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# VISCOSITY SOLUTIONS OF THE CAUCHY PROBLEM FOR SECOND-ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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Abstract. In this paper we prove the existence and uniqueness of viscosity solutions of the Cauchy problem for the second order nonlinear partial differential equations in Hilbert spaces.

## 1. Introduction

The theory of scalar partial differential equations in infinite dimensional Hilbert spaces has been developing very rapidly in recent years. The object of its study is first and second order PDE's of the form

$$G(x, u(x), Du(x), D^2u(x)) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega$ is a subset of Hilbert space $H, u$ is a real valued and $Du(x)$ and $D^2u(x)$ correspond respectively to the first and second order Fréchet derivatives of $u$. Identifying $H$ with its dual, $Du(x)$ corresponds to an element of $H$ and $D^2u(x)$ to an element of $S(H)$, the space of bounded, self-adjoint operators equipped with the operator norm. Therefore,

$$G : W \subset H \times \mathbb{R} \times H \times S(H) \mapsto \mathbb{R}$$

is appropriate. If the set $W$ is open in $H \times \mathbb{R} \times H \times S(H)$ and $G$ is locally bounded we call equation (1.1) *bounded*. It may however happen that $W$ is just dense in $H \times \mathbb{R} \times H \times S(H)$ and $G$ is not locally bounded. In such case we refer to (1.1) as to being *unbounded*.

The unbounded equations are of importance since they appear as dynamic programming equations associated with problems of optimal control and differential games. Roughly speaking, if one controls an infinite dimensional system governed by an ODE in a Hilbert space, one has to deal with a first order stationary or time dependent PDE, while controlling a system for which the state equation is a stochastic PDE gives rise to a second order stationary or time dependent PDE. "Unboundedness" arises when the state equation of a system involves unbounded operators or, in the stochastic case, for instance, so called "white noise".

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More precisely, we will study the Cauchy problem for the fully nonlinear PDE's having the form

$$\begin{gathered} u_t(t,x) + \langle Ax, Du(t,x)\rangle + F(t,x,Du(t,x),D^2u(t,x)) = 0 \\ (t,x) \in (0,T)\times H \\ u(0,x) = g(x) \quad \text{for } x \in H, \end{gathered} \tag{1.2}$$

where $H$ is a real, separable Hilbert space endowed with the inner product $\langle .,.\rangle$ and the norm $|.|$ and $A : \mathcal{D}(A) \subset H \to H$ is closed linear operator that generates an analytic $C_0$-semigroup $e^{-tA}$ on $H$. Moreover, we assume that $A$ is positive definite and self-adjoint and has compact resolvent $R(\mu, A)$.

There is an increasing interest in and a growing literatures on Hamilton-Jacobi equations in infinite dimensions. These equations were first studied by Barbu and Da Prato [1], setting the problem in classes of convex functions and using semigroup and perturbation methods. Much progress has been made recently due to the introduction of notion of viscosity solutions. We refer the reader to [4, 6, 8, 14, 15, 16] for the first order equations. As regards the second order, "bounded" equations have been investigate in [11, Parts I and III], and "unbounded" in [11, Part II], [5, 7, 9, 13, 17]. Except for [5, 6, 7, 17] the unboundedness in the studied equations was always coming from the term $\langle Ax, Du(x)\rangle$. This paper is concerned with equations that exhibit "bad behavior" in the $F$ also in $Du$ and $D^2u$ as the same as in [5, 6, 7, 17]. We notice that, [5] studied the stationary version of (1.2), [7] studied (1.2) with $F(t,x,Du(t,x),D^2u(t,x))$ has form $-\operatorname{Trace}(QD^2u(t,x)) + G(t,x,Du(t,x))$ and used a different test functions.

The plan of the paper is the following. In section 2 we give some preliminaries. In section 3 we present the definition of viscosity solutions and prove a general uniqueness and existence results for (1.2).

## 2. Preliminaries

For any Hilbert spaces $X, Y$ and $E$, we denote

$$\begin{gathered} UC(X) = \{u : X \to \mathbb{R}; u \text{ is uniformly continuous}\}, \\ BUC(X) = \{u \in UC(X); u \text{ is bounded}\}, \\ UC_x([0,T]\times X) = \{u \in C([0,T]\times X); u(t,.) \in UC(X) \text{ uniformly in } t \in [0,T]\}, \\ BUC_x([0,T]\times X) = \{u \in UC_x([0,T]\times X); u \text{ is bounded}\}. \end{gathered}$$

Let $u : (0,T)\times E \to \mathbb{R}$. If $(\hat t,\hat x) \in (0,T)\times E$ and $(a,p,Z) \in \mathbb{R}\times E\times S(E)$ we say that $(a,p,Z) \in P^{2,+}u(\hat t,\hat x)$ provided that (see [3])

$$\begin{aligned} u(t,x) \leqslant{}& u(\hat t,\hat x) + a(t-\hat t) + \langle p, x-\hat x\rangle + \frac{1}{2}\langle Z(x-\hat x), x-\hat x\rangle \\ &+ o(|x-\hat x|^2 + |t-\hat t|) \quad \text{as } (t,x) \to (\hat t,\hat x). \end{aligned}$$

The closure of $P^{2,+}$, $\bar P^{2,+}$, is defined as follows:

$$\begin{aligned} \bar P^{2,+}u(t,x) = \big\{&(a,p,Z) \in \mathbb{R}\times E\times S(E) : \exists (t_n,x_n,a_n,p_n,Z_n) \text{ in} \\ &(0,T)\times E\times\mathbb{R}\times E\times S(E) : (a_n,p_n,Z_n) \in P^{2,+}u(t_n,x_n) \\ &\text{and } (t_n,x_n,u(t_n,x_n),a_n,p_n,Z_n) \to (t,x,u(t,x),a,p,Z)\big\}. \end{aligned}$$

We are interested in the situation where $E = E_1 \times E_2$ is the product of two spaces and $u(t,x_1,x_2) = u_1(t,x_1) + u_2(t,x_2)$. Proposition below is a straightforward corollary from [2, Theorem 8.3]. In this paper, identify operator in any space is denoted by a same symbol $I$.

**Proposition 2.1.** *Let $u_i, i = 1, 2$ be upper semicontinuous on $(0, T) \times \mathbb{R}^N$ and $\varphi : (0, T) \times \mathbb{R}^{2N} \to \mathbb{R}$ be once continuously differentiable in $t$ and twice continuously differentiable in $x$. Suppose that*

$$u_1(t, x_1) + u_2(t, x_2) - \varphi(t, x_1, x_2)$$

*has a local maximum at $(\hat{t}, \hat{x}) = (\hat{t}, \hat{x}_1, \hat{x}_2) \in (0, T) \times \mathbb{R}^{2N}$ and that*

$$D^2\varphi(\hat{t}, \hat{x}) = D = D_1 - D_2,$$

*where $D_1, D_2 \in S(\mathbb{R}^N)$ and $D_1, D_2 \geq 0$. Assume, moreover, that*

$$\begin{aligned} &\exists r > 0 : \forall M > 0, \exists c > 0 \text{ such that for } i = 1, 2 \\ b_i \leqslant c \text{ if } &(b_i, q_i, Z_i) \in P^{2,+}u_i(t, x_i),\ |x_i - \hat{x}_i| + |t - \hat{t}| < r \\ &\text{and } |u_i(t, x_i)| + |q_i| + \|Z_i\| \leqslant M. \end{aligned} \tag{2.1}$$

*Then, for every $\alpha > 0$ there are $Z_1, Z_2 \in S(\mathbb{R}^N)$ such that*

(i) *$(b_i, D_{x_i}\varphi(\hat{t}, \hat{x}), Z_i) \in \bar{P}^{2,+}u_i(\hat{t}, \hat{x}_i)$, for $i = 1, 2$;*
(ii) *$-(\|D_1\| + \|D_2\|)(1 + \frac{2}{\alpha})I \leqslant \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \leqslant D + \alpha(D_1 + D_2)$;*
(iii) *$b_1 + b_2 = \varphi_t(\hat{t}, \hat{x})$.*

The norm of the symmetric matrix used above is

$$\|\phi\| = \sup\big\{|\lambda| : \lambda \text{ is an eigenvalue of } \phi\big\} = \sup\big\{|\langle \phi\xi, \xi\rangle| : |\xi| \leqslant 1\big\}.$$

**Remark 2.2.** The condition (2.1) will be satisfied if $u_i$ are the viscosity subsolutions (see [3]) of

$$(u_i)_t(t, x_i) + F(t, x_i, u_i(t, x_i), Du_i(t, x_i), D^2u_i(t, x_i)) \leqslant 0 \text{ in } (0, T) \times \mathbb{R}^N$$

with $F$ bounded on bounded sets.

