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GENERALIZED VISCOSITY SOLUTIONS OF ELLIPTIC PDES AND BOUNDARY CONDITIONS

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Abstract. Sufficient conditions are given for a generalized viscosity solution of an elliptic boundary value problem to satisfy the boundary values in the strong sense.

1. Introduction

When studying nonlinear elliptic equations of the form

$$F(\mathbf{x}, u, Du, D^2u) = 0, \quad \mathbf{x} \in \Omega, \tag{1.1}$$

one very fruitful approach may be to use the notion of viscosity solutions. We say that u is a viscosity solution if it is both a subsolution and supersolution and u is a subsolution in Ω if it is upper semicontinuous in Ω and for every $\mathbf{x}_0 \in \Omega$ the following implication holds: If $\psi \in \mathcal{C}^2(\mathbb{R}^d)$ and $u(\mathbf{x}) \le \psi(\mathbf{x}) + u(\mathbf{x}_0) - \psi(\mathbf{x}_0)$, $\mathbf{x} \in \Omega$, $|\mathbf{x} - \mathbf{x}_0| < \delta$, for some $\delta > 0$ then $F\big(\mathbf{x}_0, u(\mathbf{x}_0), D\psi(\mathbf{x}_0), D^2\psi(\mathbf{x}_0)\big) \le 0$. Supersolutions are defined symmetrically, for details and further information and references, see e.g. [4]. Since one wants to be sure that classical solutions are viscosity solutions, one has to assume that F is nonincreasing in its last argument (with the natural ordering for symmetric matrices).

One way of proving existence is to use Perron's method, introduced in the viscosity setting in [6]. That is, one proves that the supremum of a suitable set of subsolutions is the solution. For this to work one needs a subsolution $\underline{u}$ and a supersolution $\overline{u}$ such that $\underline{u} \le \overline{u}$ and a comparison result saying that a subsolution is less than or equal to a supersolution if both lie between $\underline{u}$ and $\overline{u}$. If $\underline{u}(\mathbf{x}) = \overline{u}(\mathbf{x})$ when $\mathbf{x} \in \partial\Omega$ then all functions between $\underline{u}$ and $\overline{u}$ will automatically satisfy a Dirichlet boundary condition but if this is not the case then the situation is not so simple any more. One can, however, take another approach and consider a boundary condition $G(\mathbf{x}, u(\mathbf{x})) = 0$ in the viscosity sense which means that one considers the equation $H(x, u, Du, D^2u) = 0$ in $\overline{\Omega}$ where

$$H(\mathbf{x}, r, \mathbf{p}, X) = \begin{cases} F(\mathbf{x}, r, \mathbf{p}, X), & \mathbf{x} \in \Omega, \\ G(\mathbf{x}, r), & \mathbf{x} \in \partial\Omega. \end{cases}$$

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For the use of Perron's method in this case, see also [5, Thm. 6.1]. It turns out that when dealing with subsolutions the function H should be lower semicontinuous and when considering supersolutions H should be upper semicontinuous. Thus one has to introduce upper and lower semicontinuous envelopes defined as follows: If $v : \mathcal{A} \to [-\infty, \infty]$, $\mathcal{A} \subset \mathbb{R}^N$, then $v^*(\mathbf{z}) = \lim_{s\downarrow 0} \sup\{v(\mathbf{y}) \mid y \in \mathcal{A}, \quad |\mathbf{y}-\mathbf{x}| \le s\}$ and $v_*(\mathbf{z}) = \lim_{s\downarrow 0} \inf\{u(\mathbf{y}) \mid y \in \mathcal{A}, \quad |\mathbf{y}-\mathbf{x}| \le s\}$, for $\mathbf{z} \in \mathbb{R}^N$. Thus a generalized viscosity solution of (1.1) with boundary condition $G(\mathbf{x}, u(\mathbf{x})) = 0$ should be a subsolution of $H_* = 0$ and a supersolution of $H^* = 0$. Observe that if both F and G are lower semicontinuous (otherwise replace F and G below by F_* and G_*, respectively) and Ω is open then

$$H_*(\mathbf{x}, r, \mathbf{p}, X) = \begin{cases} F(\mathbf{x}, r, \mathbf{p}, X), & \mathbf{x} \in \Omega, \\ \min\{G(\mathbf{x}, r), F(\mathbf{x}, r, \mathbf{p}, X)\}, & \mathbf{x} \in \partial\Omega. \end{cases}$$

One consequence of studying equation $H_* = 0$ is that the boundary conditions may not be satisfied at all points, and this in turn will cause grave problems when one tries to prove comparison results. Thus the purpose of this note is to study under what assumptions it follows that if u is a subsolution of $H_* = 0$, then one actually has $G(\mathbf{x}, u(\mathbf{x})) \le 0$ for all points $\mathbf{x}$ on the boundary. By symmetry one can then get corresponding results for supersolutions, because u is a supersolution of $H = 0$ if and only if $-u$ is a subsolution of $\overleftrightarrow{H} = 0$ where $\overleftrightarrow{H}(\mathbf{x}, r, \mathbf{p}, X) = -H(\mathbf{x}, -r, -\mathbf{p}, -X)$.

These results improve those that can be found in [5] (see also [2]), in particular concerning the assumptions on the domain Ω. Concerning the equations studied in e.g. [1] on sees that the assumptions in the theorems below are satisfied for the p-Laplacian equation $-\operatorname{div}(|Du|^{p-2}Du) = 0$ when $1 < p < \infty$ where one thus has $F(\mathbf{x}, r, \mathbf{q}, X) = -|\mathbf{q}|^{p-2}\operatorname{tr}(X) - (p-2)|\mathbf{q}|^{p-4}\langle \mathbf{q}, X\mathbf{q}\rangle$ when $\mathbf{q} \ne 0$ and for the infinity-Laplacian equation $-\Delta_\infty u = 0$ where $F(\mathbf{x}, r, \mathbf{q}, X) = -\langle \mathbf{q}, X\mathbf{q}\rangle$. (Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^d$.) However, the assumptions are not satisfied for the 1-Laplacian equation, nor for the minimal surface equation $-\operatorname{div}\big(\frac{1}{\sqrt{1+|Du|^2}}Du\big) = 0$ where $F(\mathbf{x}, r, \mathbf{q}, X) = -\frac{\operatorname{tr}(X)}{\sqrt{1+|\mathbf{q}|^2}} + \frac{\langle \mathbf{q}, X\mathbf{q}\rangle}{\sqrt{1+|\mathbf{q}|^2}^3}$.

