MULTIPLE SOLUTIONS FOR THE $p$-LAPLACE EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. In this note, we show the existence of at least three nontrivial solutions to the quasilinear elliptic equation

$$-\Delta_p u + |u|^{p-2}u = f(x,u)$$

in a smooth bounded domain $\Omega$ of $\mathbb{R}^N$ with nonlinear boundary conditions

$$|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = g(x,u)$$
on $\partial \Omega$. The proof is based on variational arguments.

1. Introduction

Let us consider the nonlinear elliptic problem

$$-\Delta_p u + |u|^{p-2}u = f(x,u) \quad \text{in } \Omega$$

$$|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = g(x,u) \quad \text{on } \partial \Omega,$$

(1.1)

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-laplacian and $\partial/\partial \nu$ is the outer unit normal derivative.

Problem (1.1) appears naturally in several branches of pure and applied mathematics, such as the study of optimal constants for the Sobolev trace embedding (see [5, 10, 12, 11]); the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [7, 16]); non-Newtonian fluids, reaction diffusion problems, flow through porous media, nonlinear elasticity, glaciology, etc. (see [1, 2, 3, 6]).

The purpose of this note, is to prove the existence of at least three nontrivial solutions for (1.1) under adequate assumptions on the sources terms $f$ and $g$. This result extends previous work by the author [8, 9].

Here, no oddness condition is imposed in $f$ or $g$ and a positive, a negative and a sign-changing solution are found. The proof relies on the Lusternik–Schnirelman method for non-compact manifolds (see [14]).

For a related result with Dirichlet boundary conditions, see [15] and more recently [4, 17]. The approach in this note follows the one in [15].

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Throughout this work, by (weak) solutions of (1.1) we understand critical points of the associated energy functional acting on the Sobolev space $W^{1,r}(\Omega)$:

$$
\Phi(v) = \frac{1}{p} \int_{\Omega} |\nabla v|^p + |v|^p \, dx - \int_{\Omega} F(x,v) \, dx - \int_{\partial \Omega} G(x,v) \, dS,
$$

where $F(x,u) = \int_0^u f(x,z) \, dz$, $G(x,u) = \int_0^u g(x,z) \, dz$ and $dS$ is the surface measure.

We will denote

$$
\mathcal{F}(v) = \int_{\Omega} F(x,v) \, dx \quad \text{and} \quad \mathcal{G}(v) = \int_{\partial \Omega} G(x,v) \, dS,
$$

so the functional $\Phi$ can be rewritten as

$$
\Phi(v) = \frac{1}{p} \|v\|_{W^{1,r}(\Omega)}^p - \mathcal{F}(v) - \mathcal{G}(v).
$$

2. Assumptions and statement of the results

The precise assumptions on the source terms $f$ and $g$ are as follows:

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$, is a measurable function with respect to the first argument and continuously differentiable with respect to the second argument for almost every $x \in \Omega$. Moreover, $f(x,0) = 0$ for every $x \in \Omega$.

(F2) There exist constants $p < q < p^* = Np/(N-p)$, $s > p^*/(p^*-q)$, $t = sq/(2+(q-2)s) > p^*/(p^*-2)$ and functions $a \in L^q(\Omega)$, $b \in L^t(\Omega)$, such that for $x \in \Omega$, $u,v \in \mathbb{R}$,

$$
|f_u(x,u)| \leq a(x)|u|^{q-2} + b(x),
$$

$$
|(f_u(x,u) - f_u(x,v))| \leq (a(x)(|u|^{q-2} + |v|^{q-2}) + b(x))|u - v|.
$$

(F3) There exist constants $c_1 \in (0,1/(p-1))$, $c_2 > p$, $0 < c_3 < c_4$, such that for any $u \in L^q(\Omega)$

$$
c_3 \|u\|_{L^q(\Omega)}^q \leq c_2 \int_{\Omega} F(x,u) \, dx \leq \int_{\Omega} f(x,u)u \, dx \leq c_1 \int_{\Omega} f_u(x,u)u^2 \, dx \leq c_4 \|u\|_{L^q(\Omega)}^q.
$$

(G1) $g : \partial \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function with respect to the first argument and continuously differentiable with respect to the second argument for almost every $y \in \partial \Omega$. Moreover, $g(y,0) = 0$ for every $y \in \partial \Omega$.