We say that a function $\rho : [0, +\infty) \to [0, +\infty)$ is a *modulus* if $\rho$ is continuous, nondecreasing, subadditive, and $\rho(0) = 0$. Subadditivity in particular implies that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\rho(r) \leqslant \varepsilon + C_\varepsilon r, \quad \text{for every } r \geq 0.$$

Moreover, a function $\omega : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a *local modulus* if $\omega$ is continuous, nondecreasing in both variables, subadditive in the first variable, and $\omega(0, r) = 0$, for every $r \geq 0$.

We assume the following hypothesis:

- (A1):$\mathcal{D}(A) \subset H \to H$ is a self-adjoint operator, there exists $a > 0$ such that $\langle Ax, x\rangle \geq a|x|^2$ for all $x \in \mathcal{D}(A)$, and $A^{-1}$ is compact.

**Remark 2.3.** Hypothesis (A1) implies in particular that $-A$ is the infinitesimal generator of an analytic semigroup with compact resolvent satisfying $\|e^{-tA}\| \leqslant e^{-at}$ for all $t \geq 0$ and that there is an orthonormal basis of $H$ made of eigenvectors of $A$ such that the corresponding sequence of eigenvalues diverges to $+\infty$ as $n \to \infty$. It also follows that

We also assume the *Interpolation inequality:* Let $\gamma \in (0,1], \alpha \in (0,\gamma)$. For every $\sigma > 0$, there exists $C_\sigma > 0$ such that

$$|A^\alpha z| \leqslant \sigma |A^\gamma z| + C_\sigma |z|, \quad \forall z \in \mathcal{D}(A^\gamma). \tag{2.2}$$

Let $H_1 \subset H_2 \subset \cdots$ be finite dimensional subspaces of $H$ generated by eigenvectors of $A$ such that $\overline{\cup_{N=1}^\infty H_N} = H$. Given $N \in \mathbb{N}$, denote by $P_N$ the orthogonal projection in $H$ onto $H_N$, let $Q_N = I - P_N$ and let $H_N^\perp = Q_N H$. We then have an orthogonal decomposition $H = H_N \times H_N^\perp$ and we will denote by $x_N$ an element of $H_N$ and by $x_N^\perp$ an element of $H_N^\perp$. For $x \in H$, we will write $x = (P_N x, Q_N x)$. We make the following assumptions about $F$.

- (F1) There exists $\beta \in (0,1)$ such that the function $F : [0,T] \times \mathcal{D}(A^{\frac{\beta}{2}}) \times \mathcal{D}(A^{\frac{\beta}{2}}) \times S(H) \to \mathbb{R}$ is continuous (in the topology of $[0,T] \times \mathcal{D}(A^{\frac{\beta}{2}}) \times \mathcal{D}(A^{\frac{\beta}{2}}) \times S(H)$);
- (F2) $F(t,x,p,S_1) \leqslant F(t,x,p,S_2), \forall t \in (0,T), \forall x, p \in \mathcal{D}(A^{\frac{\beta}{2}})$, and all $S_1 \geq S_2$;
- (F3) There exists a modulus $\rho$ such that
$$\begin{aligned}&|F(t,x,p,S_1) - F(t,x,q,S_2)| \\ &\leqslant \rho\big((1+|A^{\frac{\beta}{2}}x|)|A^{\frac{\beta}{2}}(p-q)| + (1+|A^{\frac{\beta}{2}}x|^2)\|S_1 - S_2\|\big),\end{aligned}$$
for all $t \in (0,T)$, all $x,p,q \in \mathcal{D}(A^{\frac{\beta}{2}})$ and all $S_1, S_2 \in S(H)$;
- (F4) There exist $0 < \eta < 1 - \beta$ and a modulus $\omega$ such that, for all $\varepsilon > 0$, all $N \geq 1$, all $t \in (0,T)$, all $x, y \in \mathcal{D}(A^{\frac{\beta}{2}})$ and $X, Y \in S(H_N)$ such that
$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{2}{\varepsilon} \begin{pmatrix} P_N A^{-\eta} P_N & -P_N A^{-\eta} P_N \\ -P_N A^{-\eta} P_N & P_N A^{-\eta} P_N \end{pmatrix} \tag{2.3}$$
we have
$$\begin{aligned}&F\Big(t, x, \frac{A^{-\eta}(x-y)}{\varepsilon}, X\Big) - F\Big(t, y, \frac{A^{-\eta}(x-y)}{\varepsilon}, Y\Big) \\ &\geq -\omega\Big(|A^{\frac{\beta}{2}}(x-y)|\big(1 + \frac{|A^{\frac{\beta}{2}}(x-y)|}{\varepsilon}\big)\Big);\end{aligned}$$
- (F5) For every $R < +\infty$, $|\lambda| \leqslant R$, $t \in (0,T)$, $x, p \in \mathcal{D}(A^{\frac{\beta}{2}})$, we have
$$\sup\big\{|F(t,x,p,S+\lambda Q_N) - F(t,x,p,S)| : \|S\| \leqslant R, S = P_N S P_N\big\} \to 0$$
as $N \to \infty$.

**Remark 2.4.** By the properties of moduli, condition (F3) guarantees that there exists a constant $C$ such that for all $t \in (0,T)$, all $x, p \in \mathcal{D}(A^{\frac{\beta}{2}})$, all $S \in S(H)$,

$$|F(t,x,p,S)| \leqslant C\Big(1 + (1+|A^{\frac{\beta}{2}}x|)|A^{\frac{\beta}{2}}p| + (1+|A^{\frac{\beta}{2}}x|^2)\|S\|\Big) + |F(t,x,0,0)|. \tag{2.4}$$

## 3. Viscosity solutions

The definition of a viscosity solution proposed here has its predecessors in [5, 6, 13].

**Definition 3.1.** A function $\psi : H \to \mathbb{R}$ is a *test function* for the equation in (1.2) if

$$\psi(t,x) = \varphi(t,x) + \delta(t)(1+|x|^2),$$

where

1. $\delta \in C^1((0,T))$ and $\delta > 0$ in $(0,T)$;
2. $\varphi \in C^{1,2}((0,T) \times H)$ and is weakly sequentially lower semicontinuous;

(3) $D\varphi(t,.) \in UC(H,H) \cap UC(\mathcal{D}(A^{\frac{1}{2}-k}), \mathcal{D}(A^{1/2}))$, for some $k = k(\varphi) > 0$ and for all $t \in (0,T)$;
(4) $D^2\varphi(t,.) \in BUC(H, S(H))$, for all $t \in (0,T)$.