2. Statement of results

We shall prove two theorems that differ only in a tradeoff between the assumptions on the domain Ω and the nonlinearity F. Note also that we do not have to assume that F and G are nondecreasing in the second variable, but this assumption is essential in comparison results. Below $\mathcal{S}(d)$ denotes the set of real symmetric $d \times d$-matrices with $X \ge Y$ for $X, Y \in \mathcal{S}(d)$ if all eigenvalues of $X - Y$ are nonnegative, and $\mathbf{n} \otimes \mathbf{n}$ is the matrix with (i, j)-element $\mathbf{n}_i\mathbf{n}_j$.

Theorem 2.1. *Assume that* $d \ge 1$ *and that*

(i) $\Omega \subset \mathbb{R}^d$ *is open;*

(ii) $\mathbf{x}_\diamond \in \partial\Omega$ *satisfies an exterior ball condition, i.e., there is a vector* $\mathbf{n}_\diamond \in \mathbb{R}^d$ *with* $|\mathbf{n}_\diamond| = 1$ *and numbers* $\rho_\diamond > 0$ *and* $\beta_\diamond > 0$ *such that* $\{\mathbf{x} \in \mathbb{R}^d \mid |\mathbf{x} - \mathbf{x}_\diamond - \rho_\diamond \mathbf{n}_\diamond| \le \rho_\diamond, |\mathbf{x} - \mathbf{x}_\diamond| \le \beta_\diamond\} \cap \overline{\Omega} = \{\mathbf{x}_\diamond\}$*;*

(iii) $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to [-\infty, \infty]$ *is lower semicontinuous and degenerate elliptic, i.e., nonincreasing in its last variable;*

(iv)
$$\liminf F\Big(\mathbf{x}, r, \lambda\mathbf{n}, -\eta\lambda^2\, \mathbf{n} \otimes \mathbf{n} + \frac{1}{\rho_\diamond}\,\lambda I\Big) > 0,$$

where the limit is taken as $\mathbf{x} \to \mathbf{x}_\diamond$, $\mathbf{x} \in \overline{\Omega}$, $\mathbf{n} \to \mathbf{n}_\diamond$, $|\mathbf{n}_\diamond| = 1$, $r \to u(\mathbf{x}_\diamond)$, $\lambda \to \infty$, *and* $\eta \to \infty$;

(v) $G : \partial\Omega \times \mathbb{R} \to [-\infty, \infty]$ *is lower semicontinuous;*

(vi) $u : \overline{\Omega} \to \mathbb{R}$ *is a subsolution of* $H_* = 0$ *in* $\overline{\Omega}$ *where*

$$H(\mathbf{x}, r, \mathbf{p}, X) = \begin{cases} F(\mathbf{x}, r, \mathbf{p}, X), & \mathbf{x} \in \Omega, \\ G(\mathbf{x}, r), & \mathbf{x} \in \partial\Omega. \end{cases}$$

Then $G(\mathbf{x}_\diamond, u(\mathbf{x}_\diamond)) \le 0$.

Theorem 2.2. *Assume that* $d \ge 1$ *and that*

(i) $\Omega \subset \mathbb{R}^d$ *is open;*

(ii) $\mathbf{x}_\diamond \in \partial\Omega$ *satisfies an exterior cone condition, i.e., there is a vector* $\mathbf{n}_\diamond \in \mathbb{R}^d$ *with* $|\mathbf{n}_\diamond| = 1$ *and numbers* $\theta_\diamond \in (0, \frac{\pi}{2}]$ *and* $\beta_\diamond > 0$ *such that* $\{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x} - \mathbf{x}_\diamond, \mathbf{n}_\diamond \rangle \ge \cos(\theta_\diamond)|\mathbf{x} - \mathbf{x}_\diamond|, |\mathbf{x} - \mathbf{x}_\diamond| \le \beta_\diamond\} \cap \overline{\Omega} = \{\mathbf{x}_\diamond\}$;

(iii) $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to [-\infty, \infty]$ *is lower semicontinuous and degenerate elliptic, i.e., nonincreasing in its last variable;*

(iv)

$$\liminf F\Big(\mathbf{x}, r, \lambda\mathbf{n}, -\mu\lambda^2(\eta\mathbf{n} \otimes \mathbf{n} - I)\Big) > 0,$$

where the limit is taken as $\mathbf{x} \to \mathbf{x}_\diamond$, $\mathbf{x} \in \overline{\Omega}$, $r \to u(\mathbf{x}_\diamond)$ $|\mathbf{n}| = 1$, $\lambda \to \infty$, $\mu \to \infty$, *and* $\eta \to \infty$;

(v) $G : \partial\Omega \times \mathbb{R} \to [-\infty, \infty]$ *is lower semicontinuous;*

(vi) $u : \overline{\Omega} \to \mathbb{R}$ *is a subsolution of* $H_* = 0$ *in* $\overline{\Omega}$ *where*

$$H(\mathbf{x}, r, \mathbf{p}, X) = \begin{cases} F(\mathbf{x}, r, \mathbf{p}, X), & \mathbf{x} \in \Omega, \\ G(\mathbf{x}, r), & \mathbf{x} \in \partial\Omega. \end{cases}$$

Then $G(\mathbf{x}_\diamond, u(\mathbf{x}_\diamond)) \le 0$.

It seems to be quite difficult to get a counterexample showing, for example, that one cannot replace (iv) in Theorem 2.2 by (iv) in Theorem 2.1) but a modification of a classical example due to Lebesgue (see e.g. [3, p. 303] where the notion of viscosity solutions is not considered) shows that one cannot hope to be able to significantly weaken the external cone condition. In this example one considers Laplace's equation $-\Delta u = 0$.