(G2) There exist constants $p < r < p_* = (N-1)p/(N-p)$, $\sigma > p_*/(p_*-r)$, $\tau = \sigma r/(2+(r-2)\sigma) > p_*/(p_*-2)$ and functions $\alpha \in L^r(\partial \Omega)$, $\beta \in L^\tau(\partial \Omega)$, such that for $y \in \partial \Omega$, $u,v \in \mathbb{R}$,

$$
|g_u(y,u)| \leq \alpha(y)|u|^r - \beta(y),
$$

$$
|(g_u(y,u) - g_u(y,v))| \leq (\alpha(y)|u|^r - \beta(y))|u - v| + \beta(y)|u - v|.
$$

(G3) There exist constants $k_1 \in (0,1/(p-1))$, $k_2 > p$, $0 < k_3 < k_4$, such that for any $u \in L^r(\partial \Omega)$

$$
k_3 \|u\|_{L^r(\partial \Omega)}^r \leq k_2 \int_{\partial \Omega} G(x,u) \, dS \leq \int_{\partial \Omega} g(x,u)u \, dS \leq k_1 \int_{\partial \Omega} g_u(x,u)u^2 \, dx \leq k_4 \|u\|_{L^r(\partial \Omega)}^r.
$$
**Remark 2.1.** Assumptions (F1)–(F3) imply, since the immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ with $1 < q < p^*$ is compact, that $F$ is $C^1$ with compact derivative. Analogously, (G1)–(G3) implies the same facts for $G$ by the compactness of the immersion $W^{1,p}(\Omega) \hookrightarrow L^r(\partial \Omega)$ for $1 < r < p^*$.

The main result of the paper reads as follows.

**Theorem 2.2.** Under assumptions (F1)–(F3), (G1)–(G3), there exist three different, nontrivial, (weak) solutions of (1.1). Moreover these solutions are, one positive, one negative and the other one has non-constant sign.

### 3. Proof of the Theorem

The proof uses the same approach as in [15]. That is, we will construct three disjoint sets $K_i \neq \emptyset$ not containing 0 such that $\Phi$ has a critical point in $K_i$. These sets will be subsets of smooth manifolds $M_i \subset W^{1,p}(\Omega)$ that will be constructed by imposing a sign restriction and a normalizing condition.

In fact, let

$$M_1 = \{ u \in W^{1,p}(\Omega) : \int_{\partial \Omega} u_+ dS > 0, \| u_+ \|_{W^{1,p}(\Omega)}^p = \langle F'(u), u_+ \rangle + \langle G'(u), u_+ \rangle \},$$

$$M_2 = \{ u \in W^{1,p}(\Omega) : \int_{\partial \Omega} u_- dS > 0, \| u_- \|_{W^{1,p}(\Omega)}^p = \langle F'(u), u_- \rangle + \langle G'(u), u_- \rangle \},$$

$$M_3 = M_1 \cap M_2,$$

where $u_+ = \max\{u, 0\}$, $u_- = \max\{-u, 0\}$ are the positive and negative parts of $u$, and $\langle \cdot, \cdot \rangle$ is the duality pairing of $W^{1,p}(\Omega)$.

Finally we define

$$K_1 = \{ u \in M_1 \mid u \geq 0 \}, \quad K_2 = \{ u \in M_2 \mid u \leq 0 \}, \quad K_3 = M_3.$$

For the proof of the main theorem, we need the following Lemmas.