**Definition 3.2.** A weakly sequentially upper (lower) semicontinuous function $u : (0,T) \times H \to \mathbb{R}$ is a *viscosity subsolution* (respectively: *viscosity supersolution*) of the equation in (1.2) if for every test function $\psi$, whenever $u - \psi$ has a local maximum (respectively: $u + \psi$ has a local minimum) at $(t,x)$ then $x \in \mathcal{D}(A^{1/2})$ and

$$\psi_t(t,x) + \langle A^{1/2}x, A^{1/2}D\psi(t,x)\rangle + F(t,x,D\psi(t,x),D^2\psi(t,x)) \leqslant 0$$

(resp.

$$-\psi_t(t,x) + \langle A^{1/2}x, -A^{1/2}D\psi(t,x)\rangle + F(t,x,-D\psi(t,x),-D^2\psi(t,x)) \geq 0).$$

A function $u$ is a *viscosity solution* of the equation in (1.2) if it is both a viscosity subsolution and a viscosity supersolution.

The main result of this paper is theorem below.

**Theorem 3.1.** *Let the Hypothesis (A1) and (F1)-(F5) hold.*
**Comparison:** *Let $u, v : (0,T) \times H \to \mathbb{R}$ be respectively a viscosity subsolution and a viscosity supersolution of the equation in* (1.2)*. Assume that there exists a constant $C$ such that*

$$u(t,x), -v(t,x), |g(x)| \leqslant C(1+|x|) \tag{3.1}$$

*and*

$$\begin{aligned} (i) \quad & \lim_{t\downarrow 0}(u(t,x)-g(x))^+ = 0 \\ (ii) \quad & \lim_{t\downarrow 0}(v(t,x)-g(x))^- = 0 \end{aligned} \tag{3.2}$$

*uniformly on the bounded subsets in $H$. Then we have that $u \leqslant v$ in $(0,T) \times H$.*
**Existence:** *Let $g \in BUC(H)$ and*

$$F_R = \sup\{|F(t,x,p,X)| : (t,x) \in [0,T] \times \mathcal{D}(A^{1/2}), |p|, \|X\| \leqslant R\} < +\infty. \tag{3.3}$$

*Then* (1.2) *has a unique solution $u \in BUC_x([0,T] \times H) \cap BUC_x([\tau,T] \times \mathcal{D}(A^{-\frac{\eta}{2}}))$ for $\tau > 0$, satisfying $\lim_{t\downarrow 0} u(t,x) = g(x)$ in $H$. Moreover, there is a modulus $m$ such that*

$$|u(t,x) - u(s, e^{-(t-s)A}x)| \leqslant m(t-s)$$

*for $0 \leqslant s \leqslant t \leqslant T$ and $x \in H$.*

Before we can attempt to prove the above theorem we would like to begin with some facts about viscosity solutions of parabolic partial differential equations in finite dimensional spaces. Those facts will be needed in the proofs of Theorem 3.1. For the definition of viscosity solutions in this case, we refer to [2].

**Proposition 3.2** ([13, Proposition 3.4])**.** *Let an upper semicontinuous function $u$ and a lower semicontinuous function $v$ on $(0,T) \times \mathbb{R}^N$ be respectively a viscosity subsolution and a viscosity supersolution of*

$$u_t(t,x) + F(t,x,Du(t,x),D^2u(t,x)) = 0 \quad \text{for } t \in (0,T), x \in \mathbb{R}^N, \tag{3.4}$$

*where $F : ([0,T] \times \mathbb{R}^N \times \mathbb{R}^N \times S(\mathbb{R}^N) \to \mathbb{R}$ is continuous and satisfies the following three conditions:*

(i) *$F(t,x,p,S_1) \leqslant F(t,x,p,S_2)$, for all $t \in (0,T)$, all $x, p \in \mathbb{R}^N$, all $S_1 \geq S_2$;*

(ii) *There exists* $\mu \in C^2(\mathbb{R}^N)$, *radial, nondecreasing, nonnegative,* $\mu \to \infty$ *as* $\|x\| \to \infty$, $D\mu, D^2\mu$ *are bounded and*

$$F(t,x,p+\alpha D\mu(x), X+\alpha D^2\mu(x)) \geq F(t,x,p,X) - \sigma(\alpha, |p|+\|X\|) \forall x,p,X, \forall \alpha \geq 0, \tag{3.5}$$

*where* $\sigma$ *is a local modulus;*

(iii) *There exists a modulus* $\omega$*: so that for all* $t \in (0,T)$*, all* $x, y \in \mathbb{R}^N$*, all* $X, Y \in S(\mathbb{R}^N)$ *such that*

$$-c_1 I \leqslant \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leqslant c_2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

*with the constants* $c_1, c_2 \geq 0$*, we have*

$$F(t,x,c_3(x-y),X) - F(t,y,c_3(x-y),-Y) \geq -\omega(|x-y|(1+(c_1+c_2+|c_3|)|x-y|)).$$

*Let* $g \in BUC(\mathbb{R}^N)$*. Then*

(i) $u(t,x) - v(t,x) \leqslant \sup_{z\in\mathbb{R}^N}(u(0,z)-v(0,z))^+$ *for all* $t \in [0,T]$ *and* $x \in \mathbb{R}^N$*.*

(ii) *If* $u(0,x) \leqslant g(x) \leqslant v(0,x)$ *and* $u, -v \leqslant M$*, then there is a modulus of continuity* $m$*, depending only on* $M, \omega$ *and a modulus of continuity of* $g$*, such that*

$$u(t,x) - v(t,y) \leqslant m(|x-y|)$$

*for all* $t \in [0,T)$ *and* $x, y \in \mathbb{R}^N$*. Moreover, if* $u = v$*, then* $u \in C([0,T] \times \mathbb{R}^N)$*.*

(iii) *If* $\sup_{t\in(0,T),x\in\mathbb{R}^N} |F(t,x,0,0)| = K < +\infty$*, then there exists a unique solution* $u \in BUC_x([0,T]\times\mathbb{R}^N)$ *of* (3.4) *such that* $u(0,x) = g(x)$ *and* $\|u\|_\infty$ *only depends on* $\|g\|_\infty$ *and* $K$*.*

The Proposition below is needed in the proof of existence.

**Proposition 3.3** ([13, Lemma 2.8])**.** *If* $F : (0,T)\times H\times H\times S(H) \to \mathbb{R}$ *is uniformly continuous on bounded sets, and satisfies (F2) and (F5) then for every* $(t,x,p,X) \in (0,T)\times H\times H\times S(H)$*,*

$$F(t, P_N x, P_N p, P_N X P_N) \to F(t,x,p,X) \quad \text{as } N \to \infty.$$

**Proof of Theorem 3.1: Comparison.** Given $\mu > 0$, define

$$u_\mu(t,x) = u(t,x) - \frac{\mu}{T-t}, \quad v_\mu(t,x) = v(t,x) + \frac{\mu}{T-t}.$$

Then $u_\mu$ and $v_\mu$ satisfy respectively

$$(u_\mu)_t(t,x) + \langle Ax, Du_\mu(t,x)\rangle + F(t,x,Du_\mu(t,x), D^2u_\mu(t,x)) \leqslant -\frac{\mu}{(T-t)^2}$$

and

$$(v_\mu)_t(t,x) + \langle Ax, Dv_\mu(t,x)\rangle + F(t,x,Dv_\mu(t,x), D^2v_\mu(t,x)) \geq \frac{\mu}{(T-t)^2}$$

For $\epsilon, \delta, \gamma > 0, 0 < t_\delta < T$ we consider the function

$$\begin{aligned}\Phi(t,s,x,y) :=& u_\mu(t,x) - v_\mu(s,y) - \frac{|A^{-\frac{\eta}{2}}(x-y)|^2}{2\varepsilon} \\ &- \delta e^{K_\mu t}(1+|x|^2) - \delta e^{K_\mu s}(1+|y|^2) - \frac{(t-s)^2}{2\gamma}.\end{aligned}$$

The constant $K_\mu$ will be chosen later.