Example 2.3. Let $d \ge 4$ and assume that ω is a nondecreasing continuous function on $[0, 1]$ with $\omega(0) = 0$ such that

$$\int_0^1 \frac{\omega(t)}{t}\,\mathrm{d}t < \infty \quad \text{but} \quad \int_0^1 \frac{\omega(t)}{t^2}\,\mathrm{d}t = +\infty.$$

Furthermore, let

$$\Omega = \{\mathbf{x} \in \mathbb{R}^d \mid |\mathbf{x}| < 1, |(x_2, \dots, x_d)| > \omega(|x_1|)^{\frac{1}{d-3}} x_1\},$$

$$\varphi(\mathbf{x}) = -\int_0^1 \frac{t^{d-3}\rho(t)^{d-2}}{|((x_1 - t), x_2, \dots, x_d)|^{d-2}}\,\mathrm{d}t, \quad \mathbf{x} \neq (s, 0, \dots, 0), \quad s \in [0,1],$$

$$F(\mathbf{x}, r, \mathbf{p}, X) = -\operatorname{tr}(X),$$

$$G(\mathbf{x}, r) = \begin{cases} r - \varphi(\mathbf{x}), & \mathbf{x} \in \partial\Omega \setminus \{\mathbf{0}\}, \\ r + \int_0^1 \frac{\omega(t)}{t}\,\mathrm{d}t + \frac{1}{d-3}, & \mathbf{x} = \mathbf{0}, \end{cases}$$

$$u(\mathbf{x}) = \begin{cases} \varphi(\mathbf{x}), & \mathbf{x} \in \overline{\Omega} \setminus \{\mathbf{0}\}, \\ -\int_0^1 \frac{\omega(t)}{t}\,\mathrm{d}t, & \mathbf{x} = \mathbf{0}. \end{cases}$$

Then all assumptions of Theorem 2.2 are satisfied except (ii) and the conclusion of Theorem 2.2 does not hold when $\mathbf{x}_\diamond = \mathbf{0}$.

For Ω in the example above to satisfy the exterior cone condition one would have to assume that $\inf_{t\in(0,1)} \omega(t) > 0$. Note also that if $\Omega = \{\mathbf{x} \in \mathbb{R}^d \mid |\mathbf{x}| < 1, |(x_2, \dots, x_d)| > (\log(|\log(x_1)|))^{-1} x_1\}$, then the exterior cone condition is not satisfied at $\mathbf{0}$ but neither can the example above be applied for any d.

In the next example we show that for the 1-Laplacian equation the claim of Theorem 2.1 (or 2.2) does not hold. This example shows that one cannot replace assumption (iv) by an assumption of the form $F(\mathbf{x}, r, \mathbf{p}, -\eta I) > 0$ for some $\eta > 0$ when $\mathbf{p} \neq 0$.

Example 2.4. Let

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1\},$$

$$F = \tilde{F}_* \quad \text{where} \quad \tilde{F}(\mathbf{x}, r, \mathbf{p}, X) = -\frac{\operatorname{tr}(X)}{|\mathbf{p}|} + \frac{\langle \mathbf{p}, X\mathbf{p}\rangle}{|\mathbf{p}|^3}, \quad \mathbf{p} \neq \mathbf{0},$$

$$G(\mathbf{x}, r) = r, \quad u(\mathbf{x}) = 1, \quad \mathbf{x} \in \overline{\Omega}.$$

Then all assumptions of Theorem 2.1 are satisfied except (iv) and the conclusion of Theorem 2.1 does not hold.

We need a special case of the following lemma, which is closely related to [4, Lemma 3.1]. Here inequalities for vectors are to be taken component wise and $\mathbf{1}$ is the vector with all components equal to 1.

Lemma 2.5. *Assume that $\mathcal{A} \subset \mathbb{R}^d$, $w : \mathcal{A} \to \mathbb{R}$ is upper semicontinuous and $\Psi : [1,\infty)^m \times \mathcal{A} \to [0,\infty)$ nondecreasing in its first m arguments and lower semicontinuous with respect to the last one. Suppose furthermore that*

$$\mathcal{N} = \cap_{\boldsymbol{\alpha} \geq \mathbf{1}} \{\mathbf{z} \in \mathcal{A} \mid \Psi(\boldsymbol{\alpha}, \mathbf{z}) = 0\} \neq \emptyset,$$

$$\lim_{\boldsymbol{\alpha} \to \boldsymbol{\infty}} \Psi(\boldsymbol{\alpha}, \mathbf{z}) = \infty, \quad \mathbf{z} \in \mathcal{A} \setminus \mathcal{N},$$

and that

$$\sup_{\mathbf{z} \in \mathcal{A}} \big(w(\mathbf{z}) - \Psi(\mathbf{1}, \mathbf{z})\big) < \infty.$$

Let $M_{\boldsymbol{\alpha}} = \sup_{\mathbf{z}\in\mathcal{A}}(w(\mathbf{z}) - \Psi(\boldsymbol{\alpha}, \mathbf{z}))$ for $\boldsymbol{\alpha} \geq 1$ and assume that $\mathbf{z}_{\boldsymbol{\alpha}} \in \mathcal{A}$ is such that

$$\lim_{\boldsymbol{\alpha} \to \boldsymbol{\infty}} \Big(M_{\boldsymbol{\alpha}} - \Big(w(\mathbf{z}_{\boldsymbol{\alpha}}) - \Psi(\boldsymbol{\alpha}, \mathbf{z}_{\boldsymbol{\alpha}})\Big)\Big) = 0.$$

