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ON THE DYNAMICS OF THE CHARACTERISTIC CURVES FOR THE LSW MODEL

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Abstract. This paper describes in a rigorous manner how the dynamics of the characteristic curves for the Lifshitz-Slyozov-Wagner (LSW) model of coarsening transforms a class of noncompactly supported initial data in functions that behave in a self-similar manner for long times.

1. Introduction

The purpose of this paper is to obtain in a rigorous way some properties for the dynamics of the characteristic curves associated with the Lifshitz-Slyozov-Wagner (LSW) model

$$\frac{\partial f(R,t)}{\partial t} + \frac{\partial}{\partial R}\Big(\big(-\frac{1}{R^2}+\frac{\Delta(t)}{R}\big)f(R,t)\Big) = 0, \quad t>0\,,\ R>0\,, \tag{1.1}$$

$$f(R,0) = f_0(R) \geq 0\,, \quad R>0\,, \tag{1.2}$$

$$\Delta(t) = \frac{\int_0^\infty f(R,t)\ dR}{\int_0^\infty f(R,t) R\, dR}\,. \tag{1.3}$$

This system was introduced to study the coarsening stage of the so-called Oswald ripening (cf. [4, 15]). The choice of $\Delta(t)$ above ensures that the volume density of the particles $\int_0^\infty f(R,t)R^3\,dR$ is preserved during their evolution. Rigorous derivations of this system using homogenization techniques, that take as starting point the Mullins-Sekerka free boundary problem, have been obtained in different scaling limits in [5, 6, 10].

The system (1.1)–(1.3) has a family of explicit self-similar solutions with the form (cf. [4]):

$$f(R,t) = \frac{1}{t^{4/3}}\Phi(\rho), \quad \rho = \frac{R}{t^{1/3}} \tag{1.4}$$

The solutions of (1.1)–(1.3) with the form (1.4) having finite volume fraction

$$\phi \equiv \int_0^\infty f(R,t)R^3\,dR < \infty$$

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are compactly supported on the space ρ. For each given value of the volume fraction filled by the particles ϕ there exists a one-parameter family of self-similar solutions of (1.1)–(1.3).

It was rigorously proved in [7] that the long time asymptotics of the compactly supported solutions of (1.1)–(1.3) depends very sensitively on the asymptotics of the initial data near the maximum radius. Additional results concerning the long time asymptotics of the LSW system (1.1)–(1.3) for compactly supported initial data can be found in [11], [12].

The long time asymptotics of the solutions of (1.1)–(1.3) for noncompactly supported initial data has been studied in [14]. In that paper was shown that there exist noncompactly supported initial data $f_0(R)$ yielding nonselfsimilar behaviour as $t \to \infty$. Moreover, in that paper were derived approximations of the noncompactly supported solutions of the LSW model for long times by means of asymptotic arguments. The resulting approximate LSW equations are a set of integrodifferential equations with two different time scales as $t \to \infty$.

The main goal of this paper is to provide some rigorous basis for some results that were just obtained in a formal manner in [14].

It was already implicit in the arguments of the seminal paper [4] that the key problem in order to understand the dynamics of the LSW model is to describe the behaviour of the characteristic curves associated to (1.1)–(1.3) near the so-called critical radius. In an informal manner we can say that the critical radius is the value of the radius for which the velocity of the characteristics in the space of radii, written in self-similar variables is close to zero. Due to this fact the characteristic curves "leak" very slowly from the region of supercritical radii to the region of subcritical radii. To compute in detail the rate of "leaking" is crucial in order to describe asymptotically the long time behaviour of noncompactly supported solutions of (1.1)–(1.3) behaving in a self-similar manner as $t \to \infty$. In particular, the analysis of the "leaking" phenomenon yields an asymptotics for the rate of change of the particles in the form

$$\frac{1}{t^2} + \frac{1}{t^2(\log(t))^2} + \frac{1}{t^2(\log(t))^2(\log(\log(t)))^2} + \dots \quad \text{as } t \to \infty \tag{1.5}$$

This type of formula were already obtained in [4]. A more detailed study of these asymptotics was made in [13]. On the other hand, the analysis in [14] provides a simpler and more general explanation for the onset of the asymptotics (1.5).

The main contribution of this paper is to develop mathematical methods that allow to analyze the behaviour of the characteristic curves near the critical radius in a fully rigorous manner. Only a small fraction of the formal arguments in [14] are proved rigorously in this paper. Nevertheless, the mathematical results in this paper show that some of the main ideas in [14] can indeed be made mathematically rigorous. Note that the results of this paper are not obtained for the full LSW model, but for a simplified problem that takes into account only the transition of the characteristic curves near the critical line. Such transition is the most relevant part in the transformation of general, noncompactly supported initial data in self-similar solutions as $t \to \infty$.

The plan of the paper is the following: In Section 2 we introduce some basic notation and formulate the main result proved in the paper. In Section 3 we compute in a formal manner the stretching generated by the dynamics of the characteristic curves. In particular we will formally obtain in this Section that the transition of the characteristic curves through the critical region can be approximated by means of the problem (4.1)-(4.4). This problem can be solved in an explicit manner, as it was noticed in [14]. This fact plays an essential role in many of the arguments in the rest of the paper. For this reason we will recall the solution of (4.1)–(4.4) in Section 4.

The rest of the paper consists studying the evolution of the characteristic curves associated to (1.1), (1.2) by means of a perturbative argument. More precisely, it will be seen in Section 5 that the transition of the characteristic curves of (1.1), (1.2) near the critical region can be approximated in some sense by means of the solution of the problem (4.1)–(4.4). To make precise the "closedness" of the solutions of these two problems it is convenient to reformulate the solutions of the problem described in Section 5 by means of an integral equations whose local solvability in time is obtained in Section 6. The local solvability of this integral equation is given in Section 7, as well as the long time asymptotics of the resulting solutions.

Some technical properties of some auxiliary functions f are given in Appendix A at the end of the paper.

2. Statement of the main result

It is convenient, in order to simplify some computations, to replace the density in the space of radii by the density in the space of volumes. We define $\bar{f}(v,t)$ by means of:

$$v = R^3$$
$$f(R,t)\,dR = \bar{f}(v,t)dv$$
$$\bar{t} = 3t$$

whence $\bar{f}(v,\bar{t}) = \frac{1}{3v^{2/3}} f(v^{1/3},t)$. Using this change of variables, equations (1.1)-(1.3) become

$$\frac{\partial \bar{f}(v,\bar{t})}{\partial \bar{t}} + \frac{\partial}{\partial v}\Big((-1+\Delta(\bar{t})v^{1/3})\bar{f}(v,\bar{t})\Big) = 0\,, \quad \bar{t}>0\,,\ v>0 \tag{2.1}$$

$$\bar{f}(v,0) = \frac{1}{3v^{2/3}} f_0(v^{1/3}) \equiv \bar{f}_0(v)\,, \quad v>0 \tag{2.2}$$

$$\Delta(\bar{t}) = \frac{\int_0^\infty \bar{f}(v,\bar{t})dv}{\int_0^\infty v^{1/3}\bar{f}(v,\bar{t})dv} \tag{2.3}$$

The analysis of these equations becomes simpler if the density $\bar{f}(v,\bar{t})$ is replaced by the distribution function

$$F(v,\bar{t}) \equiv \int_v^\infty \bar{f}(\xi,\bar{t})d\xi\,. \tag{2.4}$$

Using this new variable the system (2.1)-(2.3) becomes:

$$\frac{\partial F(v,\bar{t})}{\partial \bar{t}} + \Big(-1+\Delta(\bar{t})v^{1/3}\Big)\frac{\partial F(v,\bar{t})}{\partial v} = 0\,, \quad \bar{t}>0\,,\ v>0 \tag{2.5}$$

$$F(v,0) = F_0(v) \equiv \int_v^\infty \bar{f}_0(v)d\xi\,, \quad v>0 \tag{2.6}$$

$$\Delta(\bar{t}) = \frac{3F(0^+,\bar{t})}{\int_0^\infty v^{-2/3}F(v,\bar{t})dv}\,. \tag{2.7}$$

Global well posedness for this problem has been studied in [8] for compactly supported initial data and in [9] for noncompactly supported initial data with fast enough decay. We can assume that $\bar{f}(v,\bar{t})$ is a (nonnegative) measure satisfying $\int v\bar{f}dv < \infty$, whence $F(v,\bar{t})$ is in general a decreasing bounded function. Note that the volume preservation satisfied by the solutions of (2.5) becomes in these variables,

$$\frac{\partial}{\partial \bar{t}}\Big(\int_0^\infty v\bar{f}(v,\bar{t})dv\Big) = \frac{\partial}{\partial \bar{t}}\Big(\int_0^\infty F(v,\bar{t})dv\Big) = 0\,.$$

We introduce self-similar variables by means of:

$$F(v,t) = \frac{G(W,\tau)}{\bar{t}}, \quad W = \frac{2v}{t}, \quad \tau = \log(\bar{t}) \tag{2.8}$$

In these set of variables (2.5)-(2.7) become

$$\frac{\partial G}{\partial \tau} + (-2 + 3\lambda(\tau)W^{1/3} - W)\frac{\partial G}{\partial W} = G \tag{2.9}$$

$$\lambda(\tau) = \frac{2G(0^+,\tau)}{\int_0^\infty W^{-2/3}G(W,\tau)dW} \tag{2.10}$$

$$G(W,0) = G_0(W) \tag{2.11}$$

The self-similar solution of the LSW system having maximal support becomes, in this set of variables,

$$G_s(W) = \exp\Big(-\int_0^W \frac{d\xi}{2 - 3\xi^{1/3} + \xi}\Big), \tag{2.12}$$

$$0 \le W < 1\,,\ G_s(W) = 0 \quad \text{if } W > 0. \tag{2.13}$$

$$\lambda(\tau) = 1 \tag{2.14}$$

where we have used the normalization $G_s(0) = 1$.

Let us define $\beta(\tau) = \lambda(\tau) - 1$. Since $\int_0^\infty W^{-2/3}G(W,\tau)dW = 2$, it follows that

$$\beta(\tau) = \frac{\int_0^\infty W^{-2/3}(G(W,\tau) - G(0^+,\tau)G_s(W))dW}{\int_0^\infty W^{-2/3}G(W,\tau)dW} \tag{2.15}$$

and (2.9) might be written as

$$\frac{\partial G}{\partial \tau} + \Big(-2 + 3W^{1/3} - W + 3\beta(\tau)W^{1/3}\Big)\frac{\partial G}{\partial W} = G \tag{2.16}$$

Suppose that $G(W,\tau)$ behaves in a self-similar manner as $t \to \infty$. Then (2.15) implies that

$$\lim_{\tau\to\infty} \beta(\tau) = 0 \tag{2.17}$$

Therefore, the speed of the characteristics of (2.16) might be approximated by means of the function $(-2 + 3W^{1/3} - W)$ that has a double zero at $W = 1$. The line $W = 1$ will be termed from now on as "critical line". Due to (2.17) the characteristic curves associated to (2.16) remain trapped near the critical line for long times. For long times the only characteristic curves that contribute to the asymptotics of $G(W,\tau)$ are those starting at points $W = W_0$ with $W_0 \to \infty$. Since $G_0(W_0) \to 0$ as $W_0 \to \infty$, and the values of G increase exponentially on τ along characteristics, it follows that in order to obtain $G(W,\tau)$ of order one for $W \in (0,1)$ as $\tau \to \infty$ the characteristics starting at $W = W_0$ with $W_0 \to \infty$ must remain trapped near the critical line $W = 1$ a very precisely tuned amount of time. A detailed computation of the trapping time can be found in Section 3 below.

The purpose of this paper is to obtain detailed information in a rigorous manner on the transformations $G_0(W) \to G(W, \infty)$ induced by the transition of the characteristic curves along the critical line $W = 1$. The main result obtained in this paper is the following:

Theorem 2.1. *Suppose that $G_0(W)$ satisfies:*

$$\lim_{W_0\to\infty} \sup_{0\le\zeta\le1} \Big| \frac{G_0(W_0 e^{\Lambda(W_0)\zeta})}{G_0(W_0)} - e^{-\zeta} \Big| = 0, \tag{2.18}$$

where

$$\frac{d^k\Lambda}{dW_0^k}(W_0) \sim \frac{d^k}{dW_0^k}\big(\frac{C}{(W_0)^\alpha}\big) \quad \text{as } W_0 \to \infty \tag{2.19}$$

for some $\alpha > 0$ and $C > 0$, $k = 0, 1, 2$. Then there exists a family of functions $\beta(\tau) \in L^\infty[0, \infty)$ satisfying

$$\lim_{\tau\to\infty} \beta(\tau) = 0 \tag{2.20}$$

such that the corresponding solution of (2.9) with initial data (2.11) satisfies

$$\lim_{W\to\infty} \frac{G(W, \tau)}{G(0^+, \tau)} = G_s(W) \tag{2.21}$$

uniformly on $W \in \mathbb{R}^+$.

Condition (2.18) is reminiscent of some similar conditions on the initial data that might be found in [7], [11], [12]. In those papers it was proved that the initial data $G_0(W)$ must satisfy rather strong conditions in order to yield self-similar behaviours for long times. Roughly speaking, the conditions on those papers, as well as (2.18), impose that a suitable functional transformation of $G_0(W)$ behaves asymptotically in a self-similar manner as $W \to \infty$.

The main shortcoming of Theorem 2.1 concerning the LSW theory is that the function $\beta(\tau)$ there is not necessarily given by (2.15). Theorem 2.1 shows only that it is possible to transform initial data satisfying (2.18) into a self-similar behaviour by means of the evolution equation (2.16). In other words, the transformation induced by the trapping of the characteristics associated to (2.16) near the critical line is able to bring initial data satisfying (2.18) into self-similar behaviour. In particular, the choice of the function $\beta(\tau)$ derived in Theorem 2.1 does not yield preservation of the total volume of the particles. To obtain an unique function $\beta(\tau)$ preserving the total volume of the particles it would be needed to take into account also the dynamics of the characteristics in the regions $0 < W < 1$ and $W > 1$ as it was made at the formal level in [14]. On the other hand, as it was seen in [14], the feasibility of the transformation described in Theorem 2.1 is the key feature that must be required on G_0 in order to obtain also volume conservation.