Since the function $\Phi$ is weakly sequentially upper semicontinuous in $(0,T) \times (0,T) \times H \times H$ and (3.1), $\Phi$ has a global maximum over $[t_\delta, T) \times [t_\delta, T) \times H \times H$ at some points $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$, where $\bar{t}, \bar{s} < T$ and $\bar{x}, \bar{y}$ bounded independently of $\varepsilon$ for a fixed $\delta$. We can assume this maximum to be strict and (see [6, 8])

$$\limsup_{\delta\to 0} \limsup_{\varepsilon\to 0} \limsup_{\gamma\to 0} \delta(|\bar{x}|^2 + |\bar{y}|^2) = 0 \tag{3.6}$$

$$\limsup_{\varepsilon\to 0} \limsup_{\gamma\to 0} \frac{|A^{-\frac{\eta}{2}}(\bar{x}-\bar{y})|^2}{2\varepsilon} = 0 \quad \text{for fixed } \delta > 0. \tag{3.7}$$

$$\limsup_{\gamma\to 0} \frac{(\bar{t}-\bar{s})^2}{2\gamma} = 0 \quad \text{for fixed } \varepsilon, \delta. \tag{3.8}$$

If $u \not\leqslant v$ it then follows from (3.7), (3.8) and (3.2) that for small $\mu$ and $\delta$, and $t_\delta$ sufficiently close 0 we have $\bar{t}, \bar{s} > t_\delta$ if $\gamma$ and $\varepsilon$ sufficiently small.

We will now use a rather standard technique of reduction to finite dimensional spaces to produce appropriate test functions.

We now fix $N \in \mathbb{N}$. Then obviously

$$|A^{-\frac{\eta}{2}}(x-y)|^2 = \langle P_N A^{-\eta} P_N(x-y), x-y\rangle + |A^{-\frac{\eta}{2}} Q_N(x-y)|^2,$$

and we have

$$\begin{aligned} |A^{-\frac{\eta}{2}} Q_N(x-y)|^2 \leqslant & 2\langle Q_N A^{-\eta} Q_N(\bar{x}-\bar{y}), x-y\rangle - \langle Q_N A^{-\eta} Q_N(\bar{x}-\bar{y}), \bar{x}-\bar{y}\rangle \\ & + 2|A^{-\frac{\eta}{2}} Q_N(x-\bar{x})|^2 + 2|A^{-\frac{\eta}{2}} Q_N(y-\bar{y})|^2 \end{aligned}$$

with equality if and only if $x = \bar{x}$, $y = \bar{y}$. Therefore, if we define

$$\begin{aligned} u_1(t,x) =& u_\mu(t,x) - \frac{\langle x, Q_N A^{-\eta} Q_N(\bar{x}-\bar{y})\rangle}{\varepsilon} \\ & + \frac{\langle Q_N A^{-\eta} Q_N(\bar{x}-\bar{y}), \bar{x}-\bar{y}\rangle}{2\varepsilon} - \frac{|A^{-\frac{\eta}{2}} Q_N(x-\bar{x})|^2}{\varepsilon} - \delta e^{K_\mu t}(1+|x|^2) \end{aligned}$$

and

$$v_1(s,y) = v_\mu(s,y) - \frac{\langle y, Q_N A^{-\eta} Q_N(\bar{x}-\bar{y})\rangle}{\varepsilon} + \frac{|A^{-\frac{\eta}{2}} Q_N(y-\bar{y})|^2}{\varepsilon} + \delta e^{K_\mu s}(1+|y|^2),$$

it follows that the function

$$\tilde{\Phi}(t,s,x,y) := u_1(t,x) - v_1(s,y) - \frac{\langle P_N A^{-\eta} P_N(x-y), x-y\rangle}{2\varepsilon} - \frac{(t-s)^2}{2\gamma} \tag{3.9}$$

always satisfies $\tilde{\Phi} \leqslant \Phi$ and attains a strict global maximum over $[t_\delta, T) \times [t_\delta, T) \times H \times H$ at $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$. Moreover,

$$\tilde{\Phi}(\bar{t}, \bar{s}, \bar{x}, \bar{y}) = \Phi(\bar{t}, \bar{s}, \bar{x}, \bar{y}).$$

We now define, for $x_N, y_N \in H_N$, the functions

$$\tilde{u}_1(t, x_N) := \sup_{x_N^\perp \in H_N^\perp} u_1(t, x_N, x_N^\perp), \quad \tilde{v}_1(s, y_N) := \inf y_N^\perp \in H_N^\perp v_1(s, y_N, y_N^\perp).$$

Since the assumptions about $u, -v$ and the weakly sequentially continuity of inner product, we obtain that $\tilde{u}_1$ and $-\tilde{v}_1$ are upper semicontinuous on $(0,T) \times H_N$ (see [3]). Moreover, by definition of $u_1, v_1$ and by the form of $\tilde{\Phi}$, it follows that

$$\tilde{u}_1(\bar{t}, P_N\bar{x}) = u_1(\bar{t}, \bar{x}), \qquad \tilde{v}_1(\bar{s}, P_N\bar{y}) = v_1(\bar{s}, \bar{y}). \tag{3.10}$$

Defining now the map $\Phi_N : (0,T)\times(0,T)\times H_N \times H_N \to \mathbb{R}$ as

$$\begin{aligned}&\Phi_N(t,s,x_N,y_N)\\&:= \tilde u_1(t,x_N) - \tilde v_1(s,y_N) - \frac{\langle P_N A^{-\eta} P_N(x_N-y_N), x_N-y_N\rangle}{2\varepsilon} - \frac{(t-s)^2}{2\gamma}\\&= \sup_{x_N^\perp, y_N^\perp \in H_N^\perp} \tilde\Phi\big(t,s,(x_N,x_N^\perp),(y_N,y_N^\perp)\big).\end{aligned}$$

It is not difficult to check that $\Phi_N$ attains a strict global maximum at $(\bar t,\bar s,\bar x_N,\bar y_N) = (\bar t,\bar s,P_N\bar x,P_N\bar y)$. By the finite dimensional results (see [2]) for every $n\in\mathbb{N}$, there exist points $t^n, s^n \in (0,T); x_N^n, y_N^n \in H_N$ such that

$$t^n \to \bar t, \quad s^n \to \bar s; \quad x_N^n \to \bar x_N, \quad y_N^n \to \bar y_N, \quad \text{as } n\to\infty. \tag{3.11}$$

$$\tilde u_1(t^n,x_N^n) \to \tilde u_1(\bar t,\bar x_N), \quad \tilde v_1(s^n,y_N^n) \to \tilde v_1(\bar s,\bar y_N) \quad \text{as } n\to\infty. \tag{3.12}$$

and there exist functions $\varphi_n, \psi_n \in C^{1,2}((0,T)\times H_N)$ with uniformly continuous derivatives such that $\tilde u_1 - \varphi_n$ and $-\tilde v_1 + \psi_n$ have unique, strict, global maxima at $(t^n,x_N^n)$ and $(s^n,y_N^n)$ respectively, and

$$\begin{aligned}(\varphi_n)_t(t^n,x_N^n) &\to \frac{\bar t-\bar s}{\gamma},\\ D\varphi_n(t^n,x_N^n) &\to \frac{1}{\varepsilon}P_N A^{-\eta}P_N(\bar x_N-\bar y_N),\\ (\psi_n)_t(s^n,y_N^n) &\to \frac{\bar t-\bar s}{\gamma},\\ D\psi_n(s^n,y_N^n) &\to \frac{1}{\varepsilon}P_N A^{-\eta}P_N(\bar x_N-\bar y_N),\\ D^2\varphi_n(t^n,x_N^n) &\to X_N,\\ D^2\psi_n(s^n,y_N^n) &\to Y_N\end{aligned} \tag{3.13}$$

where $X_N, Y_N$ satisfy (2.3).

