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MULTIPLE POSITIVE SOLUTIONS FOR A CLASS OF NONRESONANT SINGULAR BOUNDARY-VALUE PROBLEMS

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Abstract. Using a specially constructed cone and the fixed point index theory, this paper shows the existence of multiple positive solutions for a class of nonresonant singular boundary-value problem of second-order differential equations. The nonexistence of positive solution is also studied.

1. Introduction

The theory of singular boundary-value problems (BVP, for short) has become an important area of investigation in previous years; see [1, 3, 5, 6, 7, 8, 9, 10, 11, 12] and references therein. We consider the nonresonant singular boundary-value problem of second-order differential equations

$$\begin{gathered} -u''(t)+\rho p(t)u(t)=\lambda f(t,u(t)), \quad t\in(0,1);\\ u(0)=u(1)=0, \end{gathered} \tag{1.1}$$

where $\rho>0$ and

$$\begin{gathered} -u''(t)+\rho p(t)u(t)=0, \quad t\in(0,1);\\ u(0)=u(1)=0 \end{gathered}$$

has only the trivial solution. Here the parameter λ belongs to $\mathbb{R}^+=[0,+\infty)$, p belongs to $C[(0,1),\mathbb{R}^+]$ with $\int_0^1 p(t)dt<+\infty$, and f belongs to $C[(0,1)\times(0,+\infty),\mathbb{R}^+]$; that is, $f(t,u)$ may be singular at $t=0,1$, and $u=0$.

In the special cases i) $p(t)=0$, $f(t,u)=p_1(t)u^{-\lambda_1}$, $\lambda_1>0$, and ii) $p(t)=0$, $f(t,u)=p_1(t)u^{\lambda_1}$, $0<\lambda_1<1$, where $p_1(t)>0$ for $t\in(0,1)$, the existence and uniqueness of positive solutions of (1.1) as $\lambda=1$ have been studied completely by Taliaferro in [8] with the shooting method and by Zhang [12] with the method of lower and upper solutions, respectively. Also a sufficient condition for the existence of $C[0,1]$ solutions of the singular problem (1.1) with $\lambda=1$ was given by O'Regan in [7] by using a continuous theorem. In the special case iii): $f(t,u)$ is quasi-homogeneous and sublinear in u, the existence of positive solutions (1.1) as $\lambda=1$ have been studied by Wei and Pang in [9, 10] with the method of lower and upper

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solutions. In the special case iv): $p(t) = 0$, $f(t,u) = p_1(t)g(u)$, $p_1(t)$ is singular only at $t = 0$ and $g(u) \geq e^u$, the existence of multiple positive solutions of (1.1) have been studied by Ha and Lee in [3] with the method of lower and upper solutions.

In the present paper, we shall construct a special cone and use fixed point index theory to investigate the existence of multiple positive solutions for (1.1), which is different from [3, 7, 8, 9, 10, 12]. Meanwhile, some results on nonexistence of positive solutions are given.

The organization of this paper is as follows. We shall introduce some lemmas and notations in the rest of this section. The main result will be stated and proved in Section 2. Finally in Section 3 some examples are worked out to demonstrate our main results.

Now we present some lemmas and notation which will be used in Section 2. First from [2, Lemmas 2.3.1 and 2.3.3] we obtain the following lemma.

Lemma 1.1. *Let P be a cone of real Banach space E, Ω be a bounded open set of E, $\theta \in \Omega$, $A : P \cap \overline{\Omega} \to P$ be completely continuous.*

(i) *If $x \neq \mu Ax$ for $x \in P \cap \partial\Omega$ and $\mu \in [0,1]$, then $i(A, P \cap \Omega, P) = 1$.*
(ii) *If $\inf_{x \in P \cap \partial\Omega} \|Ax\| > 0$ and $Ax \neq \mu x$ for $x \in P \cap \partial\Omega$ and $\mu \in (0,1]$, then $i(A, P \cap \Omega, P) = 0$.*

Next noticing that $\int_0^1 p(t)dt < +\infty$, it is not difficult from [9, 10] to obtain the following lemma.