Assumption (2.19) looks very restrictive and certainly is not the most general one possible. The main reason for this choice of $\Lambda(W_0)$ is because under this assumption it will be possible to handle easily many of the formula later. Nevertheless (2.19) could be weakened much. In any case (2.18), (2.19) covers several interesting initial data G_0. For instance if G_0 satisfies

$$G_0(W) \sim CW^B(\log(W))^D e^{-W^A} \quad \text{as } W \to \infty\,,\ A > 0,\ B, D \in \mathbb{R} \tag{2.22}$$

then formulae (2.18), (2.19) hold with

$$\Lambda(W) \sim \frac{1}{A}\frac{1}{(W)^A}$$

as $W \to \infty$.

3. A heuristic computation of the stretching generated by the evolution of the characteristics

In this Section we describe in a heuristic manner the key argument in this paper. This argument has been described in a different, but essentially equivalent manner in [14]. We recall it here for further reference.

The basic problem is to estimate the transformation induced by the characteristics of the equation (2.16) for small functions $\beta(\tau)$. Let us formulate the problem in a more precise manner. We denote as $W(\tau; W_0)$ the solution of the differential equation:

$$W_\tau = -2 + 3W^{1/3} - W + 3\beta(\tau)W^{1/3} \tag{3.1}$$

$$W(0; W_0) = W_0 \,. \tag{3.2}$$

Our goal is to obtain approximations for the solutions of this equation for small $\beta(\tau)$ uniformly valid in time. Since $\beta(\tau)$is small it follows that the evolution of $W(\tau; W_0)$ takes place in a very different manner in the regions $W - 1 \gg \sqrt{|\beta(\tau)|}$, $1 - W \gg \sqrt{|\beta(\tau)|}$ and $|W - 1| \approx \sqrt{|\beta(\tau)|}$. Indeed, in the first two regions, away from the region $W = 1$, we can approximate the equation (3.1) as:

$$W_\tau = -2 + 3W^{1/3} - W \quad \text{for } |W-1| \gg \sqrt{|\beta(\tau)|} \tag{3.3}$$

This approximation cannot be used in the region $|W - 1| \approx \sqrt{|\beta(\tau)|}$. From now on we will call critical line the line $W = 1$. Using Taylor's expansion and keeping just the leading terms we would approximate (3.1) in the region $|W - 1| \ll 1$ as:

$$W_\tau = -\frac{1}{3}(W-1)^2 + 3\beta(\tau) \tag{3.4}$$

This equation shows that $W(\tau; W_0)$ is very sensitive in this region to the precise form of $\beta(\tau)$. It follows from (3.3), (3.4) that the evolution of the characteristics away from the critical line does not depend much on $\beta(\tau)$, but this evolution is extremely sensitive to the values of $\beta(\tau)$ in the region $|W - 1| \approx \sqrt{|\beta(\tau)|}$.

To obtain a class of initial data that are transformed by means of (2.16) in functions close to self-similar solutions the key problem is to study the evolution of characteristics $W(\tau; W_0)$ with W_0 large. Therefore, we will assume from now on that $W_0 > 1$, i.e. that the characteristic curves cross the critical line during their evolution. Using (3.3) it follows that, as long as $1 - W \gg \sqrt{|\beta(\tau)|}$, we can approximate $W(\tau; W_0)$ as:

$$\log\Big(\frac{W_0}{W(\tau; W_0)}\Big) + F_{\rm ext}(W(\tau; W_0)) - F_{\rm ext}(W_0) = \tau, \tag{3.5}$$

where

$$F_{\rm ext}(W) \equiv \int_W^\infty \big[\frac{1}{2 - 3\eta^{1/3} + \eta} - \frac{1}{\eta}\big]d\eta = \int_W^\infty \frac{(3\eta^{1/3} - 2)}{(2 - 3\eta^{1/3} + \eta)\eta}d\eta \,. \tag{3.6}$$

The asymptotic behaviour of $F(W)$ as $W \to 1^+$ is

$$F_{\rm ext}(W) = \frac{3}{(W-1)} - \frac{5}{3}\log(W-1) + b_{\rm ext} + o(1) \tag{3.7}$$

where $b_{\rm ext} \in \mathbb{R}$. We can rewrite (3.5) as

$$W(\tau; W_0) = w_{\rm ext}(\tau - \log(W_0) + F_{\rm ext}(W_0) - b_{\rm ext}) \tag{3.8}$$

where

$$-\log(w_{\rm ext}(s)) + F_{\rm ext}(w_{\rm ext}(s)) = s + b_{\rm ext}\,. \tag{3.9}$$

Using (3.7) we obtain the asymptotics

$$w_{\rm ext}(s) \sim 1 + \frac{3}{s} - \frac{5\log(\frac{3}{s})}{s^2} + o(\frac{1}{s^2}) \quad \text{as } s \to \infty\,. \tag{3.10}$$

Formulae (3.8), (3.10) suggest to define the arrival time to the critical line $W = 1$ for $W(\tau; W_0)$ as:

$$T_{\rm arr} = \log(W_0) - F(W_0) + b_{\rm ext} \tag{3.11}$$

Note that the transition of the characteristic $W(\tau; W_0)$ by the region $|W - 1| \approx \sqrt{|\beta(\tau)|}$ takes place in a long time if $\beta(\tau)$ is small. Therefore, due to the asymptotics (3.10) it makes sense to approximate $W(\tau; W_0)$ near the critical line by means of the solution of the problem

$$W_{{\rm trans},\tau} = -\frac{1}{3}(W_{\rm trans} - 1)^2 + 3\beta(\tau) \tag{3.12}$$

$$W_{\rm trans}((T_{\rm arr})^+; W_0) = +\infty\,. \tag{3.13}$$

Indeed, note that the effect in $W_{\rm trans}$ of the term $\beta(\tau)$ during the range of times in which $W_{\rm trans} - 1 \gg \sqrt{|\beta(\tau)|}$ is small.

On the other hand, after the transition near the critical line has taken place, we can use again (3.3) to approximate $W(\tau; W_0)$. Suppose that $W(\bar\tau; W_0) = \bar W$. Arguing as in the region $W - 1 \gg \sqrt{|\beta(\tau)|}$ we obtain the approximation:

$$F_{\rm int}(W(\tau; W_0)) = F_{\rm int}(\bar W) + (\bar\tau - \tau), \tag{3.14}$$

where

$$F_{\rm int}(W) = \int_0^W \frac{d\eta}{2 - 3\eta^{1/3} + \eta}\,. \tag{3.15}$$

We have the asymptotics

$$F_{\rm int}(W) \sim \frac{3}{(1-W)} + \frac{5}{3}\log(1-W) + b_{\rm int} + o(1) \quad \text{as } W \to 1^-\,. \tag{3.16}$$

We then write

$$W(\tau; W_0) = w_{\rm int}(\tau - \bar\tau - F_{\rm int}(\bar W) + b_{\rm int}), \tag{3.17}$$

where

$$F_{\rm int}(w_{\rm int}(s)) = b_{\rm int} - s\,. \tag{3.18}$$

Therefore,

$$w_{\rm int}(s) \sim 1 + \frac{3}{s} - \frac{5\log(-\frac{3}{s})}{s^2} + o\big(\frac{1}{s^2}\big) \quad \text{as } s \to -\infty \tag{3.19}$$

Equations (3.17), (3.19) suggest to define the "exit time" from the transition region for $W(\tau; W_0)$ as

$$T_{\rm exit} = \bar\tau + F_{\rm int}(\bar W) - b_{\rm int}\,. \tag{3.20}$$

Using again the smallness of $\beta(\tau)$ it would be natural to assume that, to the leading order, the function $W_{\rm trans}$ that solves (3.12), (3.13) satisfies

$$W_{\rm trans}((T_{\rm exit})^{-};W_0) = -\infty\,. \tag{3.21}$$

Let us summarize. The previous argument indicates that for $\beta(\tau)$ small we can approximate $W(\tau;W_0)$ using (3.8), (3.9) if $W-1 \gg \sqrt{|\beta(\tau)|}$, (3.17), (3.18) if $1-W \gg \sqrt{|\beta(\tau)|}$ and the function $W_{\rm trans}(\tau;W_0)$ that solves (3.12), (3.13), (3.21) for $|W-1| \approx \sqrt{|\beta(\tau)|}$. Note that $W(\tau;W_0)$ is, to the leading order, independent on $\beta(\tau)$ for $|W-1| \gg \sqrt{|\beta(\tau)|}$. Therefore, the stretching of the characteristic curves $W(\tau;W_0)$, is contained, to the leading order, in the dynamics of the function $W_{\rm trans}(\tau;W_0)$. Note that for each $T_{\rm arr}$ and each function $\beta(\tau)$ there is a unique $T_{\rm exit}$. Let us write the functional relation between these quantities as

$$T_{\rm arr} = S(T_{\rm exit}), \tag{3.22}$$

where $S(\cdot)$ is an increasing function such that $S(x) < x$.

We can then examine the precise manner in which we should choose $T_{\rm arr}$, $T_{\rm exit}$ in order to transform the initial data $G_0(W)$ into a self-similar solution. Let us denote as $\bar{W}_0$ the starting value of the characteristic $W(\tau;\bar{W}_0)$ arriving at $W=0$ at the time $\tau=\bar{\tau}$. By assumption

$$\frac{G(\bar{W},\bar{\tau})}{G(0^{+},\bar{\tau})} \approx G_s(\bar{W}) = e^{-F_{\rm int}(\bar{W})}\,, \quad 0 \le \bar{W} < 1\,. \tag{3.23}$$

Then, since along the characteristic curves we have $\frac{dG}{d\tau} = G$, it follows that

$$\frac{G(\bar{W},\bar{\tau})}{G(0^{+},\bar{\tau})} = \frac{G_0(W_0)}{G_0(\bar{W}_0)}\,. \tag{3.24}$$

Combining (3.11), (3.20), (3.22) we obtain

$$\log(W_0) - F(W_0) + b_{\rm ext} = S(\bar{\tau} + F_{\rm int}(\bar{W}) - b_{\rm int}), \tag{3.25}$$

$$\log(\bar{W}_0) - F(\bar{W}_0) + b_{\rm ext} = S(\bar{\tau} - b_{\rm int})\,. \tag{3.26}$$

Subtracting these equations we obtain the following relation between W_0 and $\bar{W}_0$,

$$\frac{W_0}{\bar{W}_0} e^{-F(W_0)+F(\bar{W}_0)} = e^{[S(\bar{\tau}+F_{\rm int}(\bar{W})-b_{\rm int})-S(\bar{\tau}-b_{\rm int})]} \tag{3.27}$$

We are interested in finding the asymptotic behaviours of $G_0(W_0)$ that might be transformed into self-similar behaviours as $\bar{\tau}\to\infty$. Using (3.23)-(3.27) as well as the fact that $\lim_{W_0\to\infty} F(W_0) = 0$, we deduce that such functions $G_0(W_0)$ must satisfy:

$$\frac{G_0\Big(\bar{W}_0 e^{[S(\bar{\tau}+F_{\rm int}(\bar{W})-b_{\rm int})-S(\bar{\tau}-b_{\rm int})]}\Big)}{G_0(\bar{W}_0)} \sim e^{-F_{\rm int}(\bar{W})} \quad \text{as } \bar{W}_0 \to \infty \tag{3.28}$$

We then obtain that the initial data $G_0(W_0)$ that might be transformed in the self-similar solution (2.13) must satisfy (3.28). There are many asymptotic behaviour of the function $G_0(W_0)$ for which the transformation (3.28) can be achieved by means of functions $S(\cdot)$ satisfying $|S'(x)| \ll 1$ as $x\to\infty$. In that case, (3.28) can be written as

$$\frac{G_0\Big(\bar{W}_0 e^{S'(\bar{\tau})F_{\rm int}(\bar{W})}\Big)}{G_0(\bar{W}_0)} \sim e^{-F_{\rm int}(\bar{W})} \quad \text{as } \bar{W}_0 \to \infty\,.$$

Therefore, there exists a function $\lambda(W_0) > 0$ such that

$$\frac{G_0(W_0 e^{\Lambda(W_0)\zeta})}{G_0(W_0)} \sim e^{-\zeta} \quad \text{as } W_0 \to \infty \tag{3.29}$$

uniformly on compact sets of ζ.

On the other hand, if (3.29) holds it is possible to find an approximation of the function $S(\cdot)$ that must relate $T_{\rm arr}$, $T_{\rm exit}$. Since $S'(\bar{\tau}) \approx \Lambda(\bar{W}_0)$ and $\log(\bar{W}_0) - F(\bar{W}_0) + b_{\rm ext} \approx S(\bar{\tau} - b_{\rm int})$ we would obtain, to the leading order

$$S'(\bar{\tau}) \approx \Lambda \exp\big(- b_{\rm ext} + S(\bar{\tau})\big)\,. \tag{3.30}$$

The solution of this differential equation would provide an approximation for the function $S(\cdot)$ in (3.22). Note that for $\lambda(\cdot)$ satisfying (2.19), equation (3.30) would imply a rough asymptotics of the form

$$S(\bar{\tau}) \approx \frac{1}{\alpha} \log(\bar{\tau}) \quad \text{as } \bar{\tau} \to \infty\,. \tag{3.31}$$

The rest of this paper is devoted to a rigorous proof of the previous argument. The key difficulty consists in proving the existence of functions $\beta(\tau)$ satisfying (2.20) and solving (3.1), (3.13), (3.21) with $T_{\rm arr}$, $T_{\rm exit}$ related by means of (3.22), where S solves (3.30). It is not hard to obtain an explicit solution of this problem if (3.1) is replaced by the approximated equation (3.12). This solution will be given in the next Section. Nevertheless, the analysis of the neglected error terms is technically more involved and it will be made in the remaining Sections of the paper.