**Lemma 3.1.** There exist $c_j > 0$ such that, for every $u \in K_i$, $i = 1, 2, 3$,

$$\| u \|_{W^{1,p}(\Omega)}^p \leq c_1 \left( \int_{\Omega} f(x, u) u dx + \int_{\partial \Omega} g(x, u) u dS \right) \leq c_2 \Phi(u) \leq c_3 \| u \|_{W^{1,p}(\Omega)}^p.$$

**Proof.** Since $u \in K_i$, we have

$$\| u \|_{W^{1,p}(\Omega)}^p = \int_{\Omega} f(x, u) u dx + \int_{\partial \Omega} g(x, u) u dS.$$

This proves the first inequality. Now, by (F3) and (G3)

$$\int_{\Omega} F(x, u) dx \leq \frac{1}{k_2} \int_{\Omega} f(x, u) u dx,$$

$$\int_{\partial \Omega} G(x, u) dS \leq \frac{1}{c_2} \int_{\partial \Omega} g(x, u) u dS.$$

So, for $C = \max\{ \frac{1}{k_2}, \frac{1}{c_2} \} < \frac{1}{p}$, we have

$$\Phi(u) \leq \left( \frac{1}{p} - C \right) \| u \|_{W^{1,p}(\Omega)}^p.$$

This proves the third inequality.
To prove the middle inequality we proceed as follows:

\[
\Phi(u) = \frac{1}{p} \|u\|_{W^{1,p}(\Omega)}^p - \int_{\Omega} F(x, u) \, dx - \int_{\partial\Omega} G(x, u) \, dS
\]

\[
= \frac{1}{p} \left( \int_{\Omega} f(x, u)u \, dx + \int_{\partial\Omega} g(x, u)u \, dS \right) - \left( \int_{\Omega} F(x, u) \, dx + \int_{\partial\Omega} G(x, u) \, dS \right)
\]

\[
\geq \left( \frac{1}{p} - C \right) \left( \int_{\Omega} f(x, u)u \, dx + \int_{\partial\Omega} g(x, u)u \, dS \right).
\]

This completes the proof. □

**Lemma 3.2.** There exists \(c > 0\) such that

\[
\|u_+\|_{W^{1,p}(\Omega)} \geq c \quad \text{for } u \in K_1, \\
\|u_-\|_{W^{1,p}(\Omega)} \geq c \quad \text{for } u \in K_2, \\
\|u_+\|_{W^{1,p}(\Omega)}, \|u_-\|_{W^{1,p}(\Omega)} \geq c \quad \text{for } u \in K_3.
\]

**Proof.** By the definition of \(K_i\), by (F3) and (G3), we have

\[
\|u_\pm\|_{W^{1,p}(\Omega)}^p = \int_{\Omega} f(x, u_\pm) u_\pm \, dx + \int_{\partial\Omega} g(x, u_\pm) u_\pm \, dS
\]

\[
\leq c(\|u_\pm\|_{L^q(\Omega)}^q + \|u_\pm\|_{L^r(\partial\Omega)}^r).
\]

Now the proof follows by the Sobolev immersion Theorem and by the Sobolev trace Theorem, as \(p < q, r\). □

**Lemma 3.3.** There exists \(c > 0\) such that \(\Phi(u) \geq c\|u\|_{W^{1,p}(\Omega)}^p\) for every \(u \in W^{1,p}(\Omega)\) such that \(\|u\|_{W^{1,p}(\Omega)} \leq c\).

**Proof.** By (F3), (G3) and the Sobolev immersions we have

\[
\Phi(u) = \frac{1}{p} \|u\|_{W^{1,p}(\Omega)}^p - \mathcal{F}(u) - \mathcal{G}(u)
\]

\[
\geq \frac{1}{p} \|u\|_{W^{1,p}(\Omega)}^p - c(\|u\|_{L^q(\Omega)}^q + \|u\|_{L^r(\partial\Omega)}^r)
\]

\[
\geq \frac{1}{p} \|u\|_{W^{1,p}(\Omega)}^p - c(\|u\|_{W^{1,p}(\Omega)}^q + \|u\|_{W^{1,p}(\Omega)}^r)
\]

\[
\geq c\|u\|_{W^{1,p}(\Omega)}^p,
\]

if \(\|u\|_{W^{1,p}(\Omega)}\) is small enough, as \(p < q, r\). □

The following lemma describes the properties of the manifolds \(M_i\).