Consider finally the map $\Phi_N^n : (0,T)\times(0,T)\times H\times H \to \mathbb{R}$ defined as

$$\Phi_N^n(t,s,x,y) := u_1(t,x) - v_1(s,y) - \varphi_n(t,P_Nx) + \psi_n(s,P_Ny). \tag{3.14}$$

This map has the variables split and, by the definition of $u_1$ and $v_1$, attains its global maximum (which we can assume to be strict) at some point $(\hat t^n,\hat s^n,\hat x^n,\hat y^n)$. This point depends also on $N$ but we will drop this dependence since $N$ is now fixed. Repeating now the arguments of [5, page 409] (see also [17]) it is not difficult to show that

$$u_1(\hat t^n,\hat x^n) \to u_1(\bar t,\bar x), \quad v_1(\hat s^n,\hat y^n) \to v_1(\bar s,\bar y) \tag{3.15}$$

$$\hat t^n = t^n, \quad \hat s^n = s^n; \quad \hat x_N^n = x_N^n, \quad \hat y_N^n = y_N^n, \quad (\hat x^n,\hat y^n) \to (\bar x,\bar y) \tag{3.16}$$

as $n\to\infty$. Moreover $\bar x,\bar y \in \mathcal{D}(A^{1/2})$ and

$$A^{1/2}\hat x^n \rightharpoonup A^{1/2}\bar x, \quad A^{1/2}\hat y^n \rightharpoonup A^{1/2}\bar x \quad \text{as } n\to\infty. \tag{3.17}$$

We define

$$\psi(t,x) = \frac{\langle x, Q_N A^{-\eta}Q_N(\bar x-\bar y)\rangle}{\varepsilon} + \frac{|A^{-\frac{\eta}{2}}Q_N(x-\bar x)|^2}{\varepsilon} + \varphi_n(t,P_Nx) + \delta e^{K_\mu t}(1+|x|^2).$$

Then $\psi$ satisfies the conditions of a test function (Definition 3.1) and it follows from (3.14) and the definitions of $u_1, v_1$ that $u_\mu - \psi$ has a maximum at $(\hat{t}^n, \hat{x}^n)$. Thus we have

$$\begin{aligned} &\psi_t(\hat{t}^n, \hat{x}^n) + \langle A^{1/2}\hat{x}^n, A^{1/2}D\psi(\hat{t}^n, \hat{x}^n)\rangle \\ &+ F(\hat{t}^n, \hat{x}^n, D\psi(\hat{t}^n, \hat{x}^n), D^2\psi(\hat{t}^n, \hat{x}^n)) \leqslant -\frac{\mu}{(T-\hat{t}^n)^2}. \end{aligned} \tag{3.18}$$

We now like to pass to the limit as $n \to \infty$ in (3.18) keeping $\varepsilon, \delta, N$ fixed. Since $A^{-\frac{1-\beta}{2}}$ and $A^{-\frac{\eta}{2}}$ are compact we conclude that, as $n \to \infty$,

$$A^{\frac{1-\eta}{2}}\hat{x}^n = A^{-\frac{\eta}{2}}(A^{1/2}\hat{x}^n) \to A^{\frac{1-\eta}{2}}\bar{x}, \quad A^{\frac{\beta}{2}}(\hat{x}^n) = A^{-\frac{1-\beta}{2}}(A^{1/2}\hat{x}^n) \to A^{\frac{\beta}{2}}(\bar{x})$$

which together with the weakly semicontinuity of the norm implies

$$\liminf_{n\to\infty}\langle A^{1/2}\hat{x}^n, A^{1/2}D\psi(\hat{t}^n, \hat{x}^n)\rangle \geq \Big\langle A^{\frac{1-\eta}{2}}\bar{x}, \frac{A^{\frac{1-\eta}{2}}(\bar{x}-\bar{y})}{\varepsilon}\Big\rangle + 2\delta e^{K_\mu \bar{t}}|A^{1/2}\bar{x}|^2.$$

On the other hand, using (3.16), (3.11) and (3.13) we have that, as $n \to \infty$,

$$\begin{aligned} \psi_t(\hat{t}^n, \hat{x}^n) &\to \frac{\bar{t}-\bar{s}}{\gamma} + \delta K_\mu e^{K_\mu \bar{t}}(1+|\bar{x}|^2), \\ D\psi(\hat{t}^n, \hat{x}^n) &\to \frac{1}{\varepsilon}A^{-\eta}(\bar{x}-\bar{y}) + 2\delta e^{K_\mu \bar{t}}\bar{x}, \\ D^2\psi(\hat{t}^n, \hat{x}^n) &\to X_N + \frac{2A^{-\eta}Q_N}{\varepsilon} + 2\delta e^{K_\mu \bar{t}}I \leqslant X_N + \frac{2\|A^{-\eta}\|Q_N}{\varepsilon} + 2\delta e^{K_\mu \bar{t}}I, \end{aligned}$$

Therefore, using above results, (F1) and (F2), letting $n \to \infty$ in (3.18) yields

$$\begin{aligned} &\frac{\bar{t}-\bar{s}}{\gamma} + \delta K_\mu e^{K_\mu \bar{t}}(1+|\bar{x}|^2) + \frac{1}{\varepsilon}\big\langle A^{\frac{1-\eta}{2}}\bar{x}, A^{\frac{1-\eta}{2}}(\bar{x}-\bar{y})\big\rangle + 2\delta e^{K_\mu \bar{t}}|A^{1/2}\bar{x}|^2 \\ &+ F\Big(\bar{t}, \bar{x}, \frac{1}{\varepsilon}A^{-\eta}(\bar{x}-\bar{y}) + 2\delta e^{K_\mu \bar{t}}\bar{x}, X_N + \frac{2}{\varepsilon}\|A^{-\eta}\|Q_N + 2\delta e^{K_\mu \bar{t}}I\Big) \\ &\leqslant -\frac{\mu}{(T-\bar{t})^2}. \end{aligned} \tag{3.19}$$