4. Analysis of the simplified transition problem for the characteristic curves

In this Section we solve the transition problem (3.12), (3.13), (3.21), (3.22) that we reformulate here for convenience. Since deriving this problem we have neglected higher order terms in the Taylor's series we will term it as "Simplified Transition Problem". The solution of this problem has been found in [14], but we recall it here for convenience. Let us write $W_{\rm trans} = 1 + X$. Our goal is, given a strictly monotonic increasing function $S(\cdot)$ satisfying $S(x) < x$, for $x > 0$, to find functions $\beta(\tau)$ such that for any $T_{\rm arr} > T_0$ the unique solution of the problem

$$X_\tau = -\frac{1}{3}X^2 + 3\beta(\tau),\ T_{\rm arr} < \tau < T_{\rm exit} \tag{4.1}$$

$$X((T_{\rm arr})^+) = +\infty \tag{4.2}$$

satisfies

$$X((T_{\rm exit})^-) = -\infty \tag{4.3}$$

with

$$S(T_{\rm exit}) = T_{\rm arr}\,. \tag{4.4}$$

There are several ways of solving problem (4.1)-(4.4). The method used in this Section is convenient because it can be adapted in a perturbative manner to the transition problem one obtains if (3.12) is replaced by (3.1). We compute the solution of (4.1)–(4.4) in a basically explicit manner. In particular, we will assume that S is a smooth as required in the forthcoming computations.

We then proceed to solve (4.1)–(4.4). Let us define a function $f(\zeta) \geq T_0$ strictly monotonically increasing in ζ, defined in $\zeta \geq \zeta_0$ satisfying

$$S(f(\zeta + 1)) = f(\zeta) \tag{4.5}$$

$$f(\zeta_0) = 0, \tag{4.6}$$

where $S(\cdot)$ is as in (4.4).

Our assumptions imply the existence of the inverse function $S^{-1}(\cdot)$. Therefore, given an arbitrary function $f(\zeta)$ in $[\zeta_0, \zeta_0 + 1)$ satisfying (4.6) we obtain f defined in $[\zeta_0, \infty)$ iterating the formula

$$f(\zeta + 1) = S^{-1}(f(\zeta)). \tag{4.7}$$

In further computations we will need f to be differentiable three times. This regularity would hold if the function $f(\zeta)$ defined in $[\zeta_0, \zeta_0 + 1)$ belongs to $C^3[\zeta_0, \zeta_0 + 1]$ and satisfies the following compatibility conditions

$$f(\zeta_0) = f(\zeta_0 + 1) \tag{4.8}$$

$$f'(\zeta_0) = S'(f(\zeta_0 + 1))f'(\zeta_0 + 1) \tag{4.9}$$

$$f''(\zeta_0) = S''(f(\zeta_0 + 1))(f'(\zeta_0 + 1))^2 + S'(f(\zeta_0 + 1))f''(\zeta_0 + 1) \tag{4.10}$$

$$\begin{aligned} f'''(\zeta_0) &= S'''(f(\zeta_0 + 1))(f'(\zeta_0 + 1))^3 \\ &\quad + 3S''(f(\zeta_0 + 1))f'(\zeta_0 + 1)f''(\zeta_0 + 1) + S'(f(\zeta_0 + 1))f'''(\zeta_0 + 1)\,. \end{aligned} \tag{4.11}$$

Therefore, given $f \in C^3[\zeta_0, \zeta_0+1]$ strictly monotonically increasing in $[\zeta_0, \zeta_0+1]$ satisfying (4.8)-(4.11) we obtain, iterating (4.7), an increasing function $f \in C^3[\zeta_0, \infty)$. Moreover, since $S(x) < x$ for any x, it follows that

$$\lim_{\zeta \to \infty} f(\zeta) = \infty\,.$$

Given such a function $f(\cdot)$ we define a new function $\varphi(Y, \zeta)$ as

$$\varphi(Y, \zeta) = \frac{3Y}{f'(\zeta)} + \frac{3f''(\zeta)}{2(f'(\zeta))^2}\,. \tag{4.12}$$

Let us introduce the change of variables

$$\tau = f(\zeta) \tag{4.13}$$

$$X = \varphi(Y, \zeta) \tag{4.14}$$

Suppose that the function $\beta(\tau)$ satisfies

$$\beta(f(\zeta)) = \frac{1}{(f'(\zeta))^2}\big(\frac{1}{2}\,\{f, \zeta\} - \pi^2\big), \tag{4.15}$$

where $\{f, \zeta\}$ is the Schwartzian derivative (cf. [3]),

$$\{f, \zeta\} = \frac{f'''(\zeta)}{f'(\zeta)} - \frac{3}{2}\big(\frac{f''(\zeta)}{f'(\zeta)}\big)^2\,. \tag{4.16}$$

Under this assumption the change of variables (4.13), (4.14) transforms the original transition problem (4.1)–(4.4) into

$$\frac{dY}{d\zeta} + Y^2 + \pi^2 = 0\,, \quad \zeta > \zeta_{\rm arr} \tag{4.17}$$

$$Y((\zeta_{\rm arr})^+) = +\infty \tag{4.18}$$

$$Y((\zeta_{\rm arr} + 1)^-) = -\infty \tag{4.19}$$

where $\tau_{\rm arr} = f(\zeta_{\rm arr})$, and we have used (4.4), (4.5) to obtain (4.19).

Problem (4.17)-(4.19) can be easily solved, since the solution of (4.17), (4.18) is given by

$$Y = Y(\zeta, \zeta_0) = \pi \coth(\pi(\zeta - \zeta_{\rm arr})) \tag{4.20}$$

and it can be inmediately checked that $Y(\zeta, \zeta_0)$ satisfies (4.19).

The previous computations provide a method of finding functions $\beta(\tau)$ solving (4.1)–(4.4). Indeed, given any strictly increasing function $f \in C^3[\zeta_0, \zeta_0 + 1]$ satisfying (4.8)-(4.11) we can obtain $f \in C^3[\zeta_0, \infty)$ iterating (4.7) with the properties stated above. We then define $\beta(\tau)$ by means of (4.15). Then, the function $\beta(\tau)$ solves (4.1)-(4.4).

Remark 4.1. It has been proved in [14] that for functions S satisfying (3.31), (i.e. λ as in (2.19)), the asymptotics of $\beta(\tau)$ is

$$\beta(\tau) \sim -\frac{1}{4}\Big[\frac{1}{(\tau)^2} + \frac{1}{(\tau)^2(\log(\tau))^2} + \frac{1}{(\tau)^2(\log(\tau))^2(\log(\log(\tau)))^2} + \dots\Big]$$

as $\tau \to \infty$. This asymptotics has been previously derived in [4], [13] using different methods.

The main contribution of the rest of the paper we obtain rigorous results analogous to the ones of this Section replacing (4.1) by the complete equation (3.1) and (4.2), (4.3) by a precise formulation of the matching conditions (3.8), (3.10), (3.17), (3.19). To this end we treat this problem as a perturbation of the problem (4.1)–(4.4). Nevertheless the rigorous implementation of this perturbative argument is technically involved. The main reason for this is that, as indicated in [14] there exist infinitely many different ways of finding functions $\beta(\tau)$ that solve (4.1)-(4.4). In particular, the effect of the corrective terms (3.1) as well as the matching condition generate some corrections on $\beta(\tau)$ whose asymptotics is "beyond all the orders" if the asymptotics of $\beta(\tau)$ is computed that could yield an important effect on the behaviour of the function $S(\cdot)$ and for this reason must be studied carefully.

5. Rigorous formulation of the transition problem

5.1. **Restricting the class of functions $\beta(\tau)$.** In this Section we formulate in a precise manner the transition problem that must be solved by characteristics to understand the transformation experienced by the solution of (2.16) as the characteristics cross the critical region $W \approx 1$. In particular we give in this Section a precise meaning to the formal matching conditions (3.13), (3.21).

From now on, we will suppose that the assumptions in Theorem 2.1 hold. Let us define $S(\cdot)$ by means of

$$\int_{S_0}^{S(\tau)} \frac{d\eta}{\Lambda(e^{-b_{\rm ext}+\eta})} = \tau\,, \tag{5.1}$$

where S_0 is a large number that will be precised later. Note that $S(\tau)$ solves the ODE (cf. (3.30)):

$$S'(\tau) = \Lambda(e^{-b_{\rm ext}+S(\tau)})\,. \tag{5.2}$$

It is convenient to impose some constraints in the class of functions $\beta(\tau)$ that will be taken into account in Theorem 2.1. By assumption we restrict ourselves to the class of functions $\lambda(W)$ in Theorem 2.1. Therefore $S(\cdot)$ behaves roughly like in (3.31).

For these functions $S(\cdot)$ the function $\beta(\cdot)$ defined by means of (4.15) satisfies (cf. (8.6) in the Appendix)

$$\log(\beta(\tau)) \sim -2\log(\tau) \quad \text{as } \tau \to \infty\,. \tag{5.3}$$

Therefore, it is natural to assume that the functions $\beta(\cdot)$ belong to the class of functions verifying

$$|\beta(\tau)| \le \frac{A}{(\tau)^{2-\delta}+1} \quad \text{for } \tau \ge 0 \tag{5.4}$$

for some $\delta > 0$, $A > 0$.

5.2. **Approximating the evolution of the characteristics away from the critical line.** As a first step, we obtain a rigorous version of the approximation of $W(\tau; W_0)$ given in (3.5).

The following two Lemmas show that the effect of $\beta(\tau)W^{1/3}$ in (3.1) yields a negligible contribution to the dynamics of the characteristic curves $W(\tau; W_0)$ for W_0 large if $|W-1|$ is not too small.

Lemma 5.1. *Suppose that $W(\tau; W_0)$ is the solution of (3.1), (3.2), that $w_{\rm ext}(s)$ is as in (3.9) with $F_{\rm ext}(\cdot)$ defined in (3.6). Let us assume also (5.4). There exists a function $\varepsilon(W_0)$ satisfying $\lim_{W_0\to\infty} \varepsilon(W_0) = 0$ such that, for any W_0 large enough there holds*

$$|W(\tau; W_0) - w_{\rm ext}(\tau - T_{\rm arr})| \le \frac{\varepsilon(W_0)(w_{\rm ext}(\tau - T_{\rm arr}))^{1/3}}{1 + (\tau - T_{\rm arr})_+^2} \tag{5.5}$$

for $0 \le \tau \le T_{\rm arr} + L(T_{\rm arr})$ where $T_{\rm arr}$ is as in (3.11) and $(x)_+ = \max\{x, 0\}$ and where $L(\cdot)$ is a smooth, increasing function, satisfying

$$C_1(\zeta + 1)^3 \le L(f(\zeta)) \le C_2(\zeta + 1)^3 \tag{5.6}$$

$$L'(\tau) \ll \frac{1}{\tau} \tag{5.7}$$

for some positive constants C_1, C_2, with $f(\zeta)$ defined by means of (4.5), (4.6).

Remark 5.2. Due to (4.5), the function $f(\zeta)$ can be thought as the function $\exp(\exp(\exp(\dots\exp(\zeta))))$ for large ζ. Then, assumption (5.6) means, roughly, that $L(\tau)$ is like $\log(\log(\dots\log(\tau)))$. The estimate (5.5) means that $W(\tau; W_0)$ might be approximated by $w_{\rm ext}(\tau - T_{\rm arr})$ in a "matching region" having the width $\log(\log(\dots\log(\tau)))$. Due to this slow growth of the function L it is not hard to see that it is possible to choose it satisfying (5.7).

Proof. The proof of Lemma 5.1 is basically to a "Gronwall-like" argument for the difference $W(\tau; W_0) - w_{\rm ext}(\tau - T_{\rm arr}) \equiv Z(\tau; W_0)$. Since $W_{\rm ext} \equiv w_{\rm ext}(\tau - T_{\rm arr})$ solves (3.1) with $\beta(\cdot) \equiv 0$, and $W_{\rm ext} > 1$, it follows that

$$Z_\tau = (W_{\rm ext})^{-2/3} Z - Z + O((W_{\rm ext})^{-5/3} Z^2 + |\beta(\tau)| W_{\rm ext}^{1/3}) \tag{5.8}$$

$$Z(0; W_0) = W_0 - w_{\rm ext}(-T_{\rm arr})\,. \tag{5.9}$$

Using (3.9) and the definition of $T_{\rm arr}(W_0)$, it follows that

$$w_{\rm ext}(-T_{\rm arr}) = W_0 \exp(F_{\rm ext}(w_{\rm ext}(-T_{\rm arr})) - F_{\rm ext}(W_0))\,,$$

whence

$$\lim_{W_0\to\infty} Z(0; W_0) = 0\,. \tag{5.10}$$

Solving (5.8) we arrive at

$$
\begin{aligned}
|\frac{Z(\tau;W_0)}{Z_h(\tau;W_0)}| \le |Z(0;W_0)| + C\int_0^\tau \frac{(W_{\rm ext}(s;W_0))^{-5/3}(Z(s;W_0))^2}{Z_h(s;W_0)}ds \\
+ C\int_0^\tau \frac{|\beta(s)|(W_{\rm ext}(s;W_0))^{1/3}}{Z_h(s;W_0)}ds,
\end{aligned}
\tag{5.11}
$$

where

$$
Z_h(\tau;W_0) \equiv \exp\Big(\int_0^\tau [(W_{\rm ext}(s;W_0))^{-2/3} - 1]ds\Big)
$$

and $C > 0$ is a generic constant that might change from line to line. The asymptotics (3.10) implies

$$
0 < C_1 \le \frac{(1+(\tau - T_{\rm arr})_+^2)|Z_h(\tau;W_0)|}{e^{(-(\tau-T_{\rm arr})_- - T_{\rm arr})}} \le C_2,\ 0 \le \tau \le T_{\rm arr} + L(T_{\rm arr}),
$$

where $(x)_- = \min\{x, 0\}$.