Lemma 1.2. *(i) The boundary-value problem*

$$\begin{gathered} -u'' + \rho p(t)u = 0, \quad \text{for } t \in (0,1) \\ u(0) = 0, \quad u'(0) = 1 \end{gathered}$$

has an increasing positive solution $e_1(t) = tw_1(t) \in C[0,1] \cap C^1[0,1)$, where $w_1 \in C[0,1]$ is the unique solution of the integral equation

$$w_1(t) = 1 + \frac{\rho}{t}\int_0^t \int_0^s \tau p(\tau) w_1(\tau) d\tau ds. \tag{1.2}$$

(ii) The boundary-value problem

$$\begin{gathered} -u'' + \rho p(t)u = 0, \quad \text{for } t \in (0,1) \\ u(1) = 0, \quad u'(1) = -1 \end{gathered}$$

has a decreasing positive solution $e_2(t) = (1-t)w_2(t) \in C[0,1] \cap C^1(0,1]$, where $w_2 \in C[0,1]$ is the unique solution of the integral equation

$$w_2(t) = 1 + \frac{\rho}{1-t}\int_t^1 \int_s^1 (1-\tau) p(\tau) w_2(\tau) d\tau ds. \tag{1.3}$$

(iii) The Wronskian $\omega = \det \begin{pmatrix} e_1(t) & e_1'(t) \\ e_2(t) & e_2'(t) \end{pmatrix}$ is a positive constant.

Let $J = [0,1]$. The basic space used in this paper is $E = C[J,\mathbb{R}]$. It is well known that $C[J,\mathbb{R}]$ is a Banach space with norm $\|u\| = \max_{t\in J} |u(t)|$ ($\forall u \in C[J,\mathbb{R}]$). From Lemma 1.2, it is easy to see

$$Q := \{u \in C[J,\mathbb{R}^+] : u(t) \geq q(t)u(s),\ \forall t,s \in J\} \tag{1.4}$$

is a cone of $C[J,\mathbb{R}]$, where

$$q(t) := \frac{e_1(t)e_2(t)}{e_1(1)e_2(0)}, \quad t \in J. \tag{1.5}$$

Moreover, by (1.4)-(1.5), we have for all $u \in Q$,

$$u(t) \geq q(t)\|u\|, \quad \forall t \in J. \tag{1.6}$$

A function u is said to be a solution of (1.1) if $\lambda \geq 0$ and u satisfies (1.1). In addition, if $\lambda > 0$, $u(t) > 0$ for $t \in (0,1)$, then u is said to be a positive solution of (1.1). Obviously, if $u \in Q \setminus \{\theta\}$ is a solution of (1.1), then u is a positive solution of (1.1), where θ denotes the zero element of Banach space $C[J,\mathbb{R}]$.

2. Main Results

For convenience, we list the following assumptions.

(H1) $f \in C[(0,1)\times(0,+\infty),\mathbb{R}^+]$ and for every pair of positive numbers R and r with $R > r > 0$,

$$\int_0^1 s(1-s)f_{r,R}(s)ds < +\infty,$$

where $f_{r,R}(s) := \max\{f(s,u) : u \in [rs(1-s),R]\}$, for all $s \in (0,1)$.

(H2) For every $R > 0$, there exists $\psi_R \in C[J,\mathbb{R}^+]$ ($\psi \neq \theta$) such that $f(t,u) \geq \psi_R(t)$ for $t \in (0,1)$ and $u \in (0,R]$.

(H3) There exists an interval $[a,b] \subset (0,1)$ such that $\lim_{x\to+\infty} f(s,u)/u = +\infty$ uniformly with respect to $s \in [a,b]$.

We remark that (H2) allows $f(t,u)$ being singular at $t = 0,1$, and $u = 0$. Assumption (H3) shows that f is superlinear in u. The following theorems are our main results of this paper.

Theorem 2.1. *Assume* (H1)-(H3) *are satisfied. Then there exist positive numbers* λ^* *and* λ^{**} *with* $\lambda^* < \lambda^{**}$ *such that* (1.1) *has at least two positive solutions for* $\lambda \in (0,\lambda^*)$ *and no solution for* $\lambda > \lambda^{**}$.

To overcome difficulties arising from singularity we first consider the approximate problem

$$\begin{gathered} -u''(t) + \rho p(t)u(t) = \lambda f_n(t,u(t)), \quad t \in (0,1); \\ u(0) = u(1) = 0, \end{gathered} \tag{2.1}$$

where $f_n(t,u) =: f(t,\max\{\frac{1}{n},u\})$, $n \in \mathbb{N}$. Define an operator A_n^λ on Q by

$$(A_n^\lambda u)(t) := \lambda \int_0^1 G(t,s)f_n(s,u(s))ds, \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \frac{1}{\omega}e_1(t)e_2(s), & 0 \leq t \leq s \leq 1; \\ \frac{1}{\omega}e_1(s)e_2(t), & 0 \leq s \leq t \leq 1, \end{cases} \tag{2.3}$$

where $e_1(t)$, $e_2(t)$ and ω are defined as in Lemma 1.2.

Obviously, $u = A_n^\lambda u$ is the corresponding integral equation of (2.1). Therefore, $u \in C[J,\mathbb{R}^+]\bigcap C^2[(0,1),\mathbb{R}^+]$ is a solution of (2.1) if $u \in C[J,\mathbb{R}^+]$ is a fixe point of A_n^λ. Furthermore, u is a positive solution of (2.1) if $u \in Q \setminus \{\theta\}$ is a fixed point of A_n^λ.