If $\mathbf{z}_\diamond \in \mathcal{A}$ is a cluster point of $\mathbf{z}_{\boldsymbol\alpha}$ as $\boldsymbol\alpha \to \boldsymbol\infty$, then $\mathbf{z}_\diamond \in \mathcal{N}$ and $w(\mathbf{z}) \le w(\mathbf{z}_\diamond)$ for all $\mathbf{z} \in \mathcal{N}$ and if $\lim_{j\to\infty} \mathbf{z}_{\boldsymbol\alpha_j} = \mathbf{z}_\diamond$ then $\lim_{j\to\infty} \Psi(\boldsymbol\alpha_j, \mathbf{z}_{\boldsymbol\alpha_j}) = 0$. Moreover, if $\mathcal{A}$ is compact, then

$$\lim_{\boldsymbol\alpha\to\boldsymbol\infty} \Psi(\boldsymbol\alpha, \mathbf{z}_{\boldsymbol\alpha}) = 0.$$

If $\mathcal{A}$ is not compact then it is not necessarily true that $\lim_{\boldsymbol\alpha\to\boldsymbol\infty} \Psi(\boldsymbol\alpha, \mathbf{z}_{\boldsymbol\alpha}) = 0$ as can be seen by taking $\mathcal{A} = [0,\infty)$, $w(z) = \frac{2z}{z+1}$ and $\Psi(\alpha, z) = z\max\{0, \alpha - z\} + \frac{z}{z+1}$.

3. Proofs

Proof of Lemma 2.5. If $\boldsymbol\alpha \ge \boldsymbol\beta \ge \mathbf{1}$ then we have $\Psi(\boldsymbol\alpha, \mathbf{z}) \ge \Psi(\boldsymbol\beta, \mathbf{z})$ for all $\mathbf{z} \in \mathcal{A}$ and hence $M_{\boldsymbol\alpha} \le M_{\boldsymbol\beta}$ as well. On the other hand, $\mathcal{N}$ is nonempty and $\sup_{\mathbf{z}\in\mathcal{N}} w(\mathbf{z}) \le M_{\boldsymbol\alpha}$, so $M_{\boldsymbol\alpha}$ is bounded from below and $M_{\boldsymbol\infty} := \lim_{\boldsymbol\alpha\to\boldsymbol\infty} M_{\boldsymbol\alpha}$ exists (and is finite).

Assume next that $\mathbf{z}_{\boldsymbol\alpha_j} \to \mathbf{z}_\diamond \in \mathcal{A}$ where $\boldsymbol\alpha_j \to \boldsymbol\infty$. If $\mathbf{z}_\diamond \notin \mathcal{N}$ then there exists a vector $\boldsymbol\alpha_\diamond$ such that $\Psi(\boldsymbol\alpha_\diamond, \mathbf{z}_\diamond) \ge w(\mathbf{z}_\diamond) - M_{\boldsymbol\infty} + 2$. Since $\mathbf{z} \to \Psi(\boldsymbol\alpha_\diamond, \mathbf{z})$ is lower semicontinuous and w is upper semicontinuous we see that for sufficiently large j we have $\Psi(\boldsymbol\alpha_\diamond, \mathbf{z}_{\boldsymbol\alpha_j}) > w(\mathbf{z}_{\boldsymbol\alpha_j}) - M_{\boldsymbol\infty} + 1$. Since $\boldsymbol\alpha_j \ge \boldsymbol\alpha_\diamond$ for sufficiently large values of j and Ψ is nondecreasing in the first variables we see that $\Psi(\boldsymbol\alpha_j, \mathbf{z}_{\boldsymbol\alpha_j}) > w(\mathbf{z}_{\boldsymbol\alpha_j}) - M_{\boldsymbol\infty} + 1$ for all sufficiently large j. But this contradicts the assumption that $0 = \lim_{\boldsymbol\alpha\to\boldsymbol\infty}(M_{\boldsymbol\alpha} - w(\mathbf{z}_{\boldsymbol\alpha}) + \Psi(\boldsymbol\alpha, z_{\boldsymbol\alpha})) = M_{\boldsymbol\infty} - \lim_{\boldsymbol\alpha\to\boldsymbol\infty}(w(\mathbf{z}_{\boldsymbol\alpha}) - \Psi(\boldsymbol\alpha, z_{\boldsymbol\alpha}))$.

Since Ψ is nonnegative and w is upper semicontinuous we have

$$\begin{aligned}
w(\mathbf{z}_\diamond) &\ge \limsup_{j\to\infty} \Big(w(\mathbf{z}_{\boldsymbol\alpha_j}) - \Psi(\boldsymbol\alpha_j, \mathbf{z}_{\boldsymbol\alpha_j}) \Big) \\
&= \limsup_{j\to\infty} \Big(w(\mathbf{z}_{\boldsymbol\alpha_j}) - \Psi(\boldsymbol\alpha_j, \mathbf{z}_{\boldsymbol\alpha_j}) - M_{\boldsymbol\alpha_j} \Big) + \lim_{j\to\infty} M_{\boldsymbol\alpha_j} \\
&= M_{\boldsymbol\infty} \ge \sup_{\mathbf{z}\in\mathcal{N}} w(\mathbf{z}).
\end{aligned}$$

Since $\mathbf{z}_\diamond \in \mathcal{N}$ and $w(\mathbf{z}_\diamond) \ge \limsup_{j\to\infty} w(\mathbf{z}_{\boldsymbol\alpha_j})$ this inequality implies in addition that $\lim_{j\to\infty} \Psi(\boldsymbol\alpha_j, \mathbf{z}_{\boldsymbol\alpha_j}) = 0$.

Finally, if we assume that $\mathcal{A}$ is compact then every subsequence $(\boldsymbol\alpha_j)_{j=1}^\infty$ has a subsequence for which $\lim_{k\to\infty} \mathbf{z}_{\boldsymbol\alpha_{j_k}} \to \mathbf{z}_\diamond \in \mathcal{A}$ and the claim follows from the results already proven. □

Proof of Theorem 2.1. Suppose to the contrary that $G(\mathbf{x}_\diamond, u(\mathbf{x}_\diamond)) > 0$. Let $\mathbf{n}_\diamond$ and $\rho_\diamond$ be the unit vector and number in assumption (ii), let $\eta > 1$ be such that (possible by (iv))

$$\liminf F\Big(\mathbf{x}, r, \lambda\mathbf{n}, -\eta\lambda^2\, \mathbf{n}\otimes\mathbf{n} + \frac{1}{\rho_\diamond}\,\lambda I\Big) > 0, \tag{3.1}$$

where the limit is taken as $\mathbf{x} \to \mathbf{x}_\diamond$, $\mathbf{x} \in \overline{\Omega}$, $\mathbf{n} \to \mathbf{n}_\diamond$, $|\mathbf{n}_\diamond| = 1$, $r \to u(\mathbf{x}_\diamond)$, and $\lambda \to \infty$.