Therefore, since $w_{\rm ext} \ge 1$, the first term on the right-hand side of (5.11) combined with (5.10) might be estimated as the right-hand side of (5.5). On the other hand, using (5.4) as well as the asymptotics of $w_{\rm ext}$, we can estimate the last term in (5.11) by means of the right-hand side of (5.5) uniformly in the region $0 \le \tau \le T_{\rm arr} + L(T_{\rm arr})$ as $W_0 \to \infty$. Note that estimating this term we are using the fact that $(L(\tau))^n \ll \tau^{2-\delta}$ as $\tau \to \infty$ for any $n \in \mathbb{R}$. Finally, the term (5.11) might be estimated by means of a classical continuity argument. Indeed, (5.5) holds for small values of τ. Therefore, we can use (5.5) to estimate the second term on the right-hand side of (5.11) and since the resulting contribution is smaller than the right-hand side of (5.5) as $W_0 \to \infty$, the result follows. □

On the other hand, we can approximate the characteristics $W(\tau;W_0)$ after leaving the critical region in an analogous manner.

Lemma 5.3. *Let $W(\tau;W_0)$ be a solution of (3.1) with initial value (3.2). Suppose that at some time $\tau = \bar\tau$ we have $W(\bar\tau;W_0) = \bar W$ where $0 \le \bar W \le 1-\delta$. Let us define $w_{\rm int}(s)$ as in (3.18) with $F_{\rm int}(W)$ as in (3.15). Then, for any $\delta > 0$ there exists $\varepsilon_\delta(W_0)$ satisfying $\lim_{W_0\to\infty} \varepsilon_\delta(W_0) = 0$ such that, for any W_0 large enough and $\tau \ge 0$ there holds*

$$
\begin{gathered}
|W(\tau;W_0) - w_{\rm int}(\tau - T_{\rm exit})| \le \frac{\varepsilon_\delta(W_0)}{1+(T_{\rm exit}-\tau)_+^2}\,, \\
T_{\rm exit} - L(T_{\rm exit}) \le \tau \le \bar\tau,
\end{gathered}
\tag{5.12}
$$

where $T_{\rm exit}$ is as in (3.20) and $L(\tau)$ satisfies (5.6).

Since the proof of this result is basically analogous to the one of Lemma 5.1 we omit it.

5.3. **Relating the evolution of the characteristics $W(\tau, W_0)$ with the long time asymptotics of $G(W,\tau)$.** As a next step we formulate in a rigorous manner the main heuristic result in Section 3.

Lemma 5.4. *Suppose that $G_0(\cdot)$ satisfies the assumptions of in Theorem 2.1. Let us assume that $T_{\rm arr}$ defined in (3.11) as $T_{\rm exit}$ defined in (3.20) are related by means* *of (4.4) where $S(\cdot)$ is as in (5.1). Suppose that $G(W,\tau)$ solves (2.16) with initial data $G_0(\cdot)$. Then*

$$\lim_{\tau\to\infty}\frac{G(W,\tau)}{G(0^+,\tau)}=G_s(W) \tag{5.13}$$

uniformly on compact sets of $W\in[0,1)$, where $G_s(W)$ is as in (2.13).

Proof. Suppose that $W(\tau;W_0)$ solves (3.1), (3.2). Integration by characteristics yields

$$G(W(\tau;W_0),\tau)=G_0(W_0)\,.$$

By assumption $W(\bar\tau;W_0)=\bar W$. Let us define $\bar W_0$ as the starting point of the characteristic vanishing at time $\tau=\bar\tau$, i.e. $W(\bar\tau;\bar W_0)=0$. Therefore (3.24) holds and the definitions of $T_{\rm arr}$, $T_{\rm exit}$ in (3.11), (3.20) imply, arguing as in Section 3 the formulae (3.25), (3.26), (3.27). Henceforth,

$$\frac{G(\bar W,\bar\tau)}{G(0^+,\bar\tau)}=\frac{G_0(\bar W_0e^{[S(\bar\tau+F_{\rm int}(\bar W)-b_{\rm int})-S(\bar\tau-b_{\rm int})]+[F_{\rm ext}(W_0)-F_{\rm ext}(\bar W_0)]})}{G_0(\bar W_0)} \tag{5.14}$$

Since $S(\cdot)$ solves (5.2), the asymptotics (2.19) implies

$$S(\bar\tau+F_{\rm int}(\bar W)-b_{\rm int})-S(\bar\tau)=o(1)\quad\text{as }\bar\tau\to\infty$$

uniformly on compacts sets of $\bar W\in[0,1)$. Therefore, using again (5.2) and (2.19) we obtain

$$\begin{aligned}
&S(\bar\tau+F_{\rm int}(\bar W)-b_{\rm int})-S(\bar\tau-b_{\rm int})\\
&=\int_{\bar\tau-b_{\rm int}}^{\bar\tau+F_{\rm int}(\bar W)-b_{\rm int}}\lambda(e^{-b_{\rm ext}+S(s)})ds\\
&=\frac{CF_{\rm int}(\bar W)(1+o(1))}{(e^{-b_{\rm ext}+S(\bar\tau)})^\alpha}\\
&=\lambda(e^{-b_{\rm ext}+S(\bar\tau)})F_{\rm int}(\bar W)(1+o(1))\quad\text{as }\bar\tau\to\infty
\end{aligned} \tag{5.15}$$

Using (3.26) and (5.2) as well as the fact that $\lim_{W_0\to\infty}F_{\rm ext}(W_0)=0$, we deduce that

$$\bar W_0=e^{-b_{\rm ext}+F(\bar W_0)+S(\bar\tau-b_{\rm int})}=e^{-b_{\rm ext}+S(\bar\tau)}(1+o(1))$$

as $\bar\tau\to\infty$. Therefore, (2.19) and (5.15) yield

$$S(\bar\tau+F_{\rm int}(\bar W)-b_{\rm int})-S(\bar\tau-b_{\rm int})=\lambda(\bar W_0)F_{\rm int}(\bar W)(1+o(1))\quad\text{as }\bar\tau\to\infty\,.$$

Plugging this formula in (5.14) and using the asymptotics of $F_{\rm ext}$ as $W_0\to\infty$ we obtain

$$\frac{G(\bar W,\bar\tau)}{G(0^+,\bar\tau)}=\frac{G_0\big(\bar W_0e^{\lambda(\bar W_0)F_{\rm int}(\bar W)(1+o(1))+O(\frac{W_0-\bar W_0}{\bar W_0^{5/3}})}\big)}{G_0(\bar W_0)}\quad\text{as }\bar\tau\to\infty$$

Using (2.18) it then follows that

$$\frac{G(\bar W,\bar\tau)}{G(0^+,\bar\tau)}\sim e^{-F_{\rm int}(\bar W)}(1+o(1))=G_s(\bar W)(1+o(1))\quad\text{as }\bar\tau\to\infty$$

uniformly on compact sets of $\bar W\in[0,1)$, whence (5.13) follows. □

5.4. **Dynamics of the characteristic curves near the critical line $W = 1$.** We have then obtained that in order to transform by means of (2.16) the initial data $G_0(\cdot)$ in a function $G(W,\tau)$ behaving in a self-similar manner as $\tau \to \infty$, we need to find functions $\beta(\cdot)$ such that (4.4) holds, with $T_{\rm arr}$, $T_{\rm exit}$ as in (3.11), (3.20). As a next step we transform this question in a problem analogous to the Simplified Transition Problem (4.1)-(4.4) but with a slightly perturbed equation (4.1). More precisely, we rewrite (3.1) as

$$W_\tau = -\frac{1}{3}(W-1)^2 + \bar{h}_1(W) + 3\beta(\tau)(1+\bar{h}_2(W)), \tag{5.16}$$

where

$$\frac{|\bar{h}_1(W)|}{|W-1|^3} + \frac{|\bar{h}_2(W)|}{|W-1|} \le C \quad \text{for} \quad |W-1| \le \frac{1}{2}, \tag{5.17}$$

$$\frac{|\bar{h}_1'(W)|}{|W-1|^2} + |\bar{h}_2'(W)| \le C \quad \text{for} \quad |W-1| \le \frac{1}{2}\,. \tag{5.18}$$

We define smooth functions $h_1(W,\tau),\ h_2(W,\tau)$ as

$$h_1(W,\tau) \equiv h_1(W)\xi((W-1)L(\tau))$$
$$h_2(W,\tau) \equiv h_2(W)\xi((W-1)L(\tau))$$

where $\xi(\cdot)$ is a smooth function satisfying

$$\xi(s) = \begin{cases} 1 & \text{for } |s| \le 1, \\ 0 & \text{for } |s| \ge 2\,. \end{cases}$$

Note that (5.17), (5.18) imply

$$\begin{gathered} h_1(W,\tau) = \bar{h}_1(W), \quad \text{if } |W-1| \le \frac{6}{L(\tau)}, \\ |h_1(W,\tau)| \le \frac{K}{(L(\tau))^3}, \quad \text{if } |W-1| \ge \frac{6}{L(\tau)}, \end{gathered} \tag{5.19}$$

$$\begin{gathered} h_2(W,\tau) = \bar{h}_2(W), \quad \text{if } |W-1| \le \frac{6}{L(\tau)}, \\ |h_2(W,\tau)| \le \frac{K}{L(\tau)}, \quad \text{if } |W-1| \ge \frac{6}{L(\tau)}, \end{gathered} \tag{5.20}$$

where $K > 0$ is a fixed numerical constant. Suppose that

$$|W(\tau;W_0) - 1| \le \frac{1}{2}\frac{1}{L(\tau)} \quad \text{for} \ \ \tau \in [T_{\rm arr} + L(T_{\rm arr}), T_{\rm exit} - L(T_{\rm exit})]. \tag{5.21}$$

Then Lemmas 5.1, 5.3 as well as the asymptotics (3.10), (3.19) imply that this inequality holds for $\tau = T_{\rm arr} + L(T_{\rm arr})$, $\tau = T_{\rm exit} - L(T_{\rm exit})$. Under the assumption (5.21) would be possible to replace (5.16) by

$$\tilde{W}_\tau = -\frac{1}{3}(\tilde{W}-1)^2 + h_1(\tilde{W},\tau) + 3\beta(\tau)(1+h_2(\tilde{W},\tau))\,. \tag{5.22}$$

for $\tau \in [T_{\rm arr} + L(T_{\rm arr}), T_{\rm exit} - L(T_{\rm exit})]$. Moreover, if (5.21) holds for the solutions of (5.22), a similar inequality would also be satisfied for the solutions of (5.16).

The advantage of (5.22) with respect to (5.16) is that the functions $h_1(W,\tau)$, $h_2(W,\tau)$ are globally bounded in $\mathbb{R}$. Therefore, it is possible to define for it a transition problem with singular boundary conditions analogous to (4.2), (4.3). It turns out that such a singular boundary conditions will be shown to be convenient in order to deal with the corresponding transition problem by means of perturbations of (4.1)-(4.4).

Note that since $\beta(\tau) \to 0$ as $\tau \to \infty$ (cf. (5.4)) and due to (5.19), (5.20) the solution of (5.22) satisfying $\tilde{W}(T_{\rm arr} + L(T_{\rm arr})) = W(T_{\rm arr} + L(T_{\rm arr}); W_0)$ becomes singular for some $\tau < T_{\rm arr} + L(T_{\rm arr})$ for $T_{\rm arr}$ large enough. In a similar manner, the solution of (5.22) satisfying $\tilde{W}(T_{\rm exit} - L(T_{\rm exit})) = W(T_{\rm exit} - L(T_{\rm exit}); W_0)$ blows up for some $\tau > T_{\rm exit} - L(T_{\rm exit})$. It would be then possible to define a transition problem analogous to (4.4) between those blow-up times. Moreover, due to the fact that the function $L(\tau)$ varies very slowly we can approximate $h_1(\tilde{W}, \tau)$ as $h_1(\tilde{W}, T_{\rm arr})$, $h_1(\tilde{W}, T_{\rm exit})$in the regions $\tau \approx T_{\rm arr}$, $\tau \approx T_{\rm exit}$ respectively. On the other hand, for long times, and since $\beta(\tau) \to 0$ as $\tau \to \infty$ it would be natural to approximate (5.22), in the computations of the singularity times, by means of the simpler equations

$$\tilde{W}_\tau = -\frac{1}{3}(\tilde{W} - 1)^2 + h_1(\tilde{W}, T_{\rm arr}), \tag{5.23}$$

$$\tilde{W}_\tau = -\frac{1}{3}(\tilde{W} - 1)^2 + h_1(\tilde{W}, T_{\rm exit})\,. \tag{5.24}$$

We then define new singular times as follows. We define $T_{\rm arr,mod}$, $T_{\rm exit,mod}$ by means of

$$\int_{w_{\rm ext}(L(T_{\rm arr}))}^{\infty} \frac{d\eta}{\frac{(\eta-1)^2}{3} - h_1(\eta, T_{\rm arr})} = (T_{\rm arr} + L(T_{\rm arr}) - T_{\rm arr,mod}), \tag{5.25}$$

$$\int_{-\infty}^{w_{\rm int}(-L(T_{\rm exit}))} \frac{d\eta}{\frac{(\eta-1)^2}{3} - h_1(\eta, T_{\rm exit})} = (T_{\rm exit,mod} - T_{\rm exit} + L(T_{\rm exit}))\,. \tag{5.26}$$

Using the asymptotics of $w_{\rm ext}$, $w_{\rm int}$ in (3.10), (3.19) as well as the bounds for h_1 in (5.19)) it follows that the differences $|T_{\rm arr} - T_{\rm arr,mod}|$, $|T_{\rm exit,mod} - T_{\rm exit}|$ are smaller than $L(T_{\rm arr})$, $L(T_{\rm exit})$ respectively.