By (2.1)-(2.3), it is easy to see that A_n^λ is well defined on Q for each $n \in \mathbb{N}$ if condition (H1) holds. For the sake of proving our main results we first prove some lemmas.

Lemma 2.2. *Under condition (H1), $A_n^\lambda : Q \to Q$ is completely continuous.*

Proof. First we show $A_n^\lambda Q \subset Q$ for each $n \in \mathbb{N}$ and $\lambda > 0$. From (2.3) and the monotonicity of $e_1(t)$ and $e_2(t)$ obtained from Lemma 1.2, it follows that

$$\frac{G(t,s)}{G(\tau,s)} = \begin{cases} \frac{e_1(t)e_2(s)}{e_1(\tau)e_2(s)} \geq \frac{e_1(t)}{e_1(1)} \geq \frac{e_1(t)e_2(t)}{e_1(1)e_2(0)}, & 0 < t,\tau \leq s < 1; \\ \frac{e_1(s)e_2(t)}{e_1(s)e_2(t)} \geq \frac{e_2(t)}{e_2(0)} \geq \frac{e_1(t)e_2(t)}{e_1(1)e_2(0)}, & 0 < s \leq t,\tau < 1; \\ \frac{e_1(t)e_2(s)}{e_1(s)e_2(\tau)} \geq \frac{e_1(t)}{e_1(s)} \geq \frac{e_1(t)e_2(t)}{e_1(1)e_2(0)}, & 0 < t \leq s \leq \tau < 1; \\ \frac{e_1(s)e_2(t)}{e_1(\tau)e_2(s)} \geq \frac{e_2(t)}{e_2(s)} \geq \frac{e_1(t)e_2(t)}{e_1(1)e_2(0)}, & 0 < \tau \leq s \leq t < 1; \end{cases}$$

This equality and (2.2) guarantee that

$$\begin{aligned} (A_n^\lambda u)(t) &= \lambda \int_0^1 G(t,s) f_n(s,u(s))ds \\ &\geq \frac{e_1(t)e_2(t)}{e_1(1)e_2(0)} \cdot \lambda \int_0^1 G(\tau,s) f_n(s,u(s))ds \\ &= \frac{e_1(t)e_2(t)}{e_1(1)e_2(0)} \cdot (A_n^\lambda u)(\tau), \quad \forall t,\tau \in J, u \in Q. \end{aligned}$$

Therefore, $A_n^\lambda Q \subset Q$ for each $n \in \mathbb{N}$ and $\lambda > 0$.

Next by standard methods and Ascoli-Arzela theorem one can prove $A_n^\lambda : Q \to Q$ is completely continuous. So it is omitted. $\square$

Lemma 2.3. *Suppose conditions (H1) and (H2) hold. Then for each $r > 0$, there exists a positive number $\lambda(r)$ such that*

$$i(A_n^\lambda, Q_r, Q) = 1$$

for $\lambda \in (0, \lambda(r))$ and n sufficiently large, where $Q_r = \{u \in Q : \|u\| < r\}$.

Proof. First we show for each $u \in Q \setminus \{\theta\}$, there exists $l > 0$ such that

$$u(t) \geq lt(1-t)\|u\|, \quad \forall t \in J. \tag{2.4}$$

In fact, by (1.2)-(1.3), one can obtain $w_1(0) = 1$ and $w_2(1) = 1$. This together with the monotonicity of $e_1(t)$ and $e_2(t)$ guarantees that $\min_{t\in J} w_1(t) > 0$ and $\min_{t \in J} w_2(t) > 0$. Therefore, choose

$$l := \frac{1}{w_1(1)w_2(0)} \big(\min_{t\in J} w_1(t)\big) \cdot \big(\min_{t \in J} w_2(t)\big) > 0.$$

Immediately from (1.4)-(1.6), (2.4) follows. Next for each $r > 0$ and $n > \frac{1}{r}$, let

$$\lambda(r) := r\Big[\int_0^1 G(s,s) f_{rl,r}(s)ds\Big]^{-1}.$$

We assert $\|A_n^\lambda u\| < \|u\|$ for each $\lambda \in (0, \lambda(r))$ and $u \in \partial Q_r$. In fact, using (2.4) and $G(t,s) \le G(s,s)$ for $t, s \in J$ one can obtain

$$\begin{aligned}\|A_n^\lambda u\| &\le \lambda \int_0^1 G(s,s) f_n(s, u(s))ds \\ &\le \lambda \int_0^1 G(s,s) f_{rl,r}(s)ds \\ &< r = \|u\|, \quad \text{for } \lambda \in (0, \lambda(r)) \text{ and } u \in \partial Q_r.\end{aligned}$$

Therefore, by Lemma 1.1 we have $i(A_n^\lambda, Q_r, Q) = 1$ for $\lambda \in (0, \lambda(r))$. □

Lemma 2.4. *Suppose conditions (H1) and (H2) hold. Then for any given $\lambda \in (0, \lambda(r))$, there exists $r' \in (0, r)$ such that*

$$i(A_n^\lambda, Q_{r'}, Q) = 0$$

for n sufficiently large, where r and $\lambda(r)$ are the same as in Lemma 2.3.