Furthermore, let $\mathbf{y}_\diamond = x_\diamond + \rho_\diamond \mathbf{n}_\diamond$, and define the function Ψ by

$$\Psi(\alpha, \mathbf{x}) = \psi(\alpha(|\mathbf{x} - \mathbf{y}_\diamond| - \rho_\diamond)), \qquad \alpha \ge 1, \quad \mathbf{x} \in \overline{\Omega},$$

where ψ is some twice continuously differentiable function with $\psi'(t) \ge \frac12$, $t \ge 0$, $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi''(0) = -2\eta$. (Take for example $\psi(t) = \frac{t}{2} + \frac{1}{8\eta-2}\big(1 - \frac{1}{(1+t)^{4\eta-1}}\big)$ when $t > -\frac12$.) Let $\mathcal{A} = \overline{\Omega} \cap \{\mathbf{x} \mid |\mathbf{x} - \mathbf{x}_\diamond| \le \beta_\diamond\}$ and observe that the only point $\mathbf{x} \in \mathcal{A}$ where $\Psi(\alpha, \mathbf{x}) = 0$ is $\mathbf{x}_\diamond$. Since u is upper semicontinuous in the compact set $\mathcal{A}$ it is bounded from above there and for each $\alpha \geq 1$ there is a point $\mathbf{x}_\alpha \in \mathcal{A}$ such that

$$u(\mathbf{x}_\alpha) - \Psi(\alpha, x_\alpha) = \sup_{\mathbf{x}\in\mathcal{A}} \Big(u(\mathbf{x}) - \Psi(\alpha,\mathbf{x})\Big).$$

It follows from Lemma 2.5 that $\lim_{\alpha\to\infty} \Psi(\alpha,\mathbf{x}_\alpha) = 0$ and that $\lim_{\alpha\to\infty}\mathbf{x}_\alpha = \mathbf{x}_\diamond$. Thus we see that if α is sufficiently large, then $\mathbf{x}_\alpha$ is a local maximum point of $u(\mathbf{x}) - \Psi(\alpha,\mathbf{x})$ in $\overline{\Omega}$. Clearly we have $u(\mathbf{x}_\alpha) \geq u(\mathbf{x}_\diamond)$ and since u is upper semicontinuous we conclude that $\lim_{\alpha\to\infty} u(\mathbf{x}_\alpha) = u(\mathbf{x}_\diamond)$. Since G is lower semicontinuous this implies that if α is sufficiently large and $\mathbf{x}_\alpha \in \partial\Omega$ then $G(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha)) > 0$. The assumption that u is a subsolution of $H_* = 0$ then implies that

$$F(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), D_{\mathbf{x}}\Psi(\alpha,\mathbf{x}_\alpha), D^2_{\mathbf{x}}\Psi(\alpha,\mathbf{x}_\alpha)) \leq 0. \tag{3.2}$$

Now a calculation shows that

$$D_{\mathbf{x}}\Psi(\alpha,\mathbf{x}) = \alpha\frac{\psi'(\alpha(|\mathbf{x}-\mathbf{y}_\diamond| - \rho_\diamond))}{|\mathbf{x}-\mathbf{y}_\diamond|}(\mathbf{x}-\mathbf{y}_\diamond),$$

and

$$\begin{aligned} D^2_{\mathbf{x}}\Psi(\alpha,\mathbf{x}) &= \alpha\frac{\psi'(\alpha(|\mathbf{x}-\mathbf{y}_\diamond| - \rho_\diamond))}{|\mathbf{x}-\mathbf{y}_\diamond|} I \\ &+ \Big(\alpha^2\frac{\psi''(\alpha(|\mathbf{x}-\mathbf{y}_\diamond| - \rho_\diamond))}{|\mathbf{x}-\mathbf{y}_\diamond|^2} - \alpha\frac{\psi'(\alpha(|\mathbf{x}-\mathbf{y}_\diamond| - \rho_\diamond))}{|\mathbf{x}-\mathbf{y}_\diamond|^3}\Big)(\mathbf{x}-\mathbf{y}_\diamond)\otimes(\mathbf{x}-\mathbf{y}_\diamond). \end{aligned}$$

Now we know that $\mathbf{x}_\alpha \to \mathbf{x}_\diamond$ and $\psi(\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\diamond| - \rho_\diamond)) \to 0$ and hence $\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\diamond| - \rho_\diamond) \to 0$ as $\alpha\to\infty$. Thus we see that if we define $\mathbf{n}_\alpha = \frac{1}{|\mathbf{x}_\alpha - \mathbf{y}_\diamond|}(\mathbf{x}_\alpha - \mathbf{y}_\diamond)$ then

$$\begin{gathered} \mathbf{n}_\alpha \to \mathbf{n}_\diamond, \\ \psi'(\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\diamond| - \rho_\diamond)) \to 1, \\ \frac{\psi'(\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\diamond| - \rho_\diamond))}{|\mathbf{x}_\alpha - \mathbf{y}_\diamond|} \to \frac{1}{\rho_\diamond}, \\ \psi''(\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\diamond| - \rho_\diamond)) \to -2\eta, \end{gathered}$$

as $\alpha\to\infty$.

