ON THE $\psi$-DICHOTOMY FOR HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this article we present some conditions for the $\psi$-dichotomy of the homogeneous linear differential equation $x' = A(t)x$. Under our condition every $\psi$-integrally bounded function $f$ the nonhomogeneous linear differential equation $x' = A(t)x + f(t)$ has at least one $\psi$-bounded solution on $(0, +\infty)$.

1. Introduction

The problem of solutions being $\psi$-bounded and $\psi$-stable for systems of ordinary differential equations has been studied by many authors; see for example Akinyele [1], Avramescu [2], Constantin [3]. In particular, Diamandescu [6, 7] presented some necessary and sufficient conditions for existence of a $\psi$-bounded solution to the linear nonhomogeneous system $x' = A(t)x + f(t)$.

Denote by $R^d$ the $d$-dimensional Euclidean space. Elements in this space are denoted by $x = (x_1, x_2, \ldots, x_d)^T$ and their norm by $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_d|\}$. For real $d \times d$ matrices, we define norm $|A| = \sup_{\|x\| \leq 1} \|Ax\|$.

Let $R^+ = [0, +\infty)$ and $\psi_i : R^+ \to (0, \infty)$, $i = 1, 2, \ldots, d$ be continuous functions. Set

$$\psi = \text{diag}[\psi_1, \psi_2, \ldots, \psi_d].$$

Definition 1.1 ([6]). A function $f : R^+ \to R^d$ is said to be

- $\psi$-bounded on $R^+$ if $\psi(t)f(t)$ is bounded on $R^+$,
- $\psi$-integrable on $R^+$ if $f(t)$ is measurable and $\psi(t)f(t)$ is Lebesgue integrable on $R^+$.

In $R^d$, consider the following equations

$$x' = A(t)x + f(t) \tag{1.1}$$

$$x' = A(t)x \tag{1.2}$$

where $A(t)$ is continuous matrix on $R^+$.

By solution of (1.1), (1.2), we mean an absolutely continuous function satisfying the system for all $t \in R^+$. Let $Y(t)$ be fundamental matrix of (1.2) with $Y(0) = I_d$, the identity $d \times d$ matrix. By $X_1$ denote the subspace of $R^d$ consisting of the initial values of all $\psi$-bounded solutions of equation (1.2) and let $X_2$ be the closed subspace...
Definition 1.2. The equation (1.2) is said to have a $\psi$-exponential dichotomy if there exist positive constants $K, L, \alpha, \beta$ such that
\[
|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \leq Ke^{-\alpha(t-s)} \quad \text{for} \quad 0 \leq s \leq t,
\]
\[
|\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)| \leq Ke^{\beta(t-s)} \quad \text{for} \quad 0 \leq t \leq s.
\]
The equation (1.2) is said to have a $\psi$-ordinary dichotomy if (1.3), (1.4) hold with $\alpha = \beta = 0$.

We say that (1.2) has $\psi$-bounded grow if for some fixed $h > 0$ there exists a constant $C \geq 1$ such that every solution $x(t)$ of (1.2) is satisfied
\[
\|\psi(t)x(t)\| \leq C\|\psi(s)x(s)\| \quad \text{for} \quad 0 \leq s \leq t \leq s + h.
\]

Remark 1.3. For $\psi_i = 1$, $i = 1, 2, \ldots, d$, we obtain the notion exponential and ordinary dichotomy [4,5].

Diamandescu proved the following results.

Theorem 1.4 (6). The equation (1.1) has at least one $\psi$-bounded solution on $\mathbb{R}_+$ for every $\psi$-integrable function $f$ on $\mathbb{R}_+$ if and only if (1.2) has a $\psi$-ordinary dichotomy.

Theorem 1.5 (8). Let
\[
|\psi(t)A(t)\psi^{-1}(t)| \leq M \quad \text{for all} \quad t \geq 0,
\]
\[
|\psi(t)\psi^{-1}(s)| \leq L \quad \text{for} \quad 0 \leq s \leq t.
\]
Then (1.1) has at least one $\psi$-bounded solution on $\mathbb{R}_+$ for every $\psi$-bounded function $f$ on $\mathbb{R}_+$ if and only if (1.2) has $\psi$-exponential dichotomy.

In this paper we prove some condition of the $\psi$-dichotomy for a homogeneous linear differential equations and we concerted that with the preceding results. Finally, it is noted that the concept of $\psi$-dichotomy for linear differential equations remain valid in Banach spaces. In this case we need a few changes for the definition of $\psi$. It seems to us that the majority of the results of this paper remain true for Banach spaces.