Proof. Choose a positive number r' with $r' < \min\{r, \lambda \max_{t\in J} \int_0^1 G(t,s)\psi_r(s)ds\}$, where $\psi_r(s)$ is defined as in (H2). Now, we claim that

$$A_n^\lambda u \ne \mu u \quad \text{for } u \in \partial Q_{r'} \text{ and } \mu \in (0,1]. \tag{2.5}$$

for $n > 1/r'$. Suppose, on the contrary, there exists $u_0 \in \partial Q_{r'}$ and $\mu_0 \in (0,1]$ such that $A_n^\lambda u_0 = \mu_0 u_0$, namely,

$$u_0(t) \ge (A_n^\lambda u_0)(t) = \lambda \int_0^1 G(t,s) f_n(s, u_0(s))ds, \quad \forall t \in J.$$

Notice that $|u_0(s)| \le r' < r$ and $n > \frac{1}{r'}$ implies $f_n(s, u_0(s)) \ge \psi_r(s)$ for $s \in (0,1)$. Therefore,

$$u_0(t) \ge (A_n^\lambda u_0)(t) = \lambda \int_0^1 G(t,s)\psi_r(s)ds,$$

that is,

$$r' \ge \lambda \max_{t\in J} \int_0^1 G(t,s)\psi_r(s)ds,$$

which is in contradiction with the selection of r'. This means (2.5) holds. Thus, by Lemma 1.1 we have $i(A_n^\lambda, Q_{r'}, Q) = 0$ for $n > \frac{1}{r'}$. □

Lemma 2.5. *Suppose condition (H3) holds. Then for every $\lambda \in (0, \lambda(r))$, there exists $R > r$ such that*

$$i(A_n^\lambda, Q_R, Q) = 0$$

for all $n \in \mathbb{N}$, where $\lambda(r)$ is the same as in Lemma 2.3.

Proof. By (H3) we know there exists $R' > \max\{r, 1\}$ such that

$$\frac{f(t,u)}{u} > L := [la(1-b)\big(\lambda \min_{t\in[a,b]} \int_a^b G(t,s)ds\big)]^{-1} \quad \text{for } u > R'. \tag{2.6}$$

Let $R := 1 + \frac{R'}{la(1-b)}$. Then for $u \in \partial Q_R$, by (2.4) we have $u(t) \ge la(1-b)\|u\| > R'$ as $t \in [a,b]$. Now we show that

$$A_n^\lambda u \ne \mu u \quad \text{for } u \in \partial Q_R, \quad \text{and} \quad \mu \in (0,1]. \tag{2.7}$$

Suppose, on the contrary, there exists $u_0 \in \partial Q_R$ and $\mu_0 \in (0,1]$ such that $A_n^\lambda u_0 = \mu_0 u_0$, that is,

$$u_0(t) \geq (A_n^\lambda u_0)(t) = \lambda \int_0^1 G(t,s) f_n(s, u_0(s)) ds, \quad \forall t \in J.$$

Furthermore,

$$\begin{aligned} u_0(t) \geq (A_n^\lambda u_0)(t) &> \lambda\Big(\int_a^b G(t,s) \cdot L u_0(s) ds\Big) \\ &> \Big(\lambda \min_{t\in[a,b]} \int_a^b G(t,s) ds\Big) Lla(1-b)R = R \end{aligned}$$

for $t \in [a,b]$. This is in contradiction with $\|u_0\| = R$. This means that (2.7) holds. Therefore, by Lemma 1.1 we have $i(A_n^\lambda, Q_R, Q) = 0$ for $n \in \mathbb{N}$. $\square$

Now we are in position to prove Theorem 2.1.

