta < \infty$$

for all values of x, y, z and w, where Δ is a positive constant.

Then there exists a constant K_1 whose magnitude depends on a, b, c, d, δ and ε as well as on the functions φ, f, g and h such that every solution $(x(t), y(t), z(t), w(t))$ of the system (1.5) ultimately satisfies

$$|x(t)| \leq K_1, \quad |y(t)| \leq K_1, \quad |z(t)| \leq K_1, \quad |w(t)| \leq K_1.$$

Remark 2.10. Theorem 2.9 revises [31, Theorem 3], and improves the results of Wu and Xiong [35] except the restriction on $f(x, y)$, that is, $0 \leq f(x, y) - b \leq \eta$.

Now, the actual proof of Theorem 2.9 will rest mainly on certain properties of a piecewise continuously differentiable function $V_1 = V_1(x, y, z, w)$ defined by $V_1 = V + V_0$, where V is the function (2.1) and V_0 is defined as follows:

$$V_0(x, w) = \begin{cases} x \operatorname{sgn} w, & |w| \geq |x| \\ w \operatorname{sgn} x, & |w| \leq |x|. \end{cases} \tag{2.16}$$

The first property of V_1 is stated as follows.

Lemma 2.11. *Subject to the conditions of Theorem 2.9, there is a constant D_{14} such that*

$$V_1(x, y, z, w) \geq -D_{14} \quad \text{for } x, y, z, w \tag{2.17}$$

and

$$V_1(x, y, z, w) \to +\infty \quad \text{as } x^2 + y^2 + z^2 + w^2 \to +\infty. \tag{2.18}$$

Proof. From (2.16) we obtain $|V_0(x, w)| \leq |w|$ for all x and w. In view of the last inequality, it follows that

$$V_0(x, w) \geq -|w| \quad \text{for all } x, w.$$

Using the estimates for V and V_0 we get the estimate for V_1 as follows:

$$\begin{aligned} 2V_1 &\geq D_1 \int_0^x h(\xi) d\xi + D_2 y^2 + D_3 z^2 + D_4 w^2 - 2|w| \\ &= D_1 \int_0^x h(\xi) d\xi + D_2 y^2 + D_3 z^2 + D_4 (|w| - D_4^{-1})^2 - D_4^{-1}. \end{aligned}$$

Making use of condition (iii) of Theorem 2.1 we easily deduce that the integral on the right-hand here is non-negative and tends to infinity when x does so. Then it is evident that the expressions (2.17) and (2.18) are verified, where $D_{14} = D_4^{-1}$ which proves the lemma. □

The next property of the function V_1 is connected with its total time derivative and is contained in the following.

Lemma 2.12. *Let (x, y, z, w) be any solution of the differential system* (1.5) *and the function $v_1 = v_1(t)$ be defined by $v_1(t) = V_1(x(t), y(t), z(t), w(t))$. Then the limit*

$$\dot{v}_1^+(t) = \limsup_{h \to 0^+} \frac{v_1(t + h) - v_1(t)}{h}$$

exists and there is a constant D_{15} such that $\dot{v}_1^+(t) \le -1$ provided

$$x^2(t) + y^2(t) + z^2(t) + w^2(t) \ge D_{15}.$$

Proof. In accordance with the representation $V_1 = V + V_0$ we have a representation $v_1 = v + v_0$. The existence of $\dot{v}_1^+$ is quite immediate, since v has continuous first partial derivatives and v_0 is easily shown to be locally Lipschitizian in x and w so that the composite function $v_1 = v + v_0$ is at the least locally Lipschitizian in x, y, z and w. Subject to the assumptions of Theorem 2.1 an easy calculation from (2.16) and (1.5) shows that

$$\begin{aligned}
\dot{v}_0^+ &= \begin{cases} y \operatorname{sgn} w, & \text{if } |w| \ge |x| \\ -h(x) \operatorname{sgn} x - [\varphi(z)w + f(x,y)z & \\ +g(y) - p(t,x,y,z,w)] \operatorname{sgn} x, & \text{if } |w| \le |x| \end{cases} \\
&\le \begin{cases} y \operatorname{sgn} w, & \text{if } |w| \ge |x| \\ -h(x) \operatorname{sgn} x + D_{16}[|w| + |z| + |y| + 1], & \text{if } |w| \le |x|, \end{cases}
\end{aligned}$$

where $D_{16} = \max\big\{\frac{bc}{d}, b + \frac{c}{8d}\sqrt{\frac{\delta\varepsilon}{a}}, b + \frac{a}{8}\sqrt{\frac{\delta\varepsilon}{c}}, c + \frac{\delta}{8c}\sqrt{\frac{d}{2ac}}, \Delta\big\}$.

In view of the estimates for $\dot{v}$ and $\dot{v}_0^+$, we see that

$$\dot{v}_1^+ = \dot{v} + \dot{v}_0^+ \le -(\frac{\varepsilon c}{2})y^2 - (\frac{\delta}{8ac})z^2 - (\frac{3\varepsilon a}{4})w^2 + D_{17}(|y| + |z| + |w|)$$

if $|w| \ge |x|$, or

$$\dot{v}_1^+ = \dot{v} + \dot{v}_0^+ \le -(\frac{\varepsilon c}{2})y^2 - (\frac{\delta}{8ac})z^2 - (\frac{3\varepsilon a}{4})w^2 - h(x)\operatorname{sgn} x + D_{18}(|y| + |z| + |w|),$$

if $|w| \le |x|$. Then by an argument similar to that in the proof of theorem in [7], one may show that $\dot{v}_1^+ \le -1$ provided

$$x^2(t) + y^2(t) + z^2(t) + w^2(t) \ge D_{15}.$$

The proof of this lemma is now complete. □

Proof of Theorem 2.9. We proved through Lemma 2.11 and Lemma 2.12 that the function $V_1 = V + V_0$ has the following properties:

$$\begin{gathered}
V_1(x,y,z,w) \ge -D_{14} \text{ for all } x, y, z, w, \\
V_1(x,y,z,w) \to \infty \text{ as } x^2 + y^2 + z^2 + w^2 \to +\infty, \\
\dot{V}_1^+(t) \le -1 \text{ provided } x^2 + y^2 + z^2 + w^2 \ge D_{15}.
\end{gathered}$$

The usual Yoshizawa-type argument,that is Theorem 2.1 in Chukwu [7], applied to the above expressions this implies: For any solution $(x(t), y(t), z(t), w(t))$ of the system (1.5) we have that

$$|x(t)| \le K_1, |y(t)| \le K_1, |z(t)| \le K_1, |w(t)| \le K_1$$

for sufficiently large t. Thus the proof of Theorem 2.9 is complete. □

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