Given a function $S(\cdot)$ defined by means of (5.1) we define a new function $\tilde{S}(\cdot)$ by means of:

$$\tilde{S}(T_{\rm exit,mod}) = T_{\rm arr,mod}, \tag{5.27}$$

where $T_{\rm arr,mod}$, $T_{\rm exit,mod}$ are as in (5.25), (5.26) and $T_{\rm exit}$, $T_{\rm arr}$ are related by means of (4.4).

We now show in a rigorous manner that the effect of replacing (5.22) by (5.23) is small as $W_0 \to \infty$.

Lemma 5.5. *Let us denote as $\tilde{W}(\tau; W_0)$ the unique solution of (5.22) that verifies $\tilde{W}((T_{\rm arr,mod})^+; W_0) = +\infty$, with $T_{\rm arr,mod}$ as in (5.25) and $T_{\rm arr}$ as in (3.11). There exists $\varepsilon(W_0)$, satisfying $\lim_{W_0 \to \infty} \varepsilon(W_0) = 0$ such that*

$$|\tilde{W}(T_{\rm arr} + L(T_{\rm arr}); W_0) - w_{\rm ext}(L(T_{\rm arr}))| \le \frac{\varepsilon(W_0)}{(L(T_{\rm arr}))^2}\,. \tag{5.28}$$

Moreover, if $\tilde{W}(\tau; W_0)$ satisfies $\tilde{W}((T_{\rm exit,mod})^-; W_0) = -\infty$ with $T_{\rm exit,mod}$ as in (5.26) and $T_{\rm exit}$ as in (3.20) there exists $\varepsilon(W_0)$ such that

$$|\tilde{W}(T_{\rm exit} - L(T_{\rm exit}); W_0) - w_{\rm int}(-L(T_{\rm exit}))| \le \frac{\varepsilon(W_0)}{(L(T_{\rm exit}))^2}\,. \tag{5.29}$$

Proof. Let $\tilde{W}_{\rm app}(\tau;W_0)$ be the solution of (5.23) satisfying $\tilde{W}_{\rm app}((T_{\rm arr,mod})^+;W_0) = +\infty$. Since $\tilde{W}_{\rm app}(\tau;W_0)$ might be computed explicitly it follows from (5.25) that $\tilde{W}_{\rm app}(\tau;W_0) = w_{\rm ext}(L(T_{\rm arr}))$. Subtracting (5.22) and (5.23) we obtain

$$\begin{aligned}(\tilde{W}-\tilde{W}_{\rm app})_\tau &= -\frac{2}{3}(\tilde{W}_{\rm app}-1)(\tilde{W}-\tilde{W}_{\rm app}) - \frac{1}{3}(\tilde{W}-\tilde{W}_{\rm app})^2 \\ &\quad + \big(h_1(\tilde{W},\tau)-h_1(\tilde{W}_{\rm app},T_{\rm arr})\big) + 3\beta(\tau)(1+h_2(\tilde{W},\tau))\,,\end{aligned} \tag{5.30}$$

where, unless it is explicitly needed we will drop the dependence of the functions on W_0 for simplicity.

A local analysis of the asymptotics of $\tilde{W}(\tau)$, $\tilde{W}_{\rm app}(\tau)$ as $\tau\to(T_{\rm arr,mod})^+$ imply that $|\tilde{W}(\tau)-\tilde{W}_{\rm app}(\tau)| = O\Big((\tau-(T_{\rm arr,mod})^+)\Big)$ as $\tau\to(T_{\rm arr,mod})^+$. Therefore, integrating the equation for $\tilde{W}-\tilde{W}_{\rm app}$ we obtain

$$\begin{aligned}&\tilde{W}(\tau)-\tilde{W}_{\rm app}(\tau)\\ &= -\frac{1}{3}\int_{T_{\rm arr,mod}}^{\tau} ds\exp\Big(-\frac{2}{3}\int_s^\tau(\tilde{W}_{\rm app}(\xi)-1)d\xi\Big)\\ &\quad\times\Big[(\tilde{W}(s)-\tilde{W}_{\rm app}(s))^2+\Big(h_1(\tilde{W}(s),s)-h_1(\tilde{W}_{\rm app}(s),T_{\rm arr})\Big)\\ &\quad+3\beta(s)\big(1+h_2(\tilde{W}(s),s)\big)\Big]\,.\end{aligned}$$

Using (5.7), (5.17), (5.18) we obtain

$$\begin{aligned}&|\tilde{W}(\tau)-\tilde{W}_{\rm app}(\tau)|\\ &\le C\int_{T_{\rm arr,mod}}^{\tau}\frac{\Phi(\tau-T_{\rm arr,mod})}{\Phi(s-T_{\rm arr,mod})}\\ &\quad\times\Big[(\tilde{W}(s)-\tilde{W}_{\rm app}(s))^2+\frac{|\tilde{W}(s)-\tilde{W}_{\rm app}(s)|}{L^2}+\frac{1}{s}+|\beta(s)|\Big]ds,\end{aligned}$$

where

$$\Phi(\tau-T_{\rm arr,mod})\equiv\exp\Big(-\frac{2}{3}\int_{T_{\rm arr,mod}-1}^{\tau}(\tilde{W}_{\rm app}(\xi)-1)d\xi\Big)$$

satisfies

$$0<C_1\le\tau^2\Phi(\tau)\le C_2<+\infty,\quad \tau>0;$$

whence,

$$\begin{aligned}&|\tilde{W}(\tau)-\tilde{W}_{\rm app}(\tau)|\\ &\le C\int_{T_{\rm arr,mod}}^{\tau}\Big(\frac{s-T_{\rm arr,mod}}{\tau-T_{\rm arr,mod}}\Big)^2\\ &\quad\times\Big[(\tilde{W}(s)-\tilde{W}_{\rm app}(s))^2+\frac{|\tilde{W}(s)-\tilde{W}_{\rm app}(s)|}{(L(\tau))^2}+\frac{1}{s}+|\beta(s)|\Big]ds\,.\end{aligned} \tag{5.31}$$

Due to (5.4), (5.6) the last two terms in (5.31) can be estimated as $\varepsilon(W_0)\min\{(\tau-T_{\rm arr,mod}),\frac{1}{(\tau-T_{\rm arr,mod})^2}\}$ with $\lim_{W_0\to\infty}\varepsilon(W_0)=0$ for $T_{\rm arr,mod}\le\tau\le T_{\rm arr}+L(T_{\rm arr})$. Then, a Gronwall-like estimate yields the bound

$$|\tilde{W}(\tau)-\tilde{W}_{\rm app}(\tau)|\le C\varepsilon(W_0)\min\big\{(\tau-T_{\rm arr,mod}),\frac{1}{(\tau-T_{\rm arr,mod})^2}\big\}$$

that implies (5.28). Estimate (5.29) can be proved in a similar manner. □

Combining Lemmas 5.1, 5.3, 5.5, we obtain the following result.

Proposition 5.6. *Suppose that $W(\tau;W_0)$ is the unique solution of (3.1), (3.2). Let us assume also that $W(\bar{\tau};W_0)=\bar{W}$ where $0\le\bar{W}\le 1-\delta$ for some $\bar{\tau}>0$. Suppose that $\tilde{W}(\tau;W_0)$ is as in Lemma 5.5 and that $T_{\rm arr}$, and $T_{\rm exit}$ are as in (3.11), (3.20) respectively and are related by means of (4.4). Then there exists $\varepsilon(W_0)$ satisfying $\lim_{W_0\to\infty}\varepsilon(W_0)=0$, such that*

$$|\tilde{W}(T_{\rm arr}+L(T_{\rm arr});W_0)-W(T_{\rm arr}+L(T_{\rm arr});W_0)|\le\frac{\varepsilon(W_0)}{(L(T_{\rm arr}))^2}, \tag{5.32}$$

$$|\tilde{W}(T_{\rm exit}-L(T_{\rm exit});W_0)-W(T_{\rm exit}-L(T_{\rm exit});W_0)|\le\frac{\varepsilon(W_0)}{(L(T_{\rm exit}))^2}\,. \tag{5.33}$$

As indicated above, if (5.21) holds we would have that $W(\tau;W_0)$ solves (5.22) for $\tau\in[T_{\rm arr}+L(T_{\rm arr}),T_{\rm exit}-L(T_{\rm exit})]$. However, due to the error terms in (5.32), (5.33) we cannot ensure that $W(\tau;W_0)=\tilde{W}(\tau;W_0)$. Nevertheless, the following result holds.

Lemma 5.7. *Suppose that $W(\tau;W_0)$, $\tilde{W}(\tau;W_0)$ are as in Proposition 5.6. Let us suppose also that (5.21) holds in the interval $[T_{\rm arr}+L(T_{\rm arr}),T_{\rm exit}-L(T_{\rm exit})]$. Then there exists a function $\mu(W_0)$ such that for W_0 large enough,*

$$W(\tau;W_0+\mu(W_0))=\tilde{W}(\tau;W_0) \tag{5.34}$$

for $\tau\in[T_{\rm arr}+L(T_{\rm arr}),T_{\rm exit}-L(T_{\rm exit})]$, where

$$\frac{\mu(W_0)}{W_0}\to 0\quad \text{as } W_0\to\infty\,. \tag{5.35}$$

Moreover, suppose that $T_{\rm exit}$ is defined as in (3.20). Let us denote as $\bar{W}=W(\bar{\tau};W_0)$ Then

$$|W(\bar{\tau};W_0+\mu(W_0))-\bar{W}|\to 0\quad \text{as } W_0\to\infty \tag{5.36}$$

uniformly on compact sets of $0\le\bar{W}<1$.

Proof. The basic idea consists in estimating the derivative $\frac{\partial W(\tau;W_0)}{\partial W_0}$ for $\tau=T_{\rm arr}+L(T_{\rm arr})$. To this end, notice that, differentiating (3.1), (3.2) we obtain

$$(\frac{\partial W}{\partial W_0})_\tau=(W^{-2/3}-1+\beta(\tau)W^{-2/3})\frac{\partial W}{\partial W_0}, \tag{5.37}$$

$$\frac{\partial W}{\partial W_0}(0;W_0)=1\,. \tag{5.38}$$

Let us recall that $W_{\rm ext}(\tau,W_0)\equiv w_{\rm ext}(\tau-T_{\rm arr}(W_0))$. Using (3.8), (3.9) and (3.11) we obtain

$$(\frac{\partial W_{\rm ext}}{\partial W_0})_\tau=(W_{\rm ext}^{-2/3}-1)\frac{\partial W_{\rm ext}}{\partial W_0}, \tag{5.39}$$

$$\frac{\partial W_{\rm ext}}{\partial W_0}(0;W_0)=1\,. \tag{5.40}$$

Subtracting (5.37), (5.39) we obtain, after some computations

$$\begin{aligned}(\frac{\partial W}{\partial W_0}-\frac{\partial W_{\rm ext}}{\partial W_0})_\tau&=(W_{\rm ext}^{-2/3}-1)(\frac{\partial W}{\partial W_0}-\frac{\partial W_{\rm ext}}{\partial W_0})\\&\quad+(W^{-2/3}-W_{\rm ext}^{-2/3})\frac{\partial W}{\partial W_0}+\beta(\tau)W^{-2/3}\frac{\partial W}{\partial W_0}\,.\end{aligned} \tag{5.41}$$

We now claim that

$$|\frac{\partial W}{\partial W_0} - \frac{\partial W_{\rm ext}}{\partial W_0}| \le \varepsilon(W_0)\frac{\partial W_{\rm ext}}{\partial W_0} \tag{5.42}$$

for $0 \le \tau \le T_{\rm arr} + L(T_{\rm arr})$, where $\lim_{W_0\to\infty} \varepsilon(W_0) = 0$. Estimate (5.42) follows by means of a continuation argument. Indeed, suppose that the following estimate holds

$$\frac{\partial W}{\partial W_0} \le 2\frac{\partial W_{\rm ext}}{\partial W_0}\,. \tag{5.43}$$

This estimate is certainly valid for $\tau = 0$, and it might be shown to be extended to arbitrary values of $\tau \le T_{\rm arr} + L(T_{\rm arr})$, assuming that it is valid for previous times. On the other hand it is possible to obtain an improvement of (5.5) as follows. The term $|\beta(\tau)|W_{\rm ext}^{1/3}$ in (5.8) plays the role of "source" in that differential equation. Due to the exponential decay of $W_{\rm ext}$ it follows that this term is smaller than $CW_0^{1/3}e^{-\frac{T_{\rm arr}}{2}}$ for $0 \le \tau \le \frac{T_{\rm arr}}{2}$, and due to (5.4) it can be estimated as $\frac{CW_{\rm ext}^{1/3}}{(T_{\rm arr})^{2-\delta}}$ for $\frac{T_{\rm arr}}{2} \le \tau \le T_{\rm arr} + L(T_{\rm arr})$. Arguing as in the Proof of Lemma 5.1 it follows that similar estimates might be obtained for the difference $|W - W_{\rm ext}|$. Therefore, the term $(W^{-2/3} - W_{\rm ext}^{-2/3})$ in (5.41) can be estimated as $\frac{C}{W_{\rm ext}}$ for $0 \le \tau \le \frac{T_{\rm arr}}{2}$ and as $\frac{C}{(T_{\rm arr})^{2-\delta}}$ for $\frac{T_{\rm arr}}{2} \le \tau \le T_{\rm arr} + L(T_{\rm arr})$. A Gronwall's like argument as the one in the proof of Lemma 5.1 combined with the fact that $L(T_{\rm arr}) \ll T_{\rm arr} \ll W_{\rm ext}(\frac{T_{\rm arr}}{2}, W_0) \sim C\sqrt{W_0}$ yields the estimate (5.42), that for the particular value $\tau = T_{\rm arr} + L(T_{\rm arr})$ becomes

$$\frac{\partial W}{\partial W_0}(T_{\rm arr} + L(T_{\rm arr}); W_0) = -\frac{\partial T_{\rm arr}}{\partial W_0} w'_{\rm ext}(L(T_{\rm arr}))(1 + o(1)) \quad \text{as } W_0 \to \infty\,.$$

Since $w'_{\rm ext}(L(T_{\rm arr})) = -\frac{3}{(L(T_{\rm arr}))^2}$, and $T'_{\rm arr}(W_0) \sim \frac{1}{W_0}$ as $W_0 \to \infty$ (cf. (3.11)), we have

$$\frac{\partial W}{\partial W_0}(T_{\rm arr} + L(T_{\rm arr}); W_0) = \frac{3}{W_0}\frac{1}{(L(T_{\rm arr}))^2}(1 + o(1)) \quad \text{as } W_0 \to \infty\,. \tag{5.44}$$

Then (5.32) implies the existence of δW_0 as indicated in Lemma 5.7 satisfying $|\frac{3\delta W_0}{W_0}| \le 2\varepsilon(W_0)$, whence (5.35) follows.