2. Preliminaries

Lemma 2.1. The equation (1.2) has a $\psi$-exponential dichotomy if there exist positive constants $K', L', T, \alpha, \beta$ such that
\[
|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \leq K'e^{-\alpha(t-s)}, \quad \text{for} \quad T \leq s \leq t
\]
\[
|\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)| \leq L'e^{\beta(t-s)}, \quad \text{for} \quad T \leq t \leq s.
\]

Proof. We will show that (1.3) holds. Using a lemma of Coppel [4],
\[
|Y^{-1}(s)| \leq (2^d - 1)\frac{|Y(s)|^{d-1}}{|\det Y(s)|}.
\]

On the other hand $Y(s)$ is continuous, we deduce $|Y^{-1}(s)| \leq N_1 < +\infty$ for $0 \leq s \leq T$. It follows from the continuity of $\psi(t), \psi^{-1}(t), Y(t)$, that $|\psi(t)|, |\psi^{-1}(t)|, |Y(t)|$ are
bounded on \([0, T]\). Thus \(|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \leq N < +\infty\) for \(0 \leq s \leq T\), 
\(0 \leq t \leq T\). If \(0 \leq s \leq T \leq t\), then
\[
|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \\
\leq |\psi(t)Y(t)P_1Y^{-1}(t)\psi^{-1}(T)||\psi(T)Y(T)Y^{-1}(s)\psi^{-1}(s)| \\
\leq N|\psi(t)Y(t)P_1Y^{-1}(T)\psi^{-1}(T)| \\
\leq NK'e^{-\alpha(t-T)} \leq NK'e^\alpha e^{-\alpha(t-s)}.
\]
If \(0 \leq s \leq t \leq T\), then
\[
|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \\
\leq |\psi(t)Y(t)P_1Y^{-1}(T)\psi^{-1}(T)||\psi(T)Y(T)Y^{-1}(s)\psi^{-1}(s)| \\
\leq N^2K' \leq N^2K'e^\alpha e^{-\alpha(t-s)}.
\]
Thus the inequality \(1.3\) holds for \(K = \max\{K', NK'e^\alpha T, N^2K'e^\alpha T\}\). Similarly, 
inequality \(1.4\) holds for \(L = \max\{L', NL'e^\alpha T, N^2L'e^\alpha T\}\). \(\square\)

**Lemma 2.2.** Equation \(1.2\) has a \(\psi\)-exponential dichotomy if only if following statements are satisfied
\[
\|\psi(t)Y(t)P_1\xi\| \leq K'e^{-\alpha(t-s)}\|\psi(s)Y(s)P_1\xi\|, \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } t \geq s \geq 0 \quad (2.3) \\
\|\psi(t)Y(t)P_2\xi\| \leq L'e^{\beta(t-s)}\|\psi(s)Y(s)P_2\xi\|, \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } s \geq t \geq 0 \quad (2.4) \\
|\psi(t)Y(t)P_1Y^{-1}(t)\psi^{-1}(t)| \leq M \quad \text{for } t \geq 0 \quad (2.5)
\]
where \(K', L', M\) are positive constants.

**Proof.** If \(1.2\) has a \(\psi\)-exponential dichotomy then for any vector \(y \in \mathbb{R}^d\), we get
\[
\|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)y\| \leq K'e^{-\alpha(t-s)}\|y\| \quad \text{for } 0 \leq s \leq t.
\]
Choose \(y = \psi(s)Y(s)P_1\xi\), we obtain \(2.3\). The proof of \(2.2\) is similar. Inequality \(2.5\) evidently holds. Conversely, if inequality \(2.3), (2.4), (2.5)\) are true. For any vector \(y \in \mathbb{R}^d\), putting \(\xi = Y^{-1}(s)\psi^{-1}(s)y\) we get
\[
\|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)y\| \leq K'e^{-\alpha(t-s)}\|\psi(s)Y(s)P_1Y^{-1}(s)\psi^{-1}(s)y\| \\
\leq MK'e^{-\alpha(t-s)}\|y\| \quad \text{for } t \geq s \geq 0.
\]
Thus, we have \(1.3\). The proof of \(1.4\) is similar. \(\square\)

**Remark 2.3.** By Lemma 2.2 and in the same way as in the proof of Lemma 2.2, we can show that \(1.2\) has \(\psi\)-exponential dichotomy if there exists positive constant \(Q\) such that
\[
\|\psi(t)Y(t)P_1\xi\| \leq K'e^{-\alpha(t-s)}\|\psi(s)Y(s)P_1\xi\|, \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } t \geq s \geq Q, \quad (2.6) \\
\|\psi(t)Y(t)P_2\xi\| \leq L'e^{\beta(t-s)}\|\psi(s)Y(s)P_2\xi\|, \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } s \geq t \geq Q, \quad (2.7) \\
|\psi(t)Y(t)P_1Y^{-1}(t)\psi^{-1}(t)| \leq M \quad \text{for } t \geq Q. \quad (2.8)
\]