We now eliminate terms with $\delta$ and $N$. Using (F3) we have

$$\begin{aligned} &F\Big(\bar{t}, \bar{x}, \frac{1}{\varepsilon}A^{-\eta}(\bar{x}-\bar{y}), X_N + \frac{2}{\varepsilon}\|A^{-\eta}\|Q_N\Big) - \rho\big(d\delta e^{K_\mu \bar{t}}(1+|\bar{x}|_\beta^2)\big) \\ &\leqslant F\Big(\bar{t}, \bar{x}, \frac{1}{\varepsilon}A^{-\eta}(\bar{x}-\bar{y}) + 2\delta e^{K_\mu \bar{t}}\bar{x}, X_N + \frac{2}{\varepsilon}\|A^{-\eta}\|Q_N + 2\delta e^{K_\mu \bar{t}}I\Big) \end{aligned}$$

for some constant $d > 0$. Now, given $\tau > 0$, let $K_\tau$ be such that

$$\rho(s) \leqslant \tau + K_\tau s.$$

Applying (2.2) with $\alpha = \frac{\beta}{2}$ and $\gamma = \frac{1}{2}$, we obtain that

$$\rho\big(d\delta e^{K_\mu \bar{t}}(1+|\bar{x}|_\beta^2)\big) \leqslant \tau + 2\delta e^{K_\mu \bar{t}}|A^{1/2}\bar{x}|^2 + \delta C_\tau e^{K_\mu \bar{t}}(1+|\bar{x}|^2)$$

for some constant $C_\tau > 0$ independent of $\delta$ and $\varepsilon$. Therefore, using these results in (3.19), (F5) and choosing $K_\mu = C_\tau$, we obtain

$$\begin{aligned} &\frac{\bar{t}-\bar{s}}{\gamma} + \frac{1}{\varepsilon}\big\langle A^{\frac{1-\eta}{2}}\bar{x}, A^{\frac{1-\eta}{2}}(\bar{x}-\bar{y})\big\rangle + F\big(\bar{t}, \bar{x}, \frac{1}{\varepsilon}A^{-\eta}(\bar{x}-\bar{y}), X_N\big) \\ &\leqslant \tau + \omega_1(N; \varepsilon, \delta, \gamma) - \frac{\mu}{(T-\bar{t})^2}. \end{aligned} \tag{3.20}$$

where $\lim_{N\to\infty}\omega_1(N;\varepsilon,\delta,\gamma)=0$ if $\varepsilon,\delta,\gamma$ are fixed. Similarly, we obtain

$$\begin{aligned}&\frac{\bar t-\bar s}{\gamma}+\frac{1}{\varepsilon}\big\langle A^{\frac{1-\eta}{2}}\bar y, A^{\frac{1-\eta}{2}}(\bar x-\bar y)\big\rangle+F\big(\bar s,\bar y,\frac{1}{\varepsilon}A^{-\eta}(\bar x-\bar y),Y_N\big)\\&\geq -\tau-\omega_1(N;\varepsilon,\delta)+\frac{\mu}{(T-\bar s)^2}.\end{aligned}\tag{3.21}$$

We now subtract (3.20) from (3.21), using $(F4)$, and then let $N\to\infty$. We then conclude that

$$\begin{aligned}\frac{\mu}{(T-\bar t)^2}+\frac{\mu}{(T-\bar s)^2}\leqslant 2\tau+\omega\Big(|A^{\frac{\beta}{2}}(\bar x-\bar y)|\big(1+\frac{1}{\varepsilon}|A^{\frac{\beta}{2}}(\bar x-\bar y)|\big)\Big)\\-\frac{1}{\varepsilon}|A^{\frac{1-\eta}{2}}(\bar x-\bar y)|^2\end{aligned}\tag{3.22}$$

Set $r=|A^{\frac{1-\eta}{2}}(\bar x-\bar y)|$. Using the interpolation inequality (2.2), the fact that $|A^{\frac{\beta}{2}}(\bar x-\bar y)|\leqslant c|A^{\frac{1-\eta}{2}}(\bar x-\bar y)|$ for some $c>0$ and the property of the moduli, we have that, for all $\alpha,\sigma>0$, there exist $C_\sigma,K_\alpha>0$ such that

$$\frac{\mu}{(T-\bar t)^2}+\frac{\mu}{(T-\bar s)^2}\leqslant 2\tau+\alpha+cK_\alpha\Big(\sigma\frac{r^2}{\varepsilon}+C_\sigma\frac{|A^{-\frac{\eta}{2}}(\bar x-\bar y)|}{\varepsilon}r+r\Big)-\frac{r^2}{\varepsilon}.$$

For $\alpha$ fixed, we choose $\sigma$ such that $cK_\alpha\sigma<1$. Then, in the right -hand side of the previous inequality, we have a polynomial of order 2 in $\frac{r}{\sqrt{\varepsilon}}$ which is bounded from above and we get

$$\frac{\mu}{(T-\bar t)^2}+\frac{\mu}{(T-\bar s)^2}\leqslant 2\tau+\alpha+\frac{K_\alpha^2c^2\big(\sqrt{\varepsilon}+C_\sigma\frac{|A^{-\frac{\eta}{2}}(\bar x-\bar y)|}{\sqrt{\varepsilon}}\big)^2}{4(1-K_\alpha c\sigma)}.$$

By sending $\gamma\to0,\varepsilon\to0,\delta\to0$ and using (3.7), we obtain a contradiction, which proves that we must have $u\leqslant v$.

**Existence.** To produce a solution of (1.2) we consider approximation

$$\begin{gathered}(u_N)_t(t,x)+\langle Ax,Du_N(t,x)\rangle+F(t,x,Du_N(t,x),D^2u_N(t,x))=0,\\(t,x)\in(0,T)\times H_N,\\u_N(0,x)=g(x),\quad x\in H_N.\end{gathered}\tag{3.23}$$

Note that (3.23) satisfies the assumptions in (3.5) with constants and moduli independent of $N$. By Proposition 3.2 (iii), there is a unique solution $u_N\in BUC_x([0,T]\times H_N)$ of (3.23) such that $\|u_N\|_\infty\leqslant M$ for some $M$ which depends only on $\|g\|_\infty,F_0$ in (3.3). Moreover, since $A$ is positive definite, Proposition 3.2 (ii) provides a modulus of continuity $m_1$ such that

$$|u_N(t,x)-u_N(t,y)|\leqslant m_1(|x-y|),\quad\forall t\in(0,T),\forall x,y\in H_N.\tag{3.24}$$

We now show that for each $\tau>0$ there is a modulus $m_\tau$ such that

$$|u_N(t,x)-u_N(t,y)|\leqslant m_\tau(|A^{-\frac{\eta}{2}}(x-y)|)\quad\text{for }\tau\leqslant t\leqslant T.\tag{3.25}$$

Given $\mu>0$, set

$$u_0(t,x)=u_N(t,x)-\frac{\mu}{T-t}.$$

Let $w$ be the modulus of continuity in (F4). For every $\varepsilon>0$ let $K_\varepsilon$ be such that $w(r)\leqslant\varepsilon/2+K_\varepsilon r$. For $L>M+1$, we set

$$\psi_L(r)=2L2^{1-\frac{1}{2L}}r^{\frac{1}{2L}}.$$

The function $\psi_L \in C^2(0,\infty)$ is increasing and concave, $\psi_L'(r) \geq 1$ for $0 < r \leqslant 2, \psi_L(0) = 0, \psi_L(1) > 2(M+1)$, and

$$\psi_L(r) > L(\psi_L'(r)r + r) \quad \text{for } 0 \leqslant r \leqslant 2. \tag{3.26}$$

We will show that for every $\varepsilon > 0$ there exists $L = L_\varepsilon$ such that

$$u_0(t,x) - u_0(t,y) \leqslant \big(\psi_L(|A^{-\frac{\eta}{2}}(x-y)|) + \varepsilon\big)(1+t) \tag{3.27}$$

for all $t \in (0,T); x, y \in H_N$. Indeed, we denote by

$$\Delta := \{(x,y) \in H_N \times H_N : \ |A^{-\eta/2}(x-y)| < 1\}.$$

It is clear, from the properties of $\psi_L$, that for $(x,y) \notin \Delta$, (3.27) always satisfied independently of $L$. Assume now by contradiction that (3.27) is false. Then, given any $L > M+1$, let

$$\psi(t,x,y) = \psi_L\big((|A^{-\frac{\eta}{2}}(x-y)| + \varepsilon)\big)(1+t)$$

we have that

$$\sup_{t\in(0,T),(x,y)\in\Delta} \Big(u_0(t,x) - u_0(t,y) - \psi(t,x,y)\Big) > 0$$

(if not we are done). Then, for small $\delta > 0$,

$$\sup_{t\in(0,T),(x,y)\in\Delta} \Big(u_0(t,x) - u_0(t,y) - \psi(t,x,y) - \delta|x|^2 - \delta|y|^2\Big) > 0$$

and is attained at a point $(\bar t, \bar x, \bar y)$ with $(\bar x, \bar y) \in \Delta, \bar x \neq \bar y$. It follows from the initial condition and the definition of $u_0, \psi_L$ that $0 < \bar t < T$.