Proof of Theorem 2.1. For each $r > 0$, by Lemma 2.3-2.5, there exist three positive numbers $\lambda(r), r'$, and R with $r' < r < R$ such that

$$i(A_n^\lambda, Q_{r'}, Q) = 0, \quad i(A_n^\lambda, Q_r, Q) = 1, \quad i(A_n^\lambda, Q_R, Q) = 0 \tag{2.8}$$

for n sufficiently large. Without loss of generality, suppose (2.8) holds for $n \geq n_0$. By virtue of the excision property of the fixed point index, we get

$$i(A_n^\lambda, Q_r \setminus \overline{Q_{r'}}, Q) = 1, \quad i(A_n^\lambda, Q_R \setminus \overline{Q_r}, Q) = -1$$

for $n \geq n_0$. Therefore, using the solution property of the fixed point index, there exist $u_n \in Q_r \setminus \overline{Q_{r'}}$ and $v_n \in Q_R \setminus \overline{Q_r}$ satisfying $A_n^\lambda u_n = u_n$ and $A_n^\lambda v_n = v_n$ as $n \geq n_0$. By the proof of Lemma 2.3 we know that there is no positive fixed point on ∂Q_r. Thus, $u_n \neq v_n$. Moreover, from (2.4) it follows that

$$lr't(1-t) \leq u_n(t) < r \quad \text{and} \quad lrt(1-t) < v_n(t) \leq R, \quad \text{for } t \in J. \tag{2.9}$$

In the following we show $\{u_n(t)\}_{n\geq n_0}$ are equicontinuous on J. To see this we need to prove only that $\lim_{t\to 0+} u_n(t) = 0$ and $\lim_{t\to 1-} u_n(t) = 0$ both uniformly with respect to $n \in \{n_0, n_0+1, n_0+2, \dots\}$ and $\{u_n(t)\}_{n\geq n_0}$ are equicontinuous on any subinterval of $(0,1)$. We first claim that $\lim_{t\to 0+} u_n(t) = 0$ uniformly with respect to $n \in \{n_0, n_0+1, n_0+2, \dots\}$. According to (2.3) and Lemma 1.2, it is easy to see

$$G(t,s) \leq \begin{cases} \bar{l}s(1-t), & 0 \leq s \leq t \leq 1; \\ \bar{l}t(1-s), & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.10}$$

where $\bar{l} = \max_{t\in J} w_1(t) \cdot \max_{t\in J} w_2(t)$, $w_1(t)$ and $w_2(t)$ are stated in Lemma 1.2. On the other hand, for arbitrary $\varepsilon > 0$, by (H1) there exists $\bar{\delta} > 0$ such that

$$\lambda \bar{l} \int_0^{\bar{\delta}} s(1-s) f_{r'l,r}(s) ds \leq \frac{\varepsilon}{3}. \tag{2.11}$$

Choose $\delta \in (0, \bar{\delta})$ sufficiently small such that

$$\lambda \bar{l} \delta \bar{\delta}^{-1} \int_0^1 s(1-s) f_{r'l,r}(s) ds < \frac{\varepsilon}{3}. \tag{2.12}$$

Therefore, by (2.10)-(2.12), we know for $t \in (0,\delta)$ and $\forall n \geq n_0$ that

$$\begin{aligned}
u_n(t) &= \lambda \int_0^1 G(t,s) f_n(s,u_n(s))ds \\
&\leq \lambda \bar{l} \int_0^t s(1-t) f_{r'l,r}(s)ds + \lambda \bar{l} \int_t^1 t(1-s) f_{r'l,r}(s)ds \\
&\leq \lambda \bar{l} \int_0^t s(1-s) f_{r'l,r}(s)ds + \lambda \bar{l} \Big(\int_t^{\bar{\delta}} + \int_{\bar{\delta}}^1 \Big) t(1-s) f_{r'l,r}(s)ds \\
&\leq 2\lambda \bar{l} \int_0^{\bar{\delta}} s(1-s) f_{r'l,r}(s)ds + \frac{\lambda \bar{l} t}{\bar{\delta}} \int_{\bar{\delta}}^1 s(1-s) f_{r'l,r}(s)ds \\
&\leq 2\lambda \bar{l} \int_0^{\bar{\delta}} s(1-s) f_{r'l,r}(s)ds + \frac{\lambda \bar{l} t}{\bar{\delta}} \int_0^1 s(1-s) f_{r'l,r}(s)ds \\
&\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This implies $\lim_{t\to 0+} u_n(t) = 0$ uniformly with respect to $n \in \{n_0, n_0+1, n_0+2, \dots\}$. Similarly, one can show that $\lim_{t\to 1-} u_n(t) = 0$ uniformly with respect to $n \in \{n_0, n_0+1, n_0+2, \dots\}$ also.