To obtain (5.36) we write

$$W(\tau; W_0) = V(\tau; \bar{W})\,.$$

Note that by assumption $V(\bar{\tau}; \bar{W}) = \bar{W}$. Differentiating (3.1) we obtain

$$(\frac{\partial V}{\partial \bar{W}})_\tau = \big(V^{-2/3} - 1 + \beta(\tau)V^{-2/3}\big)\frac{\partial V}{\partial \bar{W}}, \tag{5.45}$$

$$\frac{\partial V}{\partial \bar{W}}(\bar{\tau}; \bar{W}) = 1\,. \tag{5.46}$$

Using (5.37) and (5.12) we obtain arguing as in the derivation of (5.44),

$$\frac{\partial V}{\partial \bar{W}}(T_{\rm exit} - L(T_{\rm exit}); \bar{W}) = -\frac{\partial T_{\rm exit}}{\partial \bar{W}} w'_{\rm int}(-L(T_{\rm exit}))(1 + o(1)) \quad \text{as } W_0 \to \infty\,.$$

Taking into account that $\frac{\partial T_{\rm exit}}{\partial \bar{W}} = F'_{\rm int}(\bar{W})$, and $w'_{\rm int}(-L(T_{\rm exit})) = -\frac{3}{(L(T_{\rm exit}))^2}$ is bounded in compact sets of $\bar{W} \in [0, 1)$ we obtain from (5.33) that $\tilde{W}(\tau; W_0) = V(\tau; \bar{W} + \delta\bar{W})$ with $\delta\bar{W} \to 0$ as $W_0 \to \infty$ whence Lemma 5.7. □

It is possible to obtain some regularity for $\mu(W_0)$ defined in Lemma 5.7.

Lemma 5.8. *Suppose that $\mu(W_0)$ is as in Lemma 5.7. Then*

$$\lim_{W_0\to\infty} |\frac{\partial \mu(W_0)}{\partial W_0}| = 0 .$$

Proof. Given W_0 large enough, let us write $\hat{\tau} \equiv T_{\rm arr} + L(T_{\rm arr})$. It then follows from (5.34) that

$$W(\hat{\tau}; W_0 + \mu(W_0)) = \tilde{W}(\hat{\tau}; W_0) .$$

Differentiating this formula with respect to W_0, we obtain

$$\begin{aligned} \frac{\partial \mu(W_0)}{\partial W_0} &= \Big(\frac{\partial W(\hat{\tau}; W_0 + \mu(W_0))}{\partial W_0}\Big)^{-1} \\ &\quad \times \Big[\frac{\partial \hat{\tau}}{\partial W_0}\Big(\frac{\partial \tilde{W}(\hat{\tau}; W_0)}{\partial \tau} - \frac{\partial W(\hat{\tau}; W_0 + \mu(W_0))}{\partial \tau}\Big) \\ &\quad + \frac{\partial \tilde{W}(\hat{\tau}; W_0)}{\partial W_0} - \frac{\partial W(\hat{\tau}; W_0 + \mu(W_0))}{\partial W_0}\Big]. \end{aligned}$$

Using (5.44) we obtain

$$|\frac{\partial \mu(W_0)}{\partial W_0}| \le C(L(T_{\rm arr}))^2 W_0[K + J_1 + J_2] \tag{5.47}$$

for some $C > 0$ independent on L, W_0, where

$$\begin{gathered} K \equiv \frac{\partial \hat{\tau}}{\partial W_0}\Big(\frac{\partial \tilde{W}(\hat{\tau}; W_0)}{\partial \tau} - \frac{\partial W(\hat{\tau}; W_0 + \mu(W_0))}{\partial \tau}\Big), \\ J_1 \equiv |\frac{\partial(\tilde{W}(\hat{\tau}; W_0) - w_{\rm ext}(\hat{\tau} - T_{\rm arr}))}{\partial W_0}|, \\ J_2 \equiv |\frac{\partial(W(\hat{\tau}; W_0 + \mu(W_0)) - w_{\rm ext}(\hat{\tau} - T_{\rm arr}))}{\partial W_0}| . \end{gathered}$$

To estimate J_1, we use the auxiliary function $\tilde{W}_{\rm app}(\hat{\tau}; W_0)$ defined in the proof of Lemma 5.5. Then $J_1 \le J_{1,1} + J_{1,2}$, where

$$\begin{gathered} J_{1,1} \equiv |\frac{\partial(\tilde{W}(\hat{\tau}; W_0) - \tilde{W}_{\rm app}(\hat{\tau}; W_0))}{\partial W_0}|, \\ J_{1,2} \equiv |\frac{\partial(\tilde{W}_{\rm app}(\hat{\tau}; W_0) - w_{\rm ext}(\hat{\tau} - T_{\rm arr}))}{\partial W_0}| . \end{gathered}$$

To estimate $J_{1,1}$ we recall that $\tilde{W}(\tau; W_0, L) - \tilde{W}_{\rm app}(\tau; W_0, L)$ satisfies (5.30) with $\tilde{W} - \tilde{W}_{\rm app} = O((\tau - (T_{\rm arr,mod})^+))$ as $\tau \to (T_{\rm arr,mod})^+$. Differentiating (5.30) with respect to $T_{\rm arr,mod}$ we obtain

$$\begin{aligned} &\Big(\frac{\partial(\tilde{W} - \tilde{W}_{\rm app})}{\partial T_{\rm arr,mod}}\Big)_\tau \\ &= -\frac{2}{3}(\tilde{W}_{\rm app} - 1)\frac{\partial(\tilde{W} - \tilde{W}_{\rm app})}{\partial T_{\rm arr,mod}} - \frac{2}{3}(\frac{\partial \tilde{W}_{\rm app}}{\partial T_{\rm arr,mod}})(\tilde{W} - \tilde{W}_{\rm app}) \\ &\quad - \frac{2}{3}(\tilde{W} - \tilde{W}_{\rm app})\frac{\partial(\tilde{W} - \tilde{W}_{\rm app})}{\partial T_{\rm arr,mod}} + \frac{\partial(h_1(\tilde{W}, W_0) - h_1(\tilde{W}_{\rm app}, W_0))}{\partial T_{\rm arr,mod}} \\ &\quad + 3\beta(\tau)\frac{\partial(h_2(\tilde{W}, W_0))}{\partial T_{\rm arr,mod}}, \end{aligned}$$

$$\frac{\partial(\tilde{W}-\tilde{W}_{\rm app})}{\partial T_{\rm arr,mod}} = O(1) \quad \text{as } \tau \to (T_{\rm arr,mod})^+ .$$

Note that the term $\frac{2}{3}(\frac{\partial \tilde{W}_{\rm app}}{\partial W_0})(\tilde{W}-\tilde{W}_{\rm app})$ might be bounded as $\frac{C}{(\tau-(T_{\rm arr,mod})^+)}$. The asymptotics of the linear term $-\frac{2}{3}(\tilde{W}_{\rm app}-1)\frac{\partial(\tilde{W}-\tilde{W}_{\rm app})}{\partial T_{\rm arr,mod}}$ is exactly the same as in (5.30). Using then Lemma 5.5 to estimate $(\tilde{W}-\tilde{W}_{\rm app})$ as well as Gronwall-like arguments analogous to the ones used in the proof of Lemma 5.5, as well as the fact that $\frac{dT_{\rm arr,mod}}{dW_0} \approx \frac{C}{W_0}$ we obtain

$$\lim_{W_0\to\infty} W_0 J_{1,1} = 0 . \tag{5.48}$$

We now proceed to estimate the term $J_{1,2}$. Note that $\tilde{W}_{\rm app}(\tau;W_0) = w_{\rm app}(\tau - T_{\rm arr,mod}, W_0)$, with

$$w_{\rm app,s} = -\frac{1}{3}(w_{\rm app}-1)^2 + h_1(w_{\rm app},W_0) , \quad s>0 ,$$
$$w_{\rm app}(0^+) = +\infty .$$

It then follows from (5.25) that $w_{\rm app}(L(T_{\rm arr}),W_0) = w_{\rm ext}(L(T_{\rm arr}))$, whence since the ODE satisfied for $w_{\rm app}$, $w_{\rm ext}$ is the same for $s>L$ it follows that

$$J_{1,2} = 0 . \tag{5.49}$$

Finally we estimate J_2. To this end, we argue as in the proof of Lemma 5.1. Note that $\frac{\partial(W(\hat{\tau};W_0+\mu(W_0)))}{\partial W_0} - \frac{\partial(w_{\rm ext}(\hat{\tau}-T_{\rm arr}))}{\partial W_0}$ solves (5.41) with zero initial data at $\tau=0, W$ since $\frac{\partial W}{\partial W_0}(0,W_0) = \frac{\partial(w_{\rm ext}(-T_{\rm arr}))}{\partial W_0} = 1$. We can then argue as in the proof of (5.42) to obtain

$$J_2 \le \frac{\varepsilon(W_0)}{W_0(L(\tau))^2};$$

whence

$$W_0 L^2 J_2 \to 0 \quad \text{as } W_0 \to \infty . \tag{5.50}$$

Finally we estimate K in (5.47). Using (5.16), (5.19), (5.22), (5.34) as well as the fact that for $\tau=\hat{\tau}$, $|W-1| \le \frac{6}{L(T_{\rm arr})}$ (cf. (5.21)) it follows that $K=0$. Combining this with (5.48), (5.49) and (5.50) the result follows. □

We can now prove the main result of this Section.

Theorem 5.9. *Suppose that $G_0(\cdot)$ satisfies the assumptions in Theorem 2.1. Let us define S as in (5.1). Suppose that $\beta(\tau)$ in (5.22) is chosen in such a way that the function $\tilde{W}(\tau;W_0)$ defined by means of (5.22) and*

$$\tilde{W}\big((T_{\rm arr,mod})^+;W_0\big) = +\infty. \tag{5.51}$$

Suppose that

$$\tilde{W}\big((T_{\rm exit,mod})^-;W_0\big) = -\infty , \tag{5.52}$$

where $T_{\rm arr,mod}$, $T_{\rm exit,mod}$ are related by means of (5.27). Then, $G(W,\tau)$ solution of (2.16) with initial data $G_0(\cdot)$ satisfies

$$\lim_{\tau\to\infty} \frac{G(W,\tau)}{G(0^+,\tau)} = G_s(W) \tag{5.53}$$

uniformly on compact sets of $W\in[0,1)$.

Remark 5.10. Theorem 5.9 reduces the Proof of Theorem 2.1 to the problem of finding $\beta(\tau)$ solving the Transition Problem (5.22), (5.51), (5.52).