**Lemma 2.4.** Equation \(1.2\) has \(\psi\)-bounded grow if and only if there exist positive constants \(K, \gamma\) such that
\[
|\psi(t)Y(t)Y^{-1}(s)\psi^{-1}(s)| \leq Ke^{\gamma(t-s)}, \quad \text{for } t \geq s \geq 0. \quad (2.9)
\]
Thus for consider the solution $x(t)$ of \[1.2\], with $x(0) = Y^{-1}(s)\psi^{-1}(s)\xi$. Setting $n = \left[\frac{-s}{\theta}\right]$, we get
\[
\|\psi(t)x(t)\| = \|\psi(nh + s)x(nh + s)\|
\leq C\|\psi(nh + s - h)x(nh + s - h)\|
\leq \cdots \leq C^n\|\psi(s)x(s)\|
\leq C\frac{e^{nh}}{\theta^n} \|\psi(s)x(s)\| \text{ for } 0 \leq s \leq t.
\]
Set $K = C$, $\gamma = h^{-1} \ln C$, we obtain
\[
\|\psi(t)x(t)\| \leq Ke^{\gamma(t-s)} \|\psi(s)x(s)\|.
\]
Therefore, $\|\psi(t)Y(t)Y^{-1}(t)\psi^{-1}(s)\xi\| \leq Ke^{\gamma(t-s)} \|\psi(s)\xi\|$. It follows that \[2.9\] Conversely, if \[2.9\] is true, then we can take $C = Ke^{\gamma h}$. Thus \[1.5\] is satisfied.

**Remark 2.5.** The preceding proof shows that the condition of $\psi$-bounded grow of \[1.2\] is independent of the choice of $h$.

3. THE MAIN RESULTS

**Theorem 3.1.** If \[1.2\] has a $\psi$-exponential dichotomy, then for any $0 < \theta < 1$ there exists constants $T > 0$ such that every solution $x(t)$ of \[1.2\] satisfies
\[
\|\psi(t)x(t)\| \leq \theta \sup_{\|s-t\|<T} \|\psi(s)x(s)\| \text{ for all } t \geq T. \tag{3.1}
\]

**Proof.** Set $x_1(t) = Y(t)P_1Y^{-1}(t)x(t)$, $x_2(t) = Y(t)P_2Y^{-1}(t)x(t)$. Suppose that
\[
\|\psi(s)x_2(s)\| \geq \|\psi(s)x_1(s)\|.
\]
It follows from \[2.3\] that
\[
\|\psi(s)x_1(s)\| \leq Ke^{-\alpha(t-s)}\|\psi(s)x_1(s)\| \leq Ke^{-\alpha(t-s)}\|\psi(s)x_2(s)\| \text{ for } 0 \leq s \leq t.
\]
Applying \[2.4\] for $\xi = Y^{-1}(s)x_2(s)$,
\[
\|\psi(t)x_2(t)\| = \|\psi(t)Y(t)P_2Y^{-1}(s)x_2(s)\|
\geq L^{-1}e^{\beta(t-s)}\|\psi(s)Y(s)P_2Y^{-1}(s)x_2(s)\| \text{ for } 0 \leq s \leq t.
\]
Note that $x_2(t) = Y(t)P_2Y^{-1}(t)x_2(t)$. Thus
\[
\|\psi(t)x_2(t)\| \geq L^{-1}e^{\beta(t-s)}\|\psi(s)x_2(s)\| \text{ for } 0 \leq s \leq t.
\]
Therefore,
\[
\|\psi(t)x(t)\| \geq \frac{1}{2}[L^{-1}e^{\beta(t-s)} - Ke^{-\alpha(t-s)}]\|\psi(s)x(s)\| \text{ for } 0 \leq s \leq t.
\]
Similarly, if $\|\psi(s)x_1(s)\| \geq \|\psi(s)x_2(s)\|$, then
\[
\|\psi(t)x(t)\| \geq \frac{1}{2}[K^{-1}e^{\alpha(t-s)} - L'e^{-\beta(t-s)}]\|\psi(s)x(s)\| \text{ for } 0 \leq t \leq s.
\]
For any $0 < \theta < 1$ we can choose $T > 0$ large so that $L^{-1}e^{\beta T} - Ke^{-\alpha T} \geq 2\theta^{-1}$ and $K^{-1}e^{\alpha T} - L'e^{-\beta T} \geq 2\theta^{-1}$.

Thus for $t \geq T$,
\[
\|\psi(t)x(t)\| \leq \max\{\theta\|\psi(t)x(t+T)\|, \theta\|\psi(t)x(t-T)\|\}.
\]
Then (3.1) is satisfied. □

**Definition 3.2.** The function \( f : \mathbb{R}_+ \to \mathbb{R}^d \) is said to be \( \psi \)-integrally bounded if it is measurable and Lebesgue integrals \( \int_{t}^{t+1} \| \psi(u)f(u) \| du \) are uniformly bounded for any \( t \in \mathbb{R}_+ \).

**Theorem 3.3.** Equation (1.1) has at least one \( \psi \)-bounded solution on \( \mathbb{R}_+ \) for every \( \psi \)-integrally bounded function \( f \) if and only if (1.2) has a \( \psi \)-exponential dichotomy.