Now we are in position to show $\{u_n(t)\}_{n\geq n_0}$ are equicontinuous on any subinterval $[a,b]$ of $(0,1)$. Notice that

$$u_n(t) = \frac{\lambda}{\omega} \Big(\int_0^t e_1(s)e_2(t) f_n(s,u_n(s))ds + \int_t^1 e_1(t)e_2(s) f_n(s,u_n(s))ds \Big),$$

for all $t \in (0,1)$. Thus, by Lemma 1.2, for $t \in [a,b]$, we have

$$\begin{aligned}
&|u_n'(t)| \\
&= \frac{\lambda}{\omega} \Big| \int_0^t e_1(s)e_2'(t) f_n(s,u_n(s))ds + \int_t^1 e_1'(t)e_2(s) f_n(s,u_n(s))ds \Big| \\
&\leq \frac{\lambda}{\omega} \Big(\int_0^t e_1(s)|e_2'(t)| f_{r'l,r}(s)ds + \int_t^1 |e_1'(t)| e_2(s) f_{r'l,r}(s)ds \Big) \\
&\leq \frac{\lambda}{\omega} \max_{t\in[a,b]} \{|e_1'(t)|, |e_2'(t)|\} \Big(\int_0^t \frac{e_1(s)e_2(t)}{e_2(t)} f_{r'l,r}(s)ds + \int_t^1 \frac{e_1(t)e_2(s)}{e_1(t)} f_{r'l,r}(s)ds \Big) \\
&\leq \lambda \bar{l} \max_{t\in[a,b]} \{|e_1'(t)|, |e_2'(t)|\} \cdot \max\Big\{ \frac{1}{e_1(a)}, \frac{1}{e_2(b)} \Big\} \int_0^1 s(1-s) f_{r'l,r}(s)ds < +\infty,
\end{aligned}$$

which implies $\{u_n(t)\}_{n\geq n_0}$ are equicontinuous on $[a,b]$. Similarly as above, we can get $\{v_n(t)\}_{n\geq n_0}$ are equicontinuous on $[0,1]$.

Then, the Ascoli-Arzela theorem guarantees the existence of $u, v \in C[J, \mathbb{R}^+]$ and two subsequences $\{u_{n_i}\}$ of $\{u_n\}$ and $\{v_{n_i}\}$ of $\{v_n\}$ such that $\lim_{i\to+\infty} u_{n_i}(t) = u(t)$ and $\lim_{i\to+\infty} v_{n_i}(t) = v(t)$ both uniformly with respect to $t \in J$. Moreover, by (H1), (2.9), and Lebesgue dominated convergence theorem, we obtain

$$u(t) = \lambda \int_0^1 G(t,s) f(s,u(s))ds, \quad v(t) = \lambda \int_0^1 G(t,s) f(s,v(s))ds, \quad \forall t \in J$$

with $r' \leq \|u\| \leq r \leq \|v\| \leq R$. On the other hand, similar to the proof of Lemma 2.3, it is easy to see $\|u\| < r < \|v\|$.

Choose $r = 1$. From above we know there exists $\lambda(1) > 0$ such that for each $\lambda \in (0, \lambda(1))$, (1.1) has at least two positive solutions u_λ and v_λ with $0 < \|u_\lambda\| < 1 < \|v_\lambda\|$. Let

$$\lambda^* := \sup\{\bar{\lambda} > 0 : (1.1) \text{ has at least two positive solutions as } \lambda \in (0, \bar{\lambda})\}.$$

So we get the existence of λ^* satisfying that (1.1) has multiple positive solutions as $\lambda \in (0, \lambda^*)$.

Now we are in position to prove the existence of λ^{**}. As above, still choose $r = 1$ and corresponding $\lambda(1)$, R, r'. Here we show (1.1) has no positive solution as λ sufficiently large.

First suppose $\lambda \geq \lambda^*$. If (1.1) has a positive solution u for some $\lambda \geq \lambda^*$, then by corresponding integral equation

$$u(t) = \lambda \int_0^1 G(t, s) f(s, u(s)) ds \tag{2.13}$$

and a process similar to one in the proof of Lemmas 2.4 and 2.5 (replacing λ in (2.6) with $\lambda(1)$), we obtain $r' < \|u\| < R$. This together with condition (H2) and (2.13) guarantees that $u(t) \geq \lambda \int_0^1 G(t, s)\psi_R(s) ds$, that is, $R > \|u\| \geq \lambda \cdot \max_{t\in J} \int_0^1 G(t, s)\psi_R(s) ds$, which implies $\lambda < \Big(\max_{t\in J} \int_0^1 G(t, s)\psi_R(s) ds\Big)^{-1} R$. Therefore, we obtain the existence of λ^{**}. The proof of Theorem 2.1 is complete. $\square$

If $f(t, u)$ is not singular at $u = 0$, we have the following result, under the hypothesis

(H4) $f \in C[(0, 1) \times [0, +\infty), \mathbb{R}^+]$ and for every positive number R,

$$\int_0^1 s(1 - s) f_{0,R}(s) ds < +\infty,$$

where $f_{0,R}(s) = \max\{f(s, u) : u \in [0, R]\}$, for all $s \in (0, 1)$.