If we let $\lambda = \alpha\psi'(\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\diamond| - \rho_\diamond))$ then we see that for sufficiently large α (and hence λ) we have

$$\alpha^2\psi''(\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\diamond| - \rho_\diamond)) \leq -\lambda^2\eta.$$

Thus we see (recall that $|\mathbf{x}_\alpha - \mathbf{x}_\diamond| \geq \rho_\diamond$) that

$$D^2_{\mathbf{x}}\Psi(\alpha,\mathbf{x}_\alpha) \leq \frac{\lambda}{\rho_\diamond} I - \eta\lambda^2\mathbf{n}_\alpha\otimes\mathbf{n}_\alpha,$$

Combining this result with the degenerate ellipticity of F and (3.1) we get a contradiction from inequality (3.2). □

Proof of Theorem 2.2. Suppose to the contrary that $G(\mathbf{x}_\diamond, u(\mathbf{x}_\diamond)) > 0$. In order to derive a contradiction we start by choosing a number of parameters and points.

Since G is lower semicontinuous there is a number $\epsilon > 0$ such that

$$G(\mathbf{x}, r) > 0, \quad |\mathbf{x}-\mathbf{x}_\diamond| \leq \epsilon, \quad x\in\partial\Omega, \quad |r - u(\mathbf{x}_\diamond)| \leq \epsilon. \tag{3.3}$$

By assumption (iv) there is a number k so that

$$
\begin{aligned}
&F(\mathbf{x}, r, \lambda\mathbf{n}, -\mu\lambda^2(\eta\mathbf{n}\otimes\mathbf{n} - I)) > 0 \quad \text{when} \\
&\quad \mathbf{x}\in\overline{\Omega},\ |\mathbf{x}-\mathbf{x}_\diamond| < \frac{1}{k},\ |r-u(\mathbf{x}_\diamond)| < \frac{1}{k},\ \lambda > k,\ |\mathbf{n}| = 1,\ \mu > k, \text{ and } \eta > k.
\end{aligned} \tag{3.4}
$$

Choose a number $m \ge 2$ so that

$$
m > \frac{2(k-1)}{\sin(\theta_\diamond)}, \quad \frac{2^{m+2}}{3\epsilon(m+1)(2-\sin(\theta_\diamond))^m} > k, \quad \epsilon\Big(1-\frac{\sin(\theta_\diamond)}{2}\Big)^{m+1} < \frac{1}{k}. \tag{3.5}
$$

Since u is upper semicontinuous there is a number $\delta \in (0, \min\{\epsilon, \beta_\diamond\})$ such that

$$
u(\mathbf{x}) < u(\mathbf{x}_\diamond) + 2^{-m-2}\epsilon, \quad |\mathbf{x}-\mathbf{x}_\diamond| < \delta, \quad \mathbf{x}\in\overline{\Omega}. \tag{3.6}
$$

and hence it follows from (3.3) that

$$
\text{if } |\mathbf{x}-\mathbf{x}_\diamond| \le \delta,\ x\in\partial\Omega, \text{ and } G(\mathbf{x}, u(\mathbf{x})) \le 0 \text{ then } u(\mathbf{x}) < u(\mathbf{x}_\diamond) - \epsilon. \tag{3.7}
$$

Choose $\beta \in (0, \beta_\diamond)$ so that

$$
\beta < \frac{\delta}{3}, \quad \beta < \frac{2}{5k}, \quad \beta < \frac{\epsilon(m+1)}{2^{2m+1}\, k}, \tag{3.8}
$$

and define

$$
\mathbf{y}_\beta = \mathbf{x}_\diamond + \beta\mathbf{n}_\diamond, \quad \beta\in(0,\beta_\diamond).
$$

As a test function we take

$$
\psi(\mathbf{x}) = \begin{cases} -\epsilon(2\beta)^{-m-1}\big(2\beta - |\mathbf{x}-\mathbf{y}_\beta|\big)^{m+1}, & |\mathbf{x}-\mathbf{y}_\beta| \le 2\beta, \\ 0, & |\mathbf{x}-\mathbf{y}_\beta| > 2\beta. \end{cases}
$$

Since $m \ge 2$ this function is twice continuously differentiable when $\mathbf{x} \ne \mathbf{y}_\beta$.

Since u is upper semicontinuous there is a point $\mathbf{x}_\beta \in \overline{\Omega}$, so that $|\mathbf{x}_\beta - y_\beta| \le 2\beta$ and

$$
u(\mathbf{x}_\beta) - \psi(\mathbf{x}_\beta) \ge u(\mathbf{x}) - \psi(\mathbf{x}), \quad |\mathbf{x}-\mathbf{y}_\beta| \le 2\beta, \quad \mathbf{x}\in\overline{\Omega}.
$$

If $|\mathbf{x}-\mathbf{x}_\diamond| < \delta < \beta_\diamond$ and $\mathbf{x}\in\overline{\Omega}$, then if follows from assumption (ii) that

$$
|\mathbf{x}-\mathbf{y}_\beta| \ge \beta\sin(\theta_\diamond), \tag{3.9}
$$

and hence we have

$$
u(\mathbf{x}_\diamond) - \psi(\mathbf{x}_\diamond) \le u(\mathbf{x}_\beta) - \psi(\mathbf{x}_\beta) \le u(\mathbf{x}_\beta) + \epsilon\Big(1-\frac{\sin(\theta_\diamond)}{2}\Big)^{m+1},
$$

so that we conclude, since $\psi(\mathbf{x}_\diamond) \le 0$ that

$$
u(\mathbf{x}_\beta) \ge u(\mathbf{x}_\diamond) - \epsilon\Big(1-\frac{\sin(\theta_\diamond)}{2}\Big)^{m+1}. \tag{3.10}
$$

then $|\mathbf{x}-\mathbf{x}_\diamond| < \delta$ since $\beta < \frac{\delta}{3}$ and hence by (3.6)

$$
u(\mathbf{x}) - \psi(\mathbf{x}) < u(\mathbf{x}_\diamond) + 2^{-m-2}\epsilon + 4^{-m-1}\epsilon < u(\mathbf{x}_\diamond) - \psi(\mathbf{x}_\diamond).
$$

Thus we see that we must have

$$
|\mathbf{x}_\beta - \mathbf{y}_\beta| < \frac{3}{2}\beta, \tag{3.11}
$$

that is, $\mathbf{x}_\beta$ is a local maximum point for $u-\psi$ in $\overline{\Omega}$.