**Lemma 3.4.** \(M_i\) is a \(C^{1,1}\) sub-manifold of \(W^{1,p}(\Omega)\) of co-dimension 1 (\(i = 1, 2\), 2 (\(i = 3\)) respectively. The sets \(K_i\) are complete. Moreover, for every \(u \in M_i\) we have the direct decomposition

\[
T_u W^{1,p}(\Omega) = T_u M_i \oplus \text{span}\{u_+, u_-\},
\]

where \(T_u M\) is the tangent space at \(u\) of the Banach manifold \(M\). Finally, the projection onto the first component in this decomposition is uniformly continuous on bounded sets of \(M_i\).
Proof. Let us denote
\[
\widetilde{M}_1 = \left\{ u \in W^{1,p}(\Omega) : \int_{\partial \Omega} u_+ dS > 0 \right\},
\]
\[
\widetilde{M}_2 = \left\{ u \in W^{1,p}(\Omega) : \int_{\partial \Omega} u_- dS > 0 \right\},
\]
\[
\widetilde{M}_3 = \widetilde{M}_1 \cap \widetilde{M}_2.
\]
Observe that \( \widetilde{M}_1 \subset \tilde{M}_i \).

By the Sobolev trace Theorem, the set \( \tilde{M}_i \) is open in \( W^{1,p}(\Omega) \), therefore it is enough to prove that \( M_i \) is a smooth sub-manifold of \( \tilde{M}_i \). In order to do this, we will construct a \( C^{1,1} \) function \( \varphi_i : \tilde{M}_i \rightarrow \mathbb{R}^d \) with \( d = 1 \) \( (i = 1,2) \), \( d = 2 \) \( (i = 3) \) respectively and \( M_i \) will be the inverse image of a regular value of \( \varphi_i \).

In fact, we define: For \( u \in \tilde{M}_1 \),
\[
\varphi_1(u) = ||u_+||^p_{W^{1,p}(\Omega)} - \langle F'(u), u_+ \rangle - \langle G'(u), u_+ \rangle.
\]
For \( u \in \tilde{M}_2 \),
\[
\varphi_2(u) = ||u_-||^p_{W^{1,p}(\Omega)} - \langle F'(u), u_- \rangle - \langle G'(u), u_- \rangle.
\]
For \( u \in \tilde{M}_3 \),
\[
\varphi_3(u) = (k_1(u), k_2(u)).
\]
Obviously, we have \( M_i = \varphi_i^{-1}(0) \). We need to show that 0 is a regular value for \( \varphi_i \).

To this end we compute, for \( u \in M_1 \),
\[
\langle \nabla \varphi_1(u), u_+ \rangle = p||u_+||^p_{W^{1,p}(\Omega)} - \int_{\Omega} f_u(x,u)u_+^2 + f(x,u)u_+ \, dx \]
\[
- \int_{\partial \Omega} g_u(x,u)u_+^2 + g(x,u)u_+ \, dS
\]
\[
= (p - 1) \int_{\Omega} f(x,u)u_+ \, dx - \int_{\Omega} f_u(x,u)u_+^2 \, dx
\]
\[
+ (p - 1) \int_{\partial \Omega} g(x,u)u_+ \, dS - \int_{\partial \Omega} g_u(x,u)u_+^2 \, dS.
\]
By (F3) and (G3) the last term is bounded by
\[
(p - 1 - c_1^{-1}) \int_{\Omega} f(x,u)u_+ \, dx + (p - 1 - k_1^{-1}) \int_{\partial \Omega} g(x,u)u_+ \, dS.
\]
Recall that \( c_1, k_1 < 1/(p - 1) \). Now, by Lemma \[3.1\] this is bounded by
\[
-c||u_+||^p_{W^{1,p}(\Omega)}
\]
which is strictly negative by Lemma \[3.2\]. Therefore, \( M_1 \) is a smooth sub-manifold of \( W^{1,p}(\Omega) \). The exact same argument applies to \( M_2 \).