Theorem 2.6. *Assume that conditions (H2)-(H4) hold. Then there exist two positive numbers λ^* and λ^{***} with $\lambda^* \leq \lambda^{***}$ such that*

(i) (1.1) *has at least two positive solutions for $\lambda \in (0, \lambda^*)$;*
(ii) (1.1) *has at least one positive solution for $\lambda \in (0, \lambda^{***}]$;*
(iii) (1.1) *has no solutions for $\lambda > \lambda^{***}$*

Proof. Notice that condition (H4) implies (H1). Therefore, the existence of λ^* can be obtained as in Theorem 2.1. Now we claim that the condition

$$\lambda^{***} := \sup\{\lambda \in \mathbb{R}^+ : (1.1) \text{ has at least one positive solution in } C[0, 1] \bigcap C^2(0, 1)\} \tag{2.14}$$

is required. First from the proof of Theorem 2.1, we know $\lambda^{***} \leq \lambda^{**}$. In the following we prove that (1.1) with $\lambda = \lambda^{***}$ has a positive solution $u^* \in C[0, 1] \bigcap C^2(0, 1) \bigcap Q$.

By (2.14), there exist two sequences $\{\lambda_n\}$ and $\{u_n\} \subset Q \setminus \{\theta\}$ such that $\{u_n\}$ is a positive solution of (1.1) with $\lambda = \lambda_n$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_n \to \lambda^{***}$. Without loss of generality, suppose $\lambda_n \geq \lambda^*/2$ for each $n \in \mathbb{N}$. Similar to the proof of Lemmas 2.4, 2.5 and Theorem 2.1, we can obtain that there exists two positive numbers r_1 and R_1 satisfying $r_1 \leq \|u_n\| \leq R_1$ for each $n \in \mathbb{N}$ and $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ which convergence to a function $u^* \in \overline{Q}_{R_1} \setminus Q_{r_1}$ uniformly as $t \in J$. Notice that

$$u_{n_k}(t) = \lambda_{n_k} \int_0^1 G(t,s) f(s, u_{n_k}(s)) ds, \quad \forall t \in J.$$

Letting $k \to +\infty$, by condition (H4) and Lebesgue dominated convergence theorem, we get

$$u^*(t) = \lambda^{***} \int_0^1 G(t,s) f(s, u^*(s)) ds, \quad \forall t \in J.$$

This implies that $u^*(t)$ is a positive solution of (1.1) with $\lambda = \lambda^{***}$.

Now we are in position to prove that BVP (1.1) has at least one positive solution $u_\lambda(t)$ for each $\lambda \in (0, \lambda^{***})$. Notice that for $\lambda \in (0, \lambda^{***})$,

$$\begin{gathered} -u^{*\prime\prime}(t) + \rho p(t) u^*(t) = \lambda^{***} f(t, u^*(t)) \geq \lambda f(t, u^*(t)), \quad t \in (0,1); \\ u^*(0) = u^*(1) = 0. \end{gathered}$$

This implies $u^*(t)$ is an upper solution of (1.1). On the other hand, $u(t) \equiv 0$ is a lower solution for BVP (1.1). Applying [4, Theorem 3.2] and the method used in [12, 9], one can obtain that BVP (1.1) has at least one positive solution $u_\lambda(t) \in [0, u^*(t)]$ $(t \in J)$ for each $\lambda \in (0, \lambda^{***})$. □

Remark. It is interesting to investigate what conditions should guarantee $\lambda^* = \lambda^{***}$. Ha and Lee [3] obtained $\lambda^* = \lambda^{***}$ when $p(t) = 0$, $f(t,u) = p_1(t) g(u)$, $p_1(t) > 0$ is singular only at $t = 0$ and $g(u)$ is nondecreasing with $g(u) \geq e^u$. Xu [11], also obtained $\lambda^* = \lambda^{***}$ when $p(t) = 0$, $f(t,u) = p_1(t) g(u)$, $p_1(t) > 0$ is singular at $t = 0, 1$ and $g(u)$ is nondecreasing with $g(u) \geq \delta x^m$ $(m \geq 2)$. To the best of our knowledge, it seems an open problem when $f(t,u)$ is singular at $u = 0$.

3. Examples

Example 3.1. Consider the nonresonant singular boundary-value problem

$$\begin{gathered} -u''(t) + \frac{3}{\sqrt{t}} u(t) = \lambda \Big[\frac{1}{\sqrt{t(1-t)}} \Big(u^{-4/3} + x^\alpha \sin^2 t \Big) \Big], \quad t \in (0,1); \\ u(0) = u(1) = 0, \end{gathered} \tag{3.1}$$

where $\alpha > 1$. Then there exist positive numbers λ^* and λ^{**} with $\lambda^* < \lambda^{**}$ such that BVP (3.1) has at least two positive solutions for $\lambda \in (0, \lambda^*)$ and no solution for $\lambda > \lambda^{**}$.