Proof. Due to Proposition 5.6 and Lemma 5.7 we have that the solutions of (3.1) starting at $W_0+\mu(W_0)$ at time $\tau=0$ arrives to a point $\bar{W}+\nu(\bar{W})$ at time $\tau=\bar{\tau}$, with $\frac{\mu(W_0)}{W_0}\ll 1$, $\nu(\bar{W})\ll 1$. Let us denote as $\bar{W}_0+\mu(\bar{W}_0)$ the starting point for the characteristic vanishing at time $\tau=\bar{\tau}$, i.e. the characteristic for which $\bar{W}+\nu(\bar{W})=0$ at time $\tau=\bar{\tau}$. Arguing as in Lemma 5.4 we have

$$\frac{G(\bar{W}+\nu(\bar{W}),\bar{\tau})}{G(0^+,\bar{\tau})}=\frac{G_0(W_0+\mu(W_0))}{G_0(\bar{W}_0+\mu(\bar{W}_0))}\,.$$

Since W_0, $\bar{W}_0$ are related as in Lemma 5.4 we can argue as in the Proof of that result to obtain

$$\frac{G(\bar{W},\bar{\tau})}{G(0^+,\bar{\tau})}=\frac{G_0\Big(\bar{W}_0 e^{\lambda(\bar{W}_0)F_{\rm int}(\bar{W})(1+o(1))+O(\frac{W_0-\bar{W}_0}{\bar{W}_0^{5/3}})}+\mu(W_0)\Big)}{G_0\big(\bar{W}_0+\mu(\bar{W}_0)\big)} \tag{5.54}$$

as $\bar{\tau}\to\infty$. Using Lemma 5.8 it follows that $|\mu(W_0)-\mu(\bar{W}_0)|=o(|W_0-\bar{W}_0|)$ as $\bar{\tau}\to\infty$. On the other hand, to the leading order $\frac{W_0-\bar{W}_0}{\bar{W}_0}\sim\lambda(\bar{W}_0)F_{\rm int}(\bar{W})$ (cf. (5.15)). Therefore,

$$\begin{aligned}
&\bar{W}_0 e^{\lambda(\bar{W}_0)F_{\rm int}(\bar{W})(1+o(1))+O(\frac{W_0-\bar{W}_0}{\bar{W}_0^{5/3}})}+\mu(W_0)\\
&=(\bar{W}_0+\mu(\bar{W}_0))e^{\lambda(\bar{W}_0)F_{\rm int}(\bar{W})(1+o(1))}\\
&\quad+(\mu(\bar{W}_0)-\mu(W_0))+O(\lambda(\bar{W}_0)F_{\rm int}(\bar{W}))\mu(\bar{W}_0)\\
&=(\bar{W}_0+\mu(\bar{W}_0))e^{\lambda(\bar{W}_0)F_{\rm int}(\bar{W})(1+o(1))};
\end{aligned}$$

whence (5.54) becomes

$$\frac{G(\bar{W},\bar{\tau})}{G(0^+,\bar{\tau})}=\frac{G_0((\bar{W}_0+\mu(\bar{W}_0))e^{\lambda(\bar{W}_0)F_{\rm int}(\bar{W})(1+o(1))})}{G_0(\bar{W}_0+\mu(\bar{W}_0))}\quad\text{as}\quad\bar{\tau}\to\infty\,.$$

Arguing then as in the Proof of Lemma 5.4 we obtain (5.53). □

6. Analysis of the transition problem: Local well posedness

6.1. **The Transition Problem.** The main result that we have obtained in the previous Section is that the problem of transforming an initial data $G_0(W)$ in a self-similar solution of the form (2.13) by means of the evolution (2.16) for a suitable choice of $\beta(\cdot)$ might be reduced to the Transition Problem (5.22), (5.51), (5.52). We rewrite this problem here for convenience:

Let us fix a function $S(\cdot)$ by means of (5.1). To find $\beta(\tau)$ such that $\tilde{W}(\tau;W_0)$, solution of

$$\tilde{W}_\tau=-\frac{1}{3}(\tilde{W}-1)^2+h_1(\tilde{W},\tau)+3\beta(\tau)(1+h_2(\tilde{W},\tau)), \tag{6.1}$$

$$\tilde{W}((T_{\rm arr,mod})^+;W_0)=+\infty \tag{6.2}$$

satisfies

$$\tilde{W}((T_{\rm exit,mod})^-;W_0)=-\infty \tag{6.3}$$

for any W_0 large enough, where

$$\tilde{S}(T_{\rm exit,mod})=T_{\rm arr,mod} \tag{6.4}$$

and where the function $\tilde{S}$ is defined by means of the relation $S(T_{\rm exit}) = T_{\rm arr}$ as well as (5.25), (5.26).

The key idea to solve the Transition Problem (6.1)-(6.4) might be considered, is to treat it as a perturbation of the Simplified Transition Problem (4.1)–(4.4) that was solved in an explicit manner in Section 4. We then proceed to reformulate (6.1)-(6.4) in a more convenient way.

6.2. **Reformulating the Transition Problem as a perturbation of the Simplified Transition Problem.** Arguing as in Section 4 we define a function $f(\zeta) \geq T_0$ strictly monotonically increasing in ζ, defined in $\zeta \geq \zeta_0$ satisfying

$$\tilde{S}(f(\zeta+1)) = f(\zeta), \tag{6.5}$$

$$f(\zeta_0) = T_0\,. \tag{6.6}$$

Such a function f is uniquely defined by its values in the interval $\zeta \in [\zeta_0, \zeta_0 + 1]$. Moreover, arguing as in Section 4 we would have that $f \in C^3[\zeta_0, \infty)$ if $f \in C^3[\zeta_0, \zeta_0 + 1]$ and satisfies the compatibility conditions (4.8)-(4.11).

We define a new set of variables (ζ, Y) by means of (cf. (4.13), (4.14))

$$\tau = f(\zeta), \tag{6.7}$$

$$\tilde{W} - 1 = \psi(Y, \zeta), \tag{6.8}$$

where $\psi(Y, \zeta)$ is as in (4.12). We define $\bar{\beta}(\cdot)$ as (cf. (4.15))

$$\bar{\beta}(f(\zeta)) = \frac{1}{(f'(\zeta))^2}\big(\frac{1}{2}\{f,\zeta\} - \pi^2\big)\,. \tag{6.9}$$

We will look for solutions of (6.1)-(6.4) in the form

$$\beta(f(\zeta)) = \bar{\beta}(f(\zeta)) + \mu(\zeta)\,. \tag{6.10}$$

We have

$$\frac{1}{f'(\zeta)}\big(-(Y^2+\pi^2)\frac{\partial\psi}{\partial Y} + \frac{\partial\psi}{\partial\zeta}\big) + \frac{\psi^2}{3} - 3\bar{\beta}(f(\zeta)) = 0\,. \tag{6.11}$$

On the other hand, the change of variables (6.7), (6.8) transforms (6.1) into

$$\frac{1}{f'(\zeta)}(Y_\zeta\frac{\partial\psi}{\partial Y} + \frac{\partial\psi}{\partial\zeta}) + \frac{\psi^2}{3} - h_1(\psi, f(\zeta)) - 3\beta(f(\zeta))(1 + h_2(\psi, f(\zeta))) = 0\,. \tag{6.12}$$

Subtracting (6.11) from (6.12) and using $\frac{\partial\psi}{\partial Y} = \frac{1}{f'(\zeta)}$ we obtain

$$\frac{1}{(f'(\zeta))^2}(Y_\zeta + Y^2 + \pi^2) - h_1(\psi, f(\zeta)) - 3\mu(\zeta) - 3\beta(f(\zeta))h_2(\psi, f(\zeta)) = 0\,. \tag{6.13}$$

Let us define

$$\lambda(\zeta) = 3\mu(\zeta)(f'(\zeta))^2, \tag{6.14}$$

$$\bar{R}(Y,\zeta) = (f'(\zeta))^2[h_1(\psi(Y,\zeta), f(\zeta)) + 3\beta(f(\zeta))h_2(\psi(Y,\zeta), f(\zeta))]\,. \tag{6.15}$$

Then (6.13) might be rewritten as

$$Y_\zeta + Y^2 + \pi^2 = \lambda(\zeta) + \bar{R}(Y,\zeta)\,. \tag{6.16}$$

Using (6.4), (6.5) the boundary conditions (6.2), (6.3) become

$$Y(\bar{\zeta}) = +\infty, \tag{6.17}$$

$$Y(\bar{\zeta}+1) = -\infty \tag{6.18}$$

for any $\bar{\zeta} \geq \zeta_0$. Note that the new formulation of the Transition Problem (6.16)-(6.18) is a perturbation of (4.17)-(4.19). By technical reasons it is convenient to transform (6.16)-(6.18) into a new problem without singular boundary conditions. Let us denote as $Y(\zeta, \bar{\zeta})$ the solution of (6.16)-(6.18). We define a new function $Z(\zeta, \bar{\zeta})$ as

$$Y(\zeta, \bar{\zeta}) = \pi \cot(\pi(\zeta - \bar{\zeta} - Z(\zeta, \bar{\zeta}))) . \tag{6.19}$$

Using this new function, (6.16)-(6.18) becomes

$$Z_\zeta = \frac{\sin^2(\pi(\zeta - \bar{\zeta} - Z(\zeta, \bar{\zeta})))}{\pi^2} [\lambda(\zeta) + R(Z, \zeta, \bar{\zeta})], \tag{6.20}$$

$$Z(\bar{\zeta}^+, \bar{\zeta}) = 0, \tag{6.21}$$

$$Z((\bar{\zeta} + 1)^-, \bar{\zeta}) = 0, \tag{6.22}$$

where

$$R(Z, \zeta, \bar{\zeta}) \equiv \bar{R}(\psi(\pi \cot(\pi(\zeta - \bar{\zeta} - Z(\zeta, \bar{\zeta})))), \zeta) . \tag{6.23}$$

Note that the function $R(Z, \zeta, \bar{\zeta})$ is a smooth function, bounded in compact sets of $\zeta, \bar{\zeta}$.

Let us summarize. We have transformed the Transition Problem (6.1)-(6.4) into the problem (6.20)-(6.22). Therefore, our goal is to find functions $\lambda(\zeta)$, $\zeta \geq \zeta_0$ such that the unique solution of (6.20), (6.21) satisfies (6.22). We will solve this problem transforming it in an integral equation coupled with a differential equation.

It turns out that it is possible obtain functions $\lambda(\zeta)$ solving (6.20)-(6.22) for any function $\lambda(\zeta)$ defined in $[\zeta_0, \zeta_0 + 1]$ satisfying suitable compatibility conditions. To explain in an intuitive manner the meaning of these compatibility conditions we proceed to solve (6.20)-(6.22) in the particular case $R \equiv 0$ with $\lambda(\cdot)$ small.

6.3. **Solving (6.20)-(6.22) with $R \equiv 0$ and $\lambda(\cdot)$ small.** It is illuminating to solve (6.20)-(6.22) under the assumptions stated in the heading of this Subsection, because this can be made in an explicit manner and it provides some intuition about the main arguments used later.

In the case $R \equiv 0$ the equation (6.20) becomes

$$Z_\zeta = \frac{\sin^2(\pi(\zeta - \bar{\zeta} - Z(\zeta, \bar{\zeta})))}{\pi^2} \lambda(\zeta) .$$

On the other hand, if $\lambda(\cdot)$ is small $Z(\zeta, \bar{\zeta})$ would be small too. Then the problem (6.20)-(6.22) might be approximated as follows:

Find functions $\lambda(\cdot) \in L^\infty[\zeta_0, \infty)$ such that for any $\bar{\zeta} \geq \zeta_0$ the function $Z(\zeta, \bar{\zeta})$ solution of

$$Z_\zeta = \frac{\sin^2(\pi(\zeta - \bar{\zeta}))}{\pi^2} \lambda(\zeta) \tag{6.24}$$

with initial condition (6.21) satisfies (6.22). Since, in the case $R \equiv 0$ (6.20)-(6.22) is just a reformulation of the Simplified Transition Problem solved in Section 4, the problem (6.21)-(6.24) is a linearized version of the problem studied there, and it could be studied using similar methods. However, the solution obtained here can be easily adapted to solve the whole Transition Problem (6.20)-(6.22).

The solution of (6.21), (6.24) is given by

$$Z(\zeta, \bar{\zeta}) = \frac{1}{\pi^2} \int_{\bar{\zeta}}^{\zeta} \sin^2(\pi(\xi - \bar{\zeta})) \lambda(\xi) d\xi .$$

Using (6.22), we obtain the following integral equation, for $\lambda(\cdot)$,

$$\Phi(\bar{\zeta}) \equiv \int_{\bar{\zeta}}^{\bar{\zeta}+1} \sin^2(\pi(\xi - \bar{\zeta}))\lambda(\xi)d\xi = 0\,. \tag{6.25}$$

To solve (6.25) we differentiate Φ three times with respect to $\bar{\zeta}$. We have

$$\begin{aligned}
\Phi'(\bar{\zeta}) &= \pi \int_{\bar{\zeta}}^{\bar{\zeta}+1} \sin(2\pi(\xi - \bar{\zeta}))\lambda(\xi)d\xi, \\
\Phi''(\bar{\zeta}) &= 2\pi^2 \int_{\bar{\zeta}}^{\bar{\zeta}+1} \cos(2\pi(\xi - \bar{\zeta}))\lambda(\xi)d\xi, \\
\Phi'''(\bar{\zeta}) &= 2\pi^2[\lambda(\bar{\zeta}+1) - \lambda(\bar{\zeta})] - 4\pi^3 \int_{\bar{\zeta}}^{\bar{\zeta}+1} \sin(2\pi(\xi - \bar{\zeta}))\lambda(\xi)d\xi\,.
\end{aligned}$$

Therefore,

$$\Phi'''(\bar{\zeta}) + 4\pi^2\Phi'(\bar{\zeta}) = 2\pi^2[\lambda(\bar{\zeta}+1) - \lambda(\bar{\zeta})]\,. \tag{6.26}$$

Suppose that $\lambda(\cdot)$ solves (6.25). Since $\Phi(\bar{\zeta}) = 0$ for $\bar{\zeta} \geq \zeta_0$, it follows from (6.26) that

$$\lambda(\bar{\zeta}+1) = \lambda(\bar{\zeta})\,, \quad \bar{\zeta} \geq \zeta_0\,. \tag{6.27}$$

Moreover, if $\lambda(\cdot)$ solves (6.25) we have $\Phi(\zeta_0) = \Phi'(\zeta_0) = \Phi''(\zeta_0) = 0$, i.e. $\lambda(\cdot)$ satisfies

$$\begin{aligned}
\int_{\zeta_0}^{\zeta_0+1} \sin^2(\pi(\xi - \zeta_0))\lambda(\xi)d\xi &= \int_{\zeta_0}^{\zeta_0+1} \sin(2\pi(\xi - \zeta_0))\lambda(\xi)d\xi \\
&= \int_{\zeta_0}^{\zeta_0+1} \cos(2\pi(\xi - \zeta_0))\lambda(\xi)d\xi = 0
\end{aligned} \tag{6.28}$$

If, on the contrary $\lambda(\zeta)$ is any function that satisfies (6.28) for $\zeta \in (\zeta_0, \zeta_0 + 1)$ and we extend $\lambda(\cdot)$ to $\bar{\zeta} \geq \zeta_0$ using (6.27) it would follow that the resulting $\lambda(\cdot)$ would solve (6.25), because under these assumptions (6.26) implies

$$\Phi'''(\bar{\zeta}) + 4\pi^2\Phi'(\bar{\zeta}) = 0\,, \quad \bar{\zeta} > \zeta_0\,. \tag{6.29}$$

Also (6.28) implies

$$\Phi(\zeta_0) = \Phi'(\zeta_0) = \Phi''(\zeta_0) = 0,$$

whence the fact that $\Phi(\bar{\zeta}) = 0$ for $\bar{\zeta} \geq \zeta_0$ just follows using standard uniqueness result for ODEs. We summarize this result as follows.