**Proof.** First we prove the “if” part. Suppose that (1.2) has a \( \psi \)-exponential dichotomy. Consider the function

\[
\tilde{x}(t) = \int_{0}^{t} \psi(t)Y(t)P_2Y^{-1}(s)f(s)ds - \int_{t}^{\infty} \psi(t)Y(t)P_2Y^{-1}(s)f(s)ds \\
= \int_{0}^{t} \psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)\psi(s)f(s)ds \\
- \int_{t}^{\infty} \psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)\psi(s)f(s)ds
\]

for \( t \geq 0 \). The function \( \tilde{x}(t) \) is bounded. In fact, suppose that

\[
\int_{t}^{t+1} \| \psi(s)f(s) \| ds \leq c \quad \text{for} \quad t \geq 0.
\]

Then

\[
\int_{0}^{t} e^{-\alpha(t-s)}\| \psi(s)f(s) \| ds \leq c(1 - e^{-\alpha})^{-1},
\]

\[
\int_{t}^{\infty} e^{\beta(t-s)}\| \psi(s)f(s) \| ds \leq c(1 - e^{-\beta})^{-1},
\]

by using a Lemma in Massera and Schaffer. Set

\[
x(t) = \psi^{-1}(t)\tilde{x}(t) = \int_{0}^{t} Y(t)P_2Y^{-1}(s)f(s)ds - \int_{t}^{\infty} Y(t)P_2Y^{-1}(s)f(s)ds.
\]

Then \( x(t) \) is the \( \psi \)-bounded and continuous function on \( \mathbb{R}_+ \).

\[
x'(t) = A(t)\left[ \int_{0}^{t} Y(t)P_2Y^{-1}(s)f(s)ds - \int_{t}^{\infty} Y(t)P_2Y^{-1}(s)f(s)ds \right] \\
+ Y(t)P_2Y^{-1}(t)f(t) + Y(t)P_2Y^{-1}(t)f(t) \\
= A(t)x(t) + f(t).
\]

It follows that \( x(t) \) is a solution of (1.1).

Now, we prove the “only part”. We define the set

\[
C_\psi = \{ x : \mathbb{R}_+ \to \mathbb{R}^d ; x \text{ is } \psi \text{-bounded and continuous on } \mathbb{R}_+ \}.
\]

It is well-known that \( C_\psi \) is real Banach space with the norm

\[
\| x \|_{C_\psi} = \sup_{t \geq 0} \| \psi(t)x(t) \|.
\]

First we show that (1.1) has a unique \( \psi \)-bounded solution \( x(t) \) with \( x(0) \in X_2 \) for each \( f \in C_\psi \). Further, there exists a positive constant \( r \) independent of \( f \) such that

\[
\| x \|_{C_\psi} \leq r \| f \|_{C_\psi}.
\]
We prove the existence. Suppose \( f \in C_{\psi} \). By hypothesis, there exists a \( \psi \)-bounded solution \( x(t) \) of (1.1). We denote by \( y(t) \) the solution of the Cauchy problem

\[
y' = A(t)y; \quad y(0) = -P_1x(0).
\]

This solution \( y(t) \) is \( \psi \)-bounded by definition of the subset \( X_1 \). But then \( z = x + y \) is a \( \psi \)-bounded solution of (1.1) for which

\[
P_1z(0) = P_1x(0) - P_1^2x(0) = 0.
\]

Thus \( z(0) \in X_2 \). Hence \( z(t) \) is a \( \psi \)-bounded solution of (1.1) with \( z(0) \in X_2 \).

We prove the uniqueness. Let \( x(t) \) and \( y(t) \) be the \( \psi \)-bounded solutions of equation (1.1) with \( x(0) \in X_2, y(0) \in X_2 \). Hence \( x - y \) is a \( \psi \)-bounded of (1.2) and \( x(0) - y(0) \in X_2 \). But \( x(0) - y(0) \in X_1 \), we obtain \( x(0) = y(0) \), hence \( x = y \).

We prove the inequality (3.2). Consider the map \( T : c_{\psi} \to c_{\psi} \) which is defined \( Tf \) and where \( x(0) \in X_2 \). We will show that \( T \) is continuous. Suppose that \( x_n = Tf_n, f_n \to f \) and \( x_n \to x \). For any fixed \( t \), we have

\[
\lim_{n \to \infty} \int_0^t |f_n(s) - f(s)|ds \leq \lim_{n \to \infty} \int_0^t \psi^{-1}(s)\|\psi(s)f_n(s) - \psi(s)f(s)\|ds
\]

\[
\leq \lim_{n \to \infty} \|f_n - f\|_{c_{\psi}} \int_0^t \psi^{-1}(s)ds = 0.
\]  

(3.3)

On the other hand

\[
\lim_{n \to \infty} \int_0^t A(s)x_n(s) - x(s))ds
\]

\[
\leq \lim_{n \to \infty} \int_0^t |A(s)\psi^{-1}(s)||\psi(s)x_n(s) - \psi(s)x(s)||ds
\]

\[
\leq \lim_{n \to \infty} \|x_n - x\|_{c_{\psi}} \int_0^t |A(s)\psi^{-1}(s)ds = 0.
\]  

(3.4)

From (3.3) and (3.4) we obtain

\[
x(t) - x(0) = \lim_{n \to \infty} (x_n(t) - x_n(0))
\]

\[
= \lim_{n \to \infty} \int_0^t [A(s)x_n(s) + x_n'(t) - A(s)x_n(s)ds
\]

\[
= \lim_{n \to \infty} \int_0^t [A(s)x_n(s) + f_n(s)ds \int_0^t |A(s)x(s) + f(s)|ds.
\]