Proof. BVP (3.1) can be regarded as an BVP of the form (1.1), where $\rho = 3$, $p(t) = \frac{1}{\sqrt{t}}$, and

$$f(t,u) = \frac{1}{\sqrt{t(1-t)}} \Big(u^{-4/3} + x^\alpha \sin^2 t \Big).$$

First we notice that $\int_0^1 \frac{1}{\sqrt{t}} dt < +\infty$, and $\int_0^1 t(1-t) \frac{1}{\sqrt{t}} dt = \frac{4}{15}$. By Banach fixed point theorem, it is easy to see that

$$\begin{gathered} -u''(t) + \frac{3}{\sqrt{t}} u(t) = 0, \quad t \in (0,1); \\ u(0) = u(1) = 0, \end{gathered}$$

has only the trivial solution. Next we prove that $f(t,u)$ satisfies conditions (H1)-(H3). For each pair of positive numbers R and r with $R > r > 0$, we know

$$f_{r,R}(t) \le \frac{1}{\sqrt{t(1-t)}}\Big(\big(rt(1-t)\big)^{-4/3} + R^{\alpha}\Big).$$

Then

$$\int_0^1 t(1-t) f_{r,R}(t)dt \le \int_0^1 \sqrt{t(1-t)}\Big(\big(rt(1-t)\big)^{-4/3} + R^{\alpha}\Big)dt < +\infty.$$

This means condition (H1) is satisfied. To see that (H2) holds, we notice that for each $R > 0$, one can choose $\psi_R(t) = R^{-4/3}/\sqrt{t(1-t)}$, which satisfies $\psi_R \ne \theta$ and $f(t,u) \ge \psi_R(t)$ for $t \in (0,1)$ and $u \in (0,R]$. Finally it is easy to see (H3) is satisfied since we can choose any subinterval of $[a,b] \subset (0,1)$ satisfying $\lim_{x\to+\infty} f(s,u)/u = +\infty$ uniformly with respect to $s \in [a,b]$. By Theorem 2.1, the conclusion follows. □

Analogously, using Theorem 2.6, we can prove that the following statement holds.

Example 3.2. Consider the nonresonant singular boundary-value problem

$$\begin{gathered} -u''(t) + \frac{1}{2}t^{-3/2}u(t) = \lambda\big(t(1-t)\big)^{-3/2}\big(2 + \sin u + x^{\alpha}\cos t\big), \quad t \in (0,1); \\ u(0) = u(1) = 0, \end{gathered} \tag{3.2}$$

where $\alpha > 1$. Then there exist two positive numbers λ^* and λ^{***} with $\lambda^* \le \lambda^{***}$ such that:

(i) BVP (3.2) has at least two positive solutions for $\lambda \in (0, \lambda^*)$;
(ii) BVP (3.2) has at least one positive solution for $\lambda \in (0, \lambda^{***}]$;
(iii) BVP (3.2) has no solution for $\lambda > \lambda^{***}$

References

[1] R. P. Agarwal, D. O'Regan; Nonlinear superlinear singular and nonsingular second order boundary-value problems. *J. Diff. Eqns.*, **143** (1998), 60-95.
[2] D. Guo, V. Lakshmikantham, Nonlinear problems in abstract cones, Academic Press, Inc., New York, 1988.
[3] K. Ha and Y. Lee, Existence of multiple positive solutions of singular boundary-value problems, *Nonlinear Anal.*, **28** (1997), 1429-1438.
[4] P. Hartman, Ordinary differential equations, 2nd Ed., Birkhauser, Boston, 1982.
[5] K. Lan and J. L. Webb . Positive solutions of semilinear differential equations with singularities. *J. Diff. Eqns.*, **148** (1998), 407-421.
[6] Y. Liu, Twin solutions to Singular Semipositone Problems, *J. Math. Anal. Appl.*, **286** (2003), 248-260.
[7] D. O'Regan, Singular Dirichlet boundary-value problems-I, Superlinear and nonresonance case, *Nonlinear Anal.*, **29** (1997), 221-249.
[8] S. D. Taliaferro, A nonlinear singular boundary-value problem, *Nonlinear Anal.*, **3** (1979), 897-904.
[9] Z. Wei, Positive solutions of singular Dirichlet boundary-value problems at nonresonance, *Chinese Annals of Mathematics*, **20** (1999), 543-552(in Chinese).
[10] Z. Wei, C. Pang, Positive solutions of nonresonant singular boundary-value problem of second order differential equations, *Nagoya Math. J.*, **162**(2001), 127-148.
[11] X. Xu, On some results of singular boundary-value problems, Doctorial thesis of Shandong University, 2001.
[12] Y. Zhang, Positive solutions of singular sublinear Emden-Fowler boundary-value problems, *J. Math. Anal. Appl.*, **185** (1994), 215-222.

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