To use Proposition 2.1, we denote $s = |A^{-\frac{\eta}{2}}(\bar x - \bar y)|$ and compute

$$\begin{aligned}
\psi_t(\bar t, \bar x, \bar y) &= \psi_L(s) + \varepsilon, \\
D_x\psi(\bar t, \bar x, \bar y) &= \psi_L'(s)\frac{A^{-\eta}(\bar x - \bar y)}{s}(1+\bar t) \\
D^2_{xx}\psi(\bar t, \bar x, \bar y) &= \psi_L''(s)\frac{A^{-\eta}(\bar x - \bar y)\otimes A^{-\eta}(\bar x - \bar y)}{s^2}(1+\bar t) + \psi_L'(s)\frac{P_N A^{-\eta}}{s}(1+\bar t) \\
&\quad - \psi_L'(s)\frac{A^{-\eta}(\bar x - \bar y)\otimes A^{-\eta}(\bar x - \bar y)}{s^3}(1+\bar t) \\
&= B_1 + B_2 + B_3.
\end{aligned}$$

Since $\psi_L$ is nondecreasing and concave, $B_2 \geq 0$ and $B_1, B_3 \leqslant 0$. Using this notation we have

$$\begin{aligned}
D^2\psi(\bar t, \bar x, \bar y) &= \begin{pmatrix} B_2 & -B_2 \\ -B_2 & B_2 \end{pmatrix} - \begin{pmatrix} -B_1 - B_3 & B_1 + B_3 \\ B_1 + B_3 & -B_1 - B_3 \end{pmatrix} \\
&= D_1 - D_2
\end{aligned}$$

where $D_1, D_2 \geq 0$. Proposition 2.1 applied with $\varepsilon = 1, u_1(t,x) = u_0(t,x) - \delta|x|^2$, $u_2(t,y) = -u_0(t,y) - \delta|y|^2$ tells us that there exist $a, b \in \mathbb{R}$, and matrices $X, Y \in S(\mathbb{R}^N)$ such that

$$\begin{gathered}
(a, D_x\psi(\bar t, \bar x, \bar y), X) \in \bar P^{2,+}u_1(\bar t, \bar x); \\
(-b, -D_y\psi(\bar t, \bar x, \bar y), -Y) \in \bar P^{2,-}(-u_2)(\bar t, \bar y)
\end{gathered}$$

where

$$a+b=\psi_L(s)+\varepsilon,$$
$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leqslant (1+\bar{t})\frac{2\psi'(s)}{s}\begin{pmatrix} P_N A^{-\eta} P_N & -P_N A^{-\eta} P_N \\ -P_N A^{-\eta} P_N & P_N A^{-\eta} P_N \end{pmatrix}$$

It follows from the properties of $\bar{P}^{2,+}$ and $\bar{P}^{2,-}$ that

$$(a+\frac{\mu}{(T-\bar{t})^2}, D_x\psi(\bar{t},\bar{x},\bar{y})+2\delta\bar{x}, X+2\delta I) \in \bar{P}^{2,+}u_N(\bar{t},\bar{x});$$
$$(-b+\frac{\mu}{(T-\bar{t})^2}, -D_y\psi(\bar{t},\bar{x},\bar{y})-2\delta\bar{y}, -Y-2\delta I) \in \bar{P}^{2,-}u_N(\bar{t},\bar{y}).$$

By the definition of viscosity solutions in case of finite dimensional, we obtain

$$\begin{aligned} &a+\frac{\mu}{(T-\bar{t})^2}+\frac{\psi_L'(s)}{s}(1+\bar{t})\langle A\bar{x}, A^{-\eta}(\bar{x}-\bar{y})\rangle + 2\delta\langle A^{1/2}\bar{x}, A^{1/2}\bar{x}\rangle \\ &+F(\bar{t},\bar{x},\frac{\psi_L'(s)}{s}(1+\bar{t})A^{-\eta}(\bar{x}-\bar{y})+2\delta\bar{x}, X+2\delta I) \leqslant 0 \end{aligned}$$

and

$$\begin{aligned} &-b+\frac{\mu}{(T-\bar{t})^2}+\frac{\psi_L'(s)}{s}(1+\bar{t})\langle A\bar{y}, A^{-\eta}(\bar{x}-\bar{y})\rangle - 2\delta\langle A^{1/2}\bar{y}, A^{1/2}\bar{y}\rangle \\ &+F(\bar{t},\bar{y},\frac{\psi_L'(s)}{s}(1+\bar{t})A^{-\eta}(\bar{x}-\bar{y})-2\delta\bar{y}, -Y-2\delta I) \geq 0\,. \end{aligned}$$

Repeating the arguments from the proof of comparison we obtain that

$$\begin{aligned} a+b &\leqslant -\frac{\psi_L'(s)}{s}(1+\bar{t})|A^{\frac{1-\eta}{2}}(\bar{x}-\bar{y})|^2 \\ &\quad + w\Big(|A^{\frac{\beta}{2}}(\bar{x}-\bar{y})|\big(1+\frac{\psi_L'(s)}{s}(1+\bar{t})|A^{\frac{\beta}{2}}(\bar{x}-\bar{y})|\big)\Big)+2w_1(L,\delta) \\ &\leqslant -\frac{\psi_L'(s)}{s}|A^{\frac{1-\eta}{2}}(\bar{x}-\bar{y})|^2+\frac{\varepsilon}{2} \\ &\quad + K_\varepsilon\Big(|A^{\frac{\beta}{2}}(\bar{x}-\bar{y})|\big(1+\frac{\psi_L'(s)}{s}(1+T)|A^{\frac{\beta}{2}}(\bar{x}-\bar{y})|\big)\Big)+2w_1(L,\delta) \end{aligned}$$

where $\limsup_{\delta\to 0} w_1(L,\delta)=0$. Therefore, using the interpolation inequality (2.2) with a sufficiently small $\sigma$, it follows that

$$\begin{aligned} \psi_L(s)+\varepsilon \leqslant &-\frac{\psi_L'(s)}{2s}|A^{\frac{1-\eta}{2}}(\bar{x}-\bar{y})|^2+\frac{\varepsilon}{2} \\ &+C_\varepsilon(\psi_L'(s)s+s)+\frac{c}{2}|A^{\frac{1-\eta}{2}}(\bar{x}-\bar{y})|+2w_1(L,\delta) \end{aligned}$$

where $C_\varepsilon$ depends only on $K_\varepsilon$ and the interpolation constants (but not on $L$), and $c$ is such that $|A^{\frac{1-\eta}{2}}x| \geq c|A^{-\frac{\eta}{2}}x|$ for all $x \in \mathcal{D}(A^{\frac{1-\eta}{2}})$ [5]. Thus, we eventually have

$$\psi_L(s) \leqslant C_\varepsilon(\psi_L'(s)s+s)-\frac{\varepsilon}{2}+2w_1(L,\delta),$$

which becomes, choosing $L=C_\varepsilon$ and letting $\delta\to 0$,

$$\psi_L(s) \leqslant L(\psi_L'(s)s+s)-\frac{\varepsilon}{2}.$$

This leads to a contradiction in light of (3.26). Thus we have (3.27), which implies

$$u_0(t,x)-u_0(t,y) \leqslant \psi_L(|A^{-\frac{\eta}{2}}(x-y)|)(1+T)+2M|A^{-\frac{\eta}{2}}(x-y)|$$

for all $x, y \in H_N$ and $t \in [0, T]$. We obtain the required modulus of $u_N$ by letting $\mu \to 0$.