Thus \( x(t) \) is a solution of (1.1). Since \( x(t) \) is \( \psi \)-bounded and

\[
x(0) = \lim_{n \to \infty} x_n(0) \in X_2
\]

we have \( x = Tf \). It follows from the Closed Graph Theorem that the linear map \( T \) is continuous. Hence (3.2) is proved. Now, put

\[
G(t, s) = \begin{cases} 
Y(t)P_1Y^{-1}(s) & \text{for } 0 \leq s \leq t \\
-Y(t)P_2Y^{-1}(s) & \text{for } 0 \leq t \leq s.
\end{cases}
\]

If \( \tilde{f} \in C_{\psi}, \tilde{f}(t) = 0 \) for \( t > t_1 > 0 \), then

\[
\tilde{x}(t) = \int_0^{t_1} G(t, s)\tilde{f}(s)ds
\]  

(3.5)
is a solution of (1.1). Moreover $\tilde{x} \in C_\psi$, since
\[
\psi(t)\tilde{x}(t) = \int_0^{t_1} \psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)\psi(s)\tilde{f}(s)ds \quad \text{for } t \geq t_1.
\]
On the other hand, $\tilde{x}(0) = -P_2 \int_0^{t_1} Y^{-1}(s)\tilde{f}(s)ds \in X_2$. Thus
\[
\|\tilde{x}\|_{c_\psi} \leq r\|\tilde{f}\|_{c_\psi}.
\]
Let $x$ be a nontrivial solution of (1.2) and let $\alpha(t)$ be any continuous real-valued function such that $0 \leq \alpha(t) \leq 1$ for all $t \geq 0$, $\alpha(t) = 0$ for $t \geq t_2$, $\alpha(t) = 1$ for $0 \leq t_0 \leq t \leq t_1 \leq t_2$. Set
\[
\tilde{f}(t) = \alpha(t)x(t)\|\psi(t)x(t)\|^{-1}.
\]
Then $\tilde{f} \in C_\psi$. From (3.5) and (3.6), we have
\[
\| \int_{t_0}^{t_1} \psi(t)G(t,s)x(s)\|\psi(s)x(s)\|^{-1}ds\|_{c_\psi} = r \quad \text{for } t_1 \geq t_0 \geq 0.
\]
By continuity, (3.7) remains true also in the case $t = s$. Choose $x(0) = P_1\xi, \xi \in \mathbb{R}^d$. By the arbitrary of $t_1$, from (3.7) we get
\[
\|\psi(t)Y(t)P_1\xi\| \int_{t_0}^{t} \|\psi(u)Y(u)P_1\xi\|^{-1}du \leq r \quad \text{for } t_0 \geq 0.
\]
Choose $x(0) = P_2\xi, \xi \in \mathbb{R}^d$. By the arbitrary of $t_0$, from (3.7) we get
\[
\|\psi(t)Y(t)P_2\xi\| \int_{t}^{t_1} \|\psi(u)Y(u)P_2\xi\|^{-1}du \leq r \quad \text{for } 0 \leq t \leq t_1.
\]
Next, putting $x_1(t) = Y(t)P_1Y^{-1}(s)x(s) = Y(t)P_1\xi$, we have
\[
\|\psi(t)x_1(t)\| \int_{t_0}^{t} \|\psi(u)x_1(u)\|^{-1}du \leq r \quad \text{for } t \geq t_0 \geq 0.
\]
Also putting $x_2(t) = Y(t)P_2Y^{-1}(s)x(s) = Y(t)P_2\xi$, we get
\[
\|\psi(t)x_2(t)\| \int_{t}^{t_1} \|\psi(u)x_2(u)\|^{-1}du \leq r \quad \text{for } t_1 \geq t \geq 0.
\]
It follows by integration that
\[
\int_{t_0}^{t} \|\psi(u)x_1(u)\|^{-1}du \leq e^{-r^{-1}(t-s)} \int_{t_0}^{t} \|\psi(u)x_1(u)\|^{-1}du \quad \text{for } t \geq s \geq t_0.
\]
\[
\int_{s}^{t_1} \|\psi(u)x_2(u)\|^{-1}du \leq e^{-r^{-1}(s-t)} \int_{t}^{t_1} \|\psi(u)x_2(u)\|^{-1}du \quad \text{for } t_3 \geq s \geq t.
\]
Because a $\psi$-integrable function is $\psi$-locally integrable, by Theorem 1.4 there exists a positive constant $K$ such that
\[
\|\psi(t)x_1(t)\| \leq K\|\psi(s)x(s)\| \quad \text{for } 0 \leq s \leq t,
\]
\[
\|\psi(t)x_2(t)\| \leq K\|\psi(s)x(s)\| \quad \text{for } 0 \leq t \leq s.
\]
Thus
\[
rK^{-1}\|\psi(s)x(s)\|^{-1} \leq \int_{s}^{t+s} \|\psi(u)x_1(u)\|^{-1}du \quad \text{for } s \geq 0.
\]
Using (3.10), replacing \( t_0 \) by \( s \), \( s \) by \( s + r \) we deduce

\[
\int_s^{s+r} \| \psi(u)x_1(u) \|^{-1} du \leq e^{-r^{-1}(t-r-s)} \int_s^t \| \psi(u)x_1(u) \|^{-1} du \\
\leq ee^{-r^{-1}(t-s)} \int_s^t \| \psi(u)x_1(u) \|^{-1} du \quad \text{for } t \geq s + r.
\]