Furthermore, since $\frac{3}{2}\beta < \epsilon$ we note by (3.7) that if $\mathbf{x}\in\partial\Omega$, $|\mathbf{x}-\mathbf{y}_\beta| < \frac{3}{2}\beta$ and $G(\mathbf{x}, u(\mathbf{x})) \le 0$, then $u(\mathbf{x}) < u(\mathbf{x}_\diamond) - \epsilon$ so that by (3.9) we have $u(\mathbf{x}) - \psi(\mathbf{x}) < u(\mathbf{x}_\diamond) - \epsilon + \epsilon\big(1-\frac{\sin(\theta_\diamond)}{2}\big)^{m+1} < u(\mathbf{x}_\diamond) < u(\mathbf{x}_\diamond) - \psi(\mathbf{x}_\diamond)$ so we conclude that if

$\mathbf{x}_\beta \in \partial\Omega$, then $G(\mathbf{x}_\beta, u(\mathbf{x}_\beta)) > 0$. Thus it follows from the assumption that u is a subsolution of the equation $H_* = 0$ that we in fact have

$$F\big(\mathbf{x}_\beta, u(\mathbf{x}_\beta), D\psi(\mathbf{x}_\beta), D^2\psi(\mathbf{x}_\beta)\big) \le 0.$$

It remains to show that this is a contradiction.

When we use the notation

$$\mathbf{n}_\beta = \frac{1}{|\mathbf{x}_\beta - \mathbf{y}_\beta|}(\mathbf{x}_\beta - \mathbf{y}_\beta),$$

we get

$$\begin{aligned} D\psi(\mathbf{x}_\beta) &= \epsilon(2\beta)^{-m-1}(m+1)\big(2\beta - |\mathbf{x}_\beta - \mathbf{y}_\beta|\big)^m \mathbf{n}_\beta, \\ D^2\psi(\mathbf{x}_\beta) &= \epsilon(2\beta)^{-m-1}(m+1)\big(2\beta - |\mathbf{x}_\beta - \mathbf{y}_\beta|\big)^m \frac{1}{|\mathbf{x}_\beta - \mathbf{y}_\beta|}(I - \mathbf{n}_\beta \otimes \mathbf{n}_\beta) \\ &\quad - \epsilon(2\beta)^{-m-1}m(m+1)\big(2\beta - |\mathbf{x}_\beta - \mathbf{y}_\beta|\big)^{m-1}\mathbf{n}_\beta \otimes \mathbf{n}_\beta. \end{aligned}$$

Let

$$\begin{gathered} \lambda = \epsilon(2\beta)^{-m-1}(m+1)(2\beta - |\mathbf{x}_\beta - \mathbf{y}_\beta|)^m, \\ \mu = \frac{1}{\lambda|\mathbf{x}_\beta - \mathbf{y}_\beta|}, \\ \eta = m\frac{|\mathbf{x}_\beta - \mathbf{y}_\beta|}{2\beta - |\mathbf{x}_\beta - \mathbf{y}_\beta|} + 1. \end{gathered}$$

Thus we see that

$$D\psi(\mathbf{x}_\beta) = \lambda\mathbf{n}_\beta \quad \text{and} \quad D^2\psi(\mathbf{x}_\beta) = -\mu\lambda^2(\eta\,\mathbf{n}_\beta \otimes \mathbf{n}_\beta - I).$$

From (3.9) and (3.11) a we get

$$\begin{gathered} \frac{\epsilon(m+1)(2 - \sin(\theta_\diamond))^m}{2^{m+1}\beta} \ge \lambda \ge \frac{\epsilon(m+1)}{2^{2m+1}\,\beta}, \\ \mu \ge \frac{2^{m+2}}{3\epsilon(m+1)(2 - \sin(\theta_\diamond))^m}, \\ \eta \ge m\frac{\sin(\theta_\diamond)}{2 - \sin(\theta_\diamond)} + 1. \end{gathered}$$

Now it follows from (3.5) that $\eta > k$ and $\mu > k$ and from (3.8) that $\lambda > k$. Furthermore, the last inequality in (3.5) guarantees by (3.6) and (3.10) that $|u(\mathbf{x}_\beta) - u(\mathbf{x}_\diamond)| < \frac{1}{k}$. Finally, since $|\mathbf{x}_\beta - \mathbf{y}_\beta| < \frac{3}{2}\beta$ and $\beta < \frac{2}{5k}$ by (3.8) we have $|\mathbf{x}_\beta - \mathbf{x}_\diamond| < \frac{1}{k}$. Thus it follows from (3.4) that $F\big(\mathbf{x}_\beta, u(\mathbf{x}_\beta), D\psi(\mathbf{x}_\beta), D^2\psi(\mathbf{x}_\beta)\big) > 0$ and we have a contradiction. □

Proof of the claims in Example 2.3. A straightforward calculation shows that φ is harmonic in the set $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} \ne (s, 0, \dots, 0), s \in [0, 1]\}$ and thus we see that the only point which may cause problems for the assumptions or the claim is the origin.

Using Fatou's lemma we immediately conclude that u is upper semicontinuous. If we define $k(s) = \varphi((s, s, 0, \dots, 0))$, then we deduce that $\lim_{s\to 0} k(s) = -\int_0^1 \frac{\omega(t)}{t}\,\mathrm{d}t$ by the dominated convergence theorem and the fact that $(t - s)^2 + s^2 \ge \frac{1}{2}t^2$, but $\lim_{t\uparrow 0} k'(t) = -(d-2)\int_0^1 \frac{\omega(t)}{t^2}\,\mathrm{d}t = -\infty$. Since $(s, s, 0, \dots, 0) \in \overline{\Omega}$ for s sufficiently small we conclude that there cannot be a function $\psi \in \mathcal{C}^2(\mathbb{R}^d)$ such that $u - \psi$ has a local maximum in $\overline{\Omega}$ at $\mathbf{0}$ and hence there is nothing to check in the definition of a viscosity subsolution at $\mathbf{0}$.