Next, we show that there is a modulus $m$ depending only on $m_1$ and the function $F_R$, such that

$$|u_N(t,x) - u_N(s, e^{-(t-s)A}x)| \leqslant m(t-s) \tag{3.28}$$

for $x \in H_N, 0 \leqslant t \leqslant T$. Because of (3.24) it is enough to show (3.28) for $s = 0$ since all the estimates can be reapplied at later time. To do this we begin with $g \in C^{1,1}(H)$ such that $\|g\|_\infty < \infty$. We denote the Lipschitz constant of $Dg$ by $L_{Dg}$. We use the fact that $h(t,x) = g(e^{-tAx})$ solves

$$\begin{gathered} h_t(t,x) + \langle Ax, Dh(t,x)\rangle = 0 \text{ in } (0,T] \times H_N, \\ h(0,x) = g(x) \quad \text{in } H_N \end{gathered}$$

which implies

$$u = h + tF_{\max(L_{Dg}, \|Dg\|_\infty)}, \quad v = h - tF_{\max(L_{Dg}, \|Dg\|_\infty)}$$

are respectively a viscosity supersolution and a viscosity subsolution of (3.23). To see this we note that $h$ is $C^{1,1}$ in $x, \|Dh\|_\infty \leqslant \|Dg\|_\infty$, and $L_{Dh} \leqslant L_{Dg}$. It then suffices to observe (for a subsolution $u$) the obvious fact that if for a test function $\varphi$, $u - \varphi$ has a local maximum at $(t,x)$ then $D^2\varphi(t,x) \geq -L_{Dh}I \geq -L_{Dg}I$. Hence, comparison gives

$$|u_N(t,x) - g(e^{-tA}x)| \leqslant tF_{\max(L_{Dg}, \|Dg\|_\infty)}. \tag{3.29}$$

If $g \in BUC(H)$ we can approximate it by a function $\tilde{g} \in C^{1,1}(H)$ such that $\|D\tilde{g}\|_\infty < \infty$ [10]. Hence, if $\tilde{u}_N$ solves (3.23) with $\tilde{g}$, then

$$\begin{aligned} |u_N(t,x) - g(e^{-tA}x)| &\leqslant |u_N(t,x) - \tilde{u}_N(t,x)| + |\tilde{u}_N(t,x) - \tilde{g}(e^{-tA}x)| \\ &\quad + |\tilde{g}(e^{-tA}x) - g(e^{-tA}x)| \\ &\leqslant 2\|g - \tilde{g}\|_\infty + tF_{\max(L_{Dg}, \|Dg\|_\infty)}, \end{aligned}$$

where we have used (3.29) and Proposition 3.2 (i). This gives us (3.28).

Now set $v_N(t,x) = u_N(t, P_N x)$. Since $A^{-\frac{\eta}{2}}$ is compact, (3.25) and (3.28) we have the equicontinuity of $\{v_N\}$ in the weak topology on bounded subsets of $[\tau, T] \times H$ for $\tau > 0$. The Arzela-Ascoli theorem then provides a subsequence (still denoted by $v_N$) converging uniformly on bounded subsets of $[\tau, T] \times H$ to a function $u$ that obviously satisfies the same estimates as $u_N's$ [13]. Moreover, (3.28) imply that $\lim_{t\downarrow 0} u(t,x) = g(x), x \in H$. It remains to show that $u$ solves the limiting equation in (1.2). Let $\psi(t,x) = \varphi(t,x) + \delta(t)(1 + |x|^2)$ is a test function of the equation in (1.2) and let $u(t,x) - \psi(t,x)$ have a maximum at $(\hat{t}, \hat{x})$ which we may assume to be strict. It follows that there exists a sequence $\hat{x}_N = P_N\hat{x} \to \hat{x}$ as $N \to \infty$ such that, for every $x \in H_N$,

$$v_N(t,x) - \psi(t,x) \leqslant v_N(\hat{t}, \hat{x}_N) - \psi(\hat{t}, \hat{x}).$$

Therefore, since $AP_N = P_N A$,

$$\begin{aligned} &\psi_t(\hat{t}, \hat{x}_N) + \langle A^{1/2}\hat{x}_N, A^{1/2}D\varphi(\hat{t}, \hat{x}_N)\rangle + \delta(t)|A^{1/2}\hat{x}_N|^2 \\ &+ F(\hat{t}, \hat{x}_N, P_N D\varphi(\hat{t}, \hat{x}_N) + 2\delta(\hat{t})\hat{x}_N, P_N(D^2\varphi(\hat{t}, \hat{x}_N) + 2\delta(\hat{t})I)P_N) \leqslant 0. \end{aligned} \tag{3.30}$$

Since $\hat{x}_N \in H_N$ and $\psi$ is a test function we have

$$|A^{1/2}D\varphi(\hat{t}, \hat{x}_N)| \leqslant B + C|A^{\frac{1}{2}-k}\hat{x}_N| \tag{3.31}$$

for some independent constants $B, C$. Also, by (3.31), (2.2), (2.3) and (3.3),

$$\begin{aligned}&\Big|F(\hat{t},\hat{x}_N,P_ND\varphi(\hat{t},\hat{x}_N)+\delta(\hat{t})\hat{x}_N,P_N(D^2\varphi(\hat{t},\hat{x}_N)+\delta(\hat{t})I)P_N)\Big|\\&\leqslant C_1\Big(1+|A^{\frac{\beta}{2}}\hat{x}_N|^2+|A^{\frac{1}{2}-k}\hat{x}_N|^2\Big)\\&\leqslant C_2+\frac{\delta(\hat{t})}{4}|A^{1/2}\hat{x}_N|^2.\end{aligned}$$

Using this, (3.31) and the interpolation inequality (2.2), we therefore obtain from (3.30) that $|A^{1/2}\hat{x}_N|\leqslant C_3$ for some constant $C_3$ independent of $N$. Thus, $A^{1/2}\hat{x}_N\rightharpoonup A^{1/2}\hat{x}$ (so $\hat{x}\in\mathcal{D}(A^{1/2})$) and hence

$$A^{\frac{\beta}{2}}\hat{x}_N\to A^{\frac{\beta}{2}}\hat{x},\quad\text{and}\quad A^{1/2}D\varphi(\hat{t},\hat{x}_N)\to A^{1/2}D\varphi(\hat{t},\hat{x}).$$

These convergence and Proposition 3.3 allow us to pass to the limit in (3.30) as $N\to\infty$ to conclude that

$$\begin{aligned}&(\psi)_t(\hat{t},\hat{x})+\langle A^{1/2}\hat{x},A^{1/2}D\varphi(\hat{t},\hat{x})\rangle+\delta(\hat{t})|A^{1/2}\hat{x}|^2\\&+F(\hat{t},\hat{x},D\varphi(\hat{t},\hat{x})+\delta(\hat{t})\hat{x},D^2\varphi(\hat{t},\hat{x})+\delta(\hat{t})I)\leqslant 0.\end{aligned}$$

This proves that $u$ is a viscosity subsolution. Similarly, we obtain that $u$ is a viscosity supersolution and therefore it is a viscosity solution of the equation in (1.2). The comparison gives us the uniqueness of $u$.

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