Hence

\[
r \left( \int_s^t \| \psi(u)x_1(u) \|^{-1} du \right) \leq eK \| \psi(s)x(s) \| e^{-r^{-1}(t-s)} \quad \text{for } t \geq s + r.
\]

From (3.8), replacing \( t_0 \) by \( s \), \( s \) by \( s + r \), we get

\[
\| \psi(t)x_1(t) \| \leq eK \| \psi(s)x(s) \| e^{-r^{-1}(t-s)} \quad \text{for } t \geq s + r.
\]

It is easy to see that the inequality holds also for \( s \leq t \leq s + r \). Since \( x_1(t) = Y(t)P_1Y^{-1}(s)x(s) \), it follows that

\[
\| \psi(t)Y(t)P_1Y^{-1}(s)x(t) \| \leq K'e^{-\alpha(t-s)} \quad \text{for } t \geq s \geq 0
\]

where \( K' = eK, \alpha = r^{-1} \). By the same way, using (3.9), (3.11), (3.13), we get

\[
\| \psi(t)Y(t)P_2Y^{-1}(s)x(t) \| \leq K'e^\alpha(s-t) \quad \text{for } s \geq t \geq 0.
\]

The proof is complete. \( \square \)

Now, we are going to show some conditions for (1.2) has a \( \psi \)-exponential dichotomy in the case it has \( \psi \)-bounded grow.

**Theorem 3.4.** Suppose that (1.2) has \( \psi \)-bounded grow. Equation (1.2) has a \( \psi \)-exponential dichotomy if there exists constants \( T > 0, 0 < \theta < 1 \) such that every solution of (1.2) satisfies (3.1).

**Proof.** By Remark 2.5 we shall show that (2.6), (2.7), (2.8) are satisfied for some \( Q > 0 \). We may consider \( x(t) \) is nontrivial solution of (1.2). The first we prove that every solution \( x(t) \) of (1.2) with \( x(0) \in X_1 \) satisfies

\[
\| \psi(t)x(t) \| \leq Ke^{-\alpha(t-s)} \| \psi(s)x(s) \| \quad \text{for } 0 \leq s \leq t.
\]

By Remark 2.5 we can choose \( h = T \), so that

\[
\| \psi(t)x(t) \| \leq C \| \psi(s)x(s) \| \quad \text{for } 0 \leq s \leq t \leq s + T. \quad (3.14)
\]

Hence \( \| \psi(t)x(t) \| \leq \theta \sup_{u \geq s} \| \psi(u)x(u) \| \) for \( s \geq 0 \), \( t \geq s + T \). Therefore,

\[
\sup_{u \geq s} \| \psi(u)x(u) \| > \| \psi(t)x(t) \|
\]

for \( t \geq s + T \). It follow that

\[
\sup_{u \geq s} \| \psi(u)x(u) \| = \sup_{s \leq \tau \leq s + T} \| \psi(\tau)x(\tau) \|. \quad (3.15)
\]
Hence (3.14) and (3.15) yield \( \| \psi(t)x(t) \| \leq C \| \psi(s)x(s) \| \) for \( 0 \leq s \leq t \). Set \( n = \left\lceil \frac{t-s}{\theta} \right\rceil \) then

\[
\| \psi(t)x(t) \| \\
\leq \theta \sup_{\|u-t\| \leq T} \| \psi(u)x(u) \|
\leq \theta \sup_{\|u-t\| \leq T} \{ \theta \sup_{\|u-v\| \leq T} \| \psi(v)x(v) \| \} \leq \theta^2 \sup_{\|v-t\| \leq 2T} \| \psi(v)x(v) \|
\leq \theta^n \sup_{\|v-t\| \leq nT} \| \psi(v)x(v) \| \leq \theta^n C \theta^\frac{t-s}{\theta} \| \psi(s)x(s) \|.
\]

Put \( K = \theta^{-1} C > 1 \), \( \alpha = -T^{-1} \ln \theta > 0 \), we get

\[
\| \psi(t)x(t) \| \leq K e^{-\alpha(t-s)} \| \psi(s)x(s) \| \quad \text{for} \quad 0 \leq s \leq t.
\]

Now, for each \( \xi \in \mathbb{R}^d \), consider the solution \( x(t) \) of the equation (1) with \( x(0) = P_1 \xi \).

Applying this inequality we deduce (2.6) for any \( Q \geq 0 \).

Now, suppose that \( x(t) \) is any solution of (1.2) with \( x(0) \in X_2 \).