It remains to show that G is lower semicontinuous in $\partial\Omega \times \mathbb{R}$ and it suffices to show that

$$\liminf_{x\downarrow 0} \int_0^1 \frac{t^{d-3}\omega(t)}{\sqrt{(x-t)^2+\omega(x)^{\frac{2}{d-3}}x^2}^{\,d-2}}\,\mathrm{d}t \geq \int_0^1 \frac{\omega(t)}{t}\,\mathrm{d}t + \frac{1}{d-3}. \tag{3.12}$$

By the triangle inequality and a series expansion we have

$$\begin{aligned} &\int_0^1 \frac{t^{d-3}\omega(t)}{\sqrt{(x-t)^2+\omega(x)^{\frac{2}{d-3}}x^2}^{\,d-2}}\,\mathrm{d}t \\ &\geq \int_x^1 \frac{t^{d-3}\omega(t)}{(t-x(1-\omega(x)^{\frac{1}{d-3}}))^{d-2}}\,\mathrm{d}t \\ &= \int_x^1 \frac{\omega(t)}{t}\mathrm{d}t + \sum_{n=1}^{\infty} \binom{n+d-3}{d-3}\Big(1-\omega(x)^{\frac{1}{d-3}}\Big)^n x^n \int_x^1 \frac{\omega(t)}{t^{n+1}}\mathrm{d}t. \end{aligned} \tag{3.13}$$

Since ω is nondecreasing we have

$$\int_x^1 \frac{\omega(t)}{t^{n+1}}\mathrm{d}t \geq \frac{\omega(x)}{n}(x^{-n}-1).$$

This inequality implies that

$$\begin{aligned} &\sum_{n=1}^{\infty} \binom{n+d-3}{d-3}\Big(1-\omega(x)^{\frac{1}{d-3}}\Big)^n x^n \int_x^1 \frac{\omega(t)}{t^{n+1}}\mathrm{d}t \\ &\geq \omega(x)\sum_{n=1}^{\infty} \frac{1}{n}\binom{n+d-3}{d-3}\Big(1-\omega(x)^{\frac{1}{d-3}}\Big)^n \\ &\quad - \omega(x)\sum_{n=1}^{\infty} \frac{1}{n}\binom{n+d-3}{d-3}\Big(1-\omega(x)^{\frac{1}{d-3}}\Big)^n x^n. \end{aligned} \tag{3.14}$$

Because $\frac{1}{n}\binom{n+d-3}{d-3} \geq \frac{1}{d-3}\binom{n+d-4}{d-4}$ we get

$$\begin{aligned} &\omega(x)\sum_{n=1}^{\infty} \frac{1}{n}\binom{n+d-3}{d-3}\Big(1-\omega(x)^{\frac{1}{d-3}} Big)^n \\ &\geq \frac{\omega(x)}{d-3}\sum_{n=0}^{\infty} \binom{n+d-4}{d-4}\Big(1-\omega(x)^{\frac{1}{d-3}} Big)^n - \frac{\omega(x)}{d-3} \\ &= \frac{\omega(x)}{(d-3)(1-(1-\omega(x)^{\frac{1}{d-3}}))^{d-3}} - \frac{\omega(x)}{d-3} = \frac{1-\omega(x)}{d-3}. \end{aligned}$$

This inequality, (3.13) and (3.14) imply that (3.12) holds since $\lim_{x\downarrow 0}\omega(x) = 0$ and

$$\lim_{x\downarrow 0}\int_x^1 \frac{\omega(t)}{t}\mathrm{d}t = \int_0^1 \frac{\omega(t)}{t}\mathrm{d}t,$$

$$\lim_{x\downarrow 0}\sum_{n=1}^{\infty} \frac{\omega(x)}{n}\binom{n+d-3}{d-3}\Big(1-\omega(x)^{\frac{1}{d-3}}\Big)^n x^n = 0.$$

□

Proof of the claims in Example 2.4. A straightforward calculation shows that $\tilde{F}$ is nonincreasing in its last variable when $\mathbf{p} \neq \mathbf{0}$ and then F has the same property (for all $\mathbf{p}$). Since $F(\mathbf{x}, r, \mathbf{0}, X) = -\infty$ when $X \geq 0$ it is clear that u is a subsolution of $F = 0$ in Ω so it remains to check the boundary points and we may without loss of generality assume that $\mathbf{x}_0 = (1, 0)$. Assume thus that $u(\mathbf{x}) \leq \psi(\mathbf{x})$ for all $\mathbf{x} \in \overline{\Omega}$ with $|\mathbf{x} - \mathbf{x}_0| < \delta$ for some $\delta > 0$ and $u(\mathbf{x}_0) = \psi(\mathbf{x}_0)$. Clearly $\psi_x(\mathbf{x}_0) \leq 0$ and since the function $t \mapsto \psi(\cos(t), \sin(t))$ has a local minimum at $t = 0$ we see that $\psi_y(\mathbf{x}_0) = 0$ and $\psi_{yy}(\mathbf{x}_0) \geq 0$. If $\psi_x(\mathbf{x}_0) = 0$ it follows from the fact that $\psi_{yy}(\mathbf{x}_0) \geq 0$ that $F(\mathbf{x}_0, 1, D\psi(\mathbf{x}_0), D^2\psi(\mathbf{x}_0)) = -\infty$ and if $\psi_x(\mathbf{x}_0) < 0$ it follows from that fact that $\psi_y(\mathbf{x}_0) = 0$ that

$$F(\mathbf{x}_0, 1, D\psi(\mathbf{x}_0), D^2\psi(\mathbf{x}_0)) = \tilde{F}(\mathbf{x}_0, 1, D\psi(\mathbf{x}_0), D^2\psi(\mathbf{x}_0)) = -\frac{\psi_{yy}(\mathbf{x}_0)}{|\psi_x(\mathbf{x}_0)|} \leq 0.$$

□

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Corrigendum: Posted September 5, 2006

In Example 2.4 (page 4), replace "$\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1\}$" by

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x < 1\}$$

and in the proof of the claims in Example 2.4 (page 10), replace "the function $t \mapsto \psi(\cos(t), \sin(t))$" by "the function $t \mapsto \psi(1, t)$".

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