Since trivially
\[
\langle \nabla \varphi_1(u), u_- \rangle = \langle \nabla \varphi_2(u), u_+ \rangle = 0
\]
for \( u \in M_3 \), the same conclusion holds for \( M_3 \).

To see that \( K_i \) is complete, let \( u_k \) be a Cauchy sequence in \( K_i \), then \( u_k \rightarrow u \) in \( W^{1,p}(\Omega) \). Moreover, \( (u_k)_\pm \rightarrow u_\pm \) in \( W^{1,p}(\Omega) \). Now it is easy to see, by Lemma \[3.2\] and by continuity that \( u \in K_i \).

Finally, by the first part of the proof we have the decomposition
\[
T_u W^{1,p}(\Omega) = T_u M_i \oplus \text{span}\{u_+, u_-\}.
\]
Now let \( v \in T_u W^{1,p}(\Omega) \) be a unit tangential vector, then \( v = v_1 + v_2 \) where \( v_i \) are given by

\[
v_2 = (\nabla \varphi_i(u)|\text{span}(u_+,u_-))^{-1}(\nabla \varphi_i(u), v) \in \text{span}\{u_+, u_-\},
\]

\[
v_1 = v - v_2 \in T_u M_i,
\]

From these formulas and from the estimates given in the first part of the proof, the uniform continuity follows. □

Now, we need to check the Palais-Smale condition for the functional \( \Phi \) restricted to the manifold \( M_i \).

Lemma 3.5. The functional \( \Phi|_{K_i} \) satisfies the Palais-Smale condition.

Proof. Let \( \{u_k\} \subset K_i \) be a Palais-Smale sequence, that is \( \Phi(u_k) \) is uniformly bounded and \( \nabla \Phi|_{K_i}(u_k) \to 0 \) strongly. We need to show that there exists a subsequence \( u_{k_j} \) that converges strongly in \( K_i \).

Let \( v_j \in T_{u_j} W^{1,p}(\Omega) \) be a unit tangential vector such that

\[
\langle \nabla \Phi(u_j), v_j \rangle = \langle \nabla \Phi(u_j), \|\nabla \Phi(u_j)\|_{W^{1,p}(\Omega)} \rangle.
\]

Now, by Lemma 3.4, \( v_j = w_j + z_j \) with \( w_j \in T_{u_j} M_i \) and \( z_j \in \text{span}\{(u_j)_+, (u_j)_-\} \).

Since \( \Phi(u_j) \) is uniformly bounded, by Lemma 3.1, \( u_j \) is uniformly bounded in \( W^{1,p}(\Omega) \) and hence \( w_j \) is uniformly bounded in \( W^{1,p}(\Omega) \). Therefore

\[
\|\Phi(u_j)\|_{W^{1,p}(\Omega)} = \langle \nabla \Phi(u_j), v_j \rangle = \langle \nabla \Phi|_{K_i}(u_j), v_j \rangle \to 0.
\]

As \( u_j \) is bounded in \( W^{1,p}(\Omega) \), there exists \( u \in W^{1,p}(\Omega) \) such that \( u_j \rightharpoonup u \), weakly in \( W^{1,p}(\Omega) \). As it is well known that the unrestricted functional \( \Phi \) satisfies the Palais-Smale condition (cf. [9] and [13]), the lemma follows. See [15] for the details. □

We obtain immediately the following result.

Lemma 3.6. Let \( u \in K_i \) be a critical point of the restricted functional \( \Phi|_{K_i} \). Then \( u \) is also a critical point of the unrestricted functional \( \Phi \) and hence a weak solution to (1.1).

With all this preparatives, the proof of the Theorem follows easily.

Proof of the Theorem. The proof now is a standard application of the Lusternik–Schnirelman method for non-compact manifolds. See [14]. □

References


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