May be consider \( \| \psi(0)x(0) \| = 1 \). We can define a sequence \( t_n \to +\infty \) by

\[
\| \psi(t_n)x(t_n) \| = \theta^{-n} C, \quad \| \psi(t_n)x(t_n) \| < \theta^{-n} C \quad \text{for} \quad 0 \leq t \leq t_n.
\]

Since \( \| \psi(t_n)x(t_n) \| \leq C \) for \( 0 \leq t \leq T \) and \( \| \psi(t_1)x(t_1) \| = C \theta^{-1} > C \) we get \( T < t_1 \).

Consequently,

\[
T < t_1 < t_2 < \cdots < t_n < \cdots.
\]

From

\[
\| \psi(t_n)x(t_n) \| \leq \theta \sup_{0 \leq u \leq t_n + T} \| \psi(u)x(u) \|
\]

and

\[
\| \psi(t_n)x(t_n) \| \leq \theta^{-1} \| \psi(t_n)x(t_n) \| \quad \text{for} \quad 0 \leq t \leq t_n
\]

we get \( t_{n+1} < t_n + T \). Suppose that \( 0 \leq s \leq t \) and \( t_m \leq t \leq t_{m+1}, t_n \leq s \leq t_{n+1} \) (\( 1 \leq m \leq n \)). Then

\[
\| \psi(t)x(t) \| < \theta^{-m-1} C = \theta^{-m} \| \psi(t_{n+1})x(t_{n+1}) \|
\leq C \theta^{-1} \theta^{n-m+1} \| \psi(s)x(s) \|
\leq C \theta^{-1} \theta^{\frac{n-s}{\theta}} \| \psi(s)x(s) \|.
\]

Thus \( \| \psi(t)x(t) \| \leq K e^{-\alpha(s-t)} \| \psi(s)x(s) \| \) for \( t_1 \leq t < s \).

For any unit vector \( \xi \in X_2 \), let \( x(t, \xi) \) be the solution of (1.2) with \( x(0)x(0) = \xi \).

Then \( x(t, \xi) \) is unbounded, and hence there is a value \( t = t_1(\xi) \) such that

\[
\| \psi(t_1)x(t_1) \| = \theta^{-1} C.
\]

We will show that the values \( t_1(\xi) \) are bounded. In fact, otherwise there exists a sequence of unit vector \( \xi_k \in X_2 \) such that \( t^k_1 = t_1(\xi_k) \to +\infty \) as \( k \to +\infty \).

By the compactness of the unit sphere in \( X_2 \) we may suppose that \( \xi_k \to \xi \) as \( k \to +\infty \), where \( \xi \) is a unit vector. Then \( x(t, \xi_k) \to x(t, \xi) \) for every \( t \geq 0 \). Since \( \| \psi(t)x(t, \xi_k) \| < \theta^{-1} C \) for \( 0 \leq t \geq t^k_1 \) and \( t^k_1 \to +\infty \) we get

\[
\| \psi(t)x(t, \xi) \| \leq \theta^{-1} C \quad \text{for} \quad t \geq 0
\]

which is a contradiction because \( \xi \in X_2 \). Thus there exists \( Q > 0 \) such that \( t_1(\xi) \) for all unit vector \( \xi \) and every solution \( x(t) \) of equation (1.2) with \( x(0) \in X_2 \) satisfies

\[
\| \psi(t)x(t) \| \leq K e^{-\alpha(s-t)} \| \psi(s)x(s) \| \quad \text{for} \quad Q < t \leq s.
\]
Thus \(|\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)| \leq Le^{|t-s|}\), for \(Q \leq t \leq s\). Thus (2.7) is proved. Note that (2.8) is proved in [8, Theorem 2.1, estimate (12)]. So the proof is complete. 

From Theorem 3.1 and Theorem 3.4, we have the following result.

**Corollary 3.5.** Suppose that (1.2) has \(\psi\)-bounded grow. Then equation (1.2) has a \(\psi\)-exponential dichotomy if and only if there exists constants \(T > 0\), \(0 < \theta < 1\) such that every solution of (1.2) is satisfied (3.1).

**Theorem 3.6.** Suppose that (1.2) has \(\psi\)-bounded grow. Then (1.1) has at least one \(\psi\)-bounded solution on \(\mathbb{R}_+\) for every \(\psi\)-bounded function \(f\) on \(\mathbb{R}_+\) if and only if (1.2) has \(\psi\)-exponential dichotomy.

**Proof.** Diamandescu presented this Theorem. In the proof [8, Theorem 1.2], the author proved that \(|\psi(t)A(t)\psi^{-1}(t)| \leq M\) for all \(t \geq 0\) and \(|\psi(t)\psi^{-1}(s)| \leq L\) for \(t \geq s \geq 0\) deduce (2.9). Throughout the proof, he only used condition (2.9). By lemma 2.4, condition (2.9) is satisfied if and only if (1.2) has \(\psi\)-bounded grow. The proof is complete.

Now, consider the perturbed equation

\[
x'(t) = [A(t) + B(t)]x(t)
\]

where \(B(t)\) is a \(d \times d\) continuous matrix function on \(\mathbb{R}_+\). We have the following result.

**Theorem 3.7.** (a) Suppose that (1.2) has a \(\psi\)-exponential dichotomy. If \(\delta = \sup_{t \geq 0} |\psi(t)B(t)\psi^{-1}(t)|\) is sufficiently small, then (3.16) has a \(\psi\)-exponential dichotomy.

(b) Suppose that (1.2) has a \(\psi\)-exponential dichotomy or \(\psi\)-ordinary dichotomy. If \(\int_0^\infty |\psi(t)B(t)\psi^{-1}(t)|dt < \infty\), then (3.16) has a \(\psi\)-ordinary dichotomy.

**Proof.** (a) By Theorem 3.3 it suffices to show that the equation

\[
x'(t) = [A(t) + B(t)]x(t) + f(t)
\]

has at least a \(\psi\)-bounded solution for every \(\psi\)-integrally bounded \(f\) function. Denote \(Y(t), P_1, P_2\) as in the proof of the Theorem 3.3.

Consider the map \(T : C_\psi \to C_\psi\) which is defined by

\[
Tz(t) = \int_0^t Y(t)P_1Y^{-1}(s)[B(s)z(s) + f(s)]ds
- \int_t^\infty Y(t)P_2Y^{-1}(s)[B(s)z(s) + f(s)]ds.
\]
It is easy verified that \( Tz \in C_\psi \). More ever if \( z_1, z_2 \in C_\psi \) then
\[
\|Tz_1 - Tz_2\| \\
\leq \int_0^t |\psi(t)Y(t)P_1 Y^{-1}(s)\psi^{-1}(s)||\psi(s)B(s)\psi^{-1}(s)||\psi(s)z_1(s) - \psi(s)z_2(s)|\,ds \\
+ \int_t^\infty |\psi(t)Y(t)P_2 Y^{-1}(s)\psi^{-1}(s)||\psi(s)B(s)\psi^{-1}(s)||\psi(s)z_1(s) - \psi(s)z_2(s)|\,ds \\
\leq K\delta\|z_1 - z_2\|C_\psi + \int_0^t e^{-\alpha(t-s)}\,ds + L\delta\|z_1 - z_2\|C_\psi \int_t^\infty e^{\beta(t-s)}\,ds \\
\leq \delta(K\alpha^{-1} + L\beta^{-1})\|z_1 - z_2\|C_\psi.
\]

Hence, by the contraction principle, if \( \delta(K\alpha^{-1} + L\beta^{-1}) < 1 \), then the mapping \( T \) has a unique fixed point. Denoting this fixed point by \( z \), we have
\[
z(t) = \int_0^t Y(t)P_1 Y^{-1}(s)|B(s)z(s) + f(s)|\,ds - \int_t^\infty Y(t)P_2 Y^{-1}(s)|B(s)z(s) + f(s)|\,ds.
\]

It follows that \( z(t) \) is a solution of \((3.17)\).

(b) We can assume that \((1.2)\) has a \( \psi \)-ordinary dichotomy. By Theorem 1.4 it suffices to show that \((3.17)\) has at least a \( \psi \)-bounded solution for every \( \psi \)-integrable \( f \). From \( \int_0^\infty |\psi(t)B(t)\psi^{-1}(t)|\,dt < \infty \), it follows that
\[
k = K\int_T^\infty |\psi(t)B(t)\psi^{-1}(t)|\,dt < 1
\]
for a sufficiently large and positive \( T \). Let \( C_{T,\psi} \) be the Banach space of all \( \psi \)-bounded and continuous functions \( z(t) \) on \([T, \infty)\) equipped with the norm
\[
\|z\|_{C_{T,\psi}} = \sup_{t \geq T} \|\psi(t)z(t)\|.
\]

Consider the map \( T : C_{T,\psi} \to C_{T,\psi} \) which is defined by
\[
Tz(t) = \int_T^t Y(t)P_1 Y^{-1}(s)|B(s)z(s) + f(s)|\,ds - \int_t^\infty Y(t)P_2 Y^{-1}(s)|B(s)z(s) + f(s)|\,ds.
\]

It is easy to check that \( Tz \in C_{T,\psi} \). Moreover if \( z_1, z_2 \in C_{T,\psi} \) then
\[
\|Tz_1 - Tz_2\|_{C_{T,\psi}} \leq K\int_T^\infty |\psi(s)B(s)\psi^{-1}(s)||\psi(s)z_1(s) - \psi(s)z_2(s)|\,ds \\
\leq k\|z_1 - z_2\|_{C_{T,\psi}}.
\]

It follows from the contraction principle that the equation \( Tz = z \) has a unique solution \( \tilde{z} \in C_{T,\psi} \). Denote by \( y \) the solution of \((3.16)\), which is extension of \( \tilde{z} \) on \( \mathbb{R}_+ \). Clearly \( y \) is a \( \psi \)-bounded solution of \((3.16)\). The proof is complete. \( \square \)

We remark that \((1.2)\) has a \( \psi \)-ordinary dichotomy with \( P_1 = I_d \) if and only if it is \( \psi \)-uniformly stable. Theorem 3.7 follows [7, Theorem 3.4].

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References


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