Proposition 6.1. *A function $\lambda(\cdot)$ solves the integral equation* (6.25) *if and only if $\lambda(\cdot)$ satisfies* (6.27), (6.28).

The idea explained in this Subsection is the same that will be used to solve the Transition Problem (6.20)-(6.22). Note that for the simplified version just considered we have obtained a set of compatibility conditions (6.28). A similar set of compatibility conditions arises in the study of (6.20)-(6.22). We find them in next Subsection.

6.4. **Transition problem: Compatibility conditions.** To study (6.20)-(6.22) we begin by finding a set of compatibility conditions analogous to (6.28). Note that (6.20)-(6.22) imply

$$\Phi(\bar{\zeta}) \equiv \int_{\bar{\zeta}}^{\bar{\zeta}+1} \frac{\sin^2(\pi(\zeta - \bar{\zeta} - Z(\zeta, \bar{\zeta})))}{\pi^2} [\lambda(\zeta) + R(Z(\zeta, \bar{\zeta}), \zeta, \bar{\zeta})] d\zeta = 0\,, \tag{6.30}$$

where $\bar{\zeta} \geq \zeta_0$. We then argue as in Subsection 6.3. Note that a first compatibility condition for $\lambda(\zeta)$ in $\zeta \in (\zeta_0, \zeta_0 + 1)$ is

$$\int_{\zeta_0}^{\zeta_0+1} \sin^2(\pi(\zeta - \zeta_0 - Z(\zeta, \zeta_0)))[\lambda(\zeta) + R(Z, \zeta, \zeta_0)] d\zeta = 0 \tag{6.31}$$

with $Z(\zeta, \zeta_0)$ defined by means of (6.20), (6.21) with $\bar{\zeta} = \zeta_0$. Differentiating $\Phi(\bar{\zeta})$ in (6.30), and choosing $\bar{\zeta} = \zeta_0$ we obtain

$$\begin{aligned} &-\pi \int_{\zeta_0}^{\zeta_0+1} \sin(2\pi(\zeta - \zeta_0 - Z(\zeta, \zeta_0)))\big(1 + \frac{\partial Z(\zeta, \zeta_0)}{\partial \zeta_0}\big)[\lambda(\zeta) + R(Z, \zeta, \zeta_0)] d\zeta \\ &+ \int_{\zeta_0}^{\zeta_0+1} \sin^2(\pi(\zeta - \zeta_0 - Z(\zeta, \zeta_0)))[\frac{\partial R}{\partial \zeta_0}(Z, \zeta, \zeta_0) + \frac{\partial R}{\partial Z}\frac{\partial Z}{\partial \zeta_0}(\zeta, \zeta_0)] d\zeta = 0 \end{aligned} \tag{6.32}$$

We can simplify (6.32) after computing an equation for $\frac{\partial Z(\zeta, \zeta_0)}{\partial \zeta_0}$. Differentiating (6.20), with respect to $\bar{\zeta}$, we obtain

$$\begin{aligned} (\frac{\partial Z}{\partial \bar{\zeta}})_\zeta = &-\frac{\sin(2\pi(\zeta - \bar{\zeta} - Z(\zeta, \bar{\zeta})))}{\pi}[\lambda(\zeta) + R(Z, \zeta, \bar{\zeta})](1 + \frac{\partial Z}{\partial \bar{\zeta}}) \\ &+ \frac{\sin^2(\pi(\zeta - \bar{\zeta} - Z(\zeta, \bar{\zeta})))}{\pi^2} g(\zeta, \bar{\zeta}), \end{aligned} \tag{6.33}$$

where

$$g(\zeta, \bar{\zeta}) \equiv \frac{\partial R}{\partial \bar{\zeta}} + \frac{\partial R}{\partial Z}\frac{\partial Z}{\partial \bar{\zeta}}\,. \tag{6.34}$$

On the other hand, differentiating (6.21) we arrive at

$$\frac{\partial Z}{\partial \zeta}(\bar{\zeta}^+, \bar{\zeta}) + \frac{\partial Z}{\partial \bar{\zeta}}(\bar{\zeta}^+, \bar{\zeta}) = 0\,.$$

Note that (6.20) implies $\frac{\partial Z}{\partial \zeta}(\bar{\zeta}^+, \bar{\zeta}) = 0$. Then

$$\frac{\partial Z}{\partial \bar{\zeta}}(\bar{\zeta}^+, \bar{\zeta}) = 0\,. \tag{6.35}$$

Integrating (6.33), (6.35), we obtain

$$\begin{aligned} &\frac{\partial Z}{\partial \bar{\zeta}}(\zeta, \bar{\zeta}) \\ &= \int_{\bar{\zeta}}^{\zeta} e^{-\psi(\zeta, \bar{\zeta}) + \psi(\eta, \bar{\zeta})} \Big[-\frac{\sin(2\pi(\eta - \bar{\zeta} - Z(\eta, \bar{\zeta})))}{\pi}[\lambda(\eta) + R(Z(\eta, \bar{\zeta}), \eta, \bar{\zeta})] \\ &\quad + \frac{\sin^2(\pi(\eta - \bar{\zeta} - Z(\eta, \bar{\zeta})))}{\pi^2} g(\eta, \bar{\zeta})\Big] d\eta, \end{aligned} \tag{6.36}$$

where

$$\psi(\zeta, \bar{\zeta}) \equiv \frac{1}{\pi}\int_{\bar{\zeta}}^{\zeta} \sin(2\pi(\eta - \bar{\zeta} - Z(\eta, \bar{\zeta})))[\lambda(\eta) + R(Z(\eta, \bar{\zeta}), \eta, \bar{\zeta})] d\eta\,. \tag{6.37}$$

We can rewrite (6.36) as

$$\frac{\partial Z}{\partial\bar\zeta}(\zeta,\bar\zeta)+1=e^{-\psi(\zeta,\bar\zeta)}\Big(1+\int_{\bar\zeta}^{\zeta}e^{\psi(\eta,\bar\zeta)}\Big[\frac{\sin^2(\pi(\eta-\bar\zeta-Z(\eta,\bar\zeta)))}{\pi^2}g(\eta,\bar\zeta)\Big]d\eta\Big) \quad (6.38)$$

substituting (6.38) with $\bar\zeta=\zeta_0$ in (6.32) and using (6.37), we obtain

$$\begin{aligned}
&-\int_{\zeta_0}^{\zeta_0+1}\frac{\partial\psi(\zeta,\zeta_0)}{\partial\zeta}e^{-\psi(\zeta,\zeta_0)}d\zeta-\frac{1}{\pi^2}\int_{\zeta_0}^{\zeta_0+1}\frac{\partial\psi(\zeta,\zeta_0)}{\partial\zeta}e^{-\psi(\zeta,\zeta_0)}\\
&\times\int_{\zeta_0}^{\zeta}e^{\psi(\eta,\zeta_0)}\sin^2(\pi(\eta-\zeta_0-Z(\eta,\zeta_0)))g(\eta,\zeta_0)\,d\eta\,d\zeta\\
&+\frac{1}{\pi^2}\int_{\zeta_0}^{\zeta_0+1}\sin^2(\pi(\zeta-\zeta_0-Z(\zeta,\zeta_0)))g(\zeta,\zeta_0)d\zeta=0\,.
\end{aligned}$$

After some integrations by parts we arrive to the following compatibility condition

$$(1-e^{\psi(\zeta_0+1,\zeta_0)})+\frac{1}{\pi^2}\int_{\zeta_0}^{\zeta_0+1}e^{\psi(\zeta,\zeta_0)}\sin^2(\pi(\zeta-\zeta_0-Z(\zeta,\zeta_0)))g(\zeta,\zeta_0)d\zeta=0\,. \quad (6.39)$$

Actually (6.39) holds for any $\bar\zeta\ge\zeta_0$ if ζ_0 is replaced by $\bar\zeta$. We obtain an additional compatibility condition that must be satisfied by $\lambda(\cdot)$ differentiating the resulting equation with respect to $\bar\zeta$ and particularizing the value $\bar\zeta=\zeta_0$. Equivalently we can just differentiate with respect to ζ_0 in (6.39) to obtain

$$\begin{aligned}
&-e^{\psi(\zeta_0+1,\zeta_0)}\frac{d(\psi(\zeta_0+1,\zeta_0))}{d\zeta_0}\\
&+\frac{1}{\pi^2}\int_{\zeta_0}^{\zeta_0+1}e^{\psi(\zeta,\zeta_0)}\frac{\partial\psi(\zeta,\zeta_0)}{\partial\zeta_0}\sin^2(\pi(\zeta-\zeta_0-Z(\zeta,\zeta_0)))g(\zeta,\zeta_0)d\zeta\\
&-\frac{1}{\pi}\int_{\zeta_0}^{\zeta_0+1}e^{\psi(\zeta,\zeta_0)}\sin(2\pi(\zeta-\zeta_0-Z(\zeta,\zeta_0)))(1+\frac{\partial Z(\zeta,\zeta_0)}{\partial\zeta_0})g(\zeta,\zeta_0)d\zeta\\
&+\frac{1}{\pi^2}\int_{\zeta_0}^{\zeta_0+1}e^{\psi(\zeta,\zeta_0)}\sin^2(\pi(\zeta-\zeta_0-Z(\zeta,\zeta_0)))\frac{\partial g(\zeta,\zeta_0)}{\partial\zeta_0}d\zeta=0\,.
\end{aligned} \quad (6.40)$$

Using (6.37), we obtain

$$\begin{aligned}
&\frac{d(\psi(\zeta_0+1,\zeta_0))}{d\zeta_0}\\
&=-2\int_{\zeta_0}^{\zeta_0+1}\cos(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0)))(1+\frac{\partial Z(\eta,\zeta_0)}{\partial\zeta_0})\\
&\times\big[\lambda(\eta)+R(Z(\eta,\zeta_0),\eta,\zeta_0)\big]d\eta\\
&\quad+\frac{1}{\pi}\int_{\zeta_0}^{\zeta_0+1}\sin(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0)))g(\eta,\zeta_0)d\eta\,.
\end{aligned}$$

Therefore, we can use (6.38) to transform (6.40) into

$$\begin{aligned}
&-2e^{\psi(\zeta_0+1,\zeta_0)}\int_{\zeta_0}^{\zeta_0+1}e^{-\psi(\eta,\zeta_0)}\cos(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0)))\\
&\times\big[\lambda(\eta)+R(Z(\eta,\zeta_0),\eta,\zeta_0)\big]d\eta+K_1(\zeta_0)+K_2(\zeta_0)=0,
\end{aligned} \quad (6.41)$$

where

$$
\begin{aligned}
&K_1(\zeta_0)\\
&\equiv -2e^{\psi(\zeta_0+1,\zeta_0)}\int_{\zeta_0}^{\zeta_0+1}\int_{\bar\zeta}^{\eta} e^{\psi(\theta,\bar\zeta)-\psi(\eta,\zeta_0)}\cos(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0)))\\
&\quad\times\big[\frac{\sin^2(\pi(\theta-\bar\zeta-Z(\theta,\bar\zeta)))}{\pi^2}g(\theta,\bar\zeta)\big][\lambda(\eta)+R(Z(\eta,\zeta_0),\eta,\zeta_0)]d\theta d\eta\\
&\quad+\frac{e^{\psi(\zeta_0+1,\zeta_0)}}{\pi}\int_{\zeta_0}^{\zeta_0+1}\sin(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0)))g(\eta,\zeta_0)d\eta\\
&\quad+\frac{1}{\pi^2}\int_{\zeta_0}^{\zeta_0+1}e^{\psi(\zeta,\zeta_0)}\frac{\partial\psi(\zeta,\zeta_0)}{\partial\zeta_0}\sin^2(\pi(\zeta-\zeta_0-Z(\zeta,\zeta_0)))g(\zeta,\zeta_0)d\zeta\\
&\quad-\frac{1}{\pi}\int_{\zeta_0}^{\zeta_0+1}e^{\psi(\zeta,\zeta_0)}\sin(2\pi(\zeta-\zeta_0-Z(\zeta,\zeta_0)))(1+\frac{\partial Z(\zeta,\zeta_0)}{\partial\zeta_0})g(\zeta,\zeta_0)d\zeta\,,
\end{aligned}
\tag{6.42}
$$

$$
K_2(\zeta_0)\equiv\frac{1}{\pi^2}\int_{\zeta_0}^{\zeta_0+1}e^{\psi(\zeta,\zeta_0)}\sin^2(\pi(\zeta-\zeta_0-Z(\zeta,\zeta_0)))\frac{\partial g(\zeta,\zeta_0)}{\partial\zeta_0}d\zeta\,. \tag{6.43}
$$

In the definitions of $K_1(\zeta_0)$, $K_2(\zeta_0)$ we have included in $K_1(\zeta_0)$, $K_2(\zeta_0)$ all the terms depending on $g(\zeta,\zeta_0)$ and in $\frac{\partial g(\zeta,\zeta_0)}{\partial\zeta_0}$ respectively.

Equations (6.31), (6.39) and (6.41) are the compatibility conditions that we will need to assume on $\lambda(\cdot)$ in order to solve (6.20)-(6.22). We remark that these conditions reduce to (6.28) if $R\equiv 0$ and only linear terms on $\lambda(\cdot)$ are kept, as could be expected.

The rest of this Section is devoted to obtaining local existence and uniqueness results for (6.20)-(6.22) given a function $\lambda(\cdot)$ defined in $[\zeta_0,\zeta_0+1]$ satisfying the compatibility conditions (6.31), (6.39) and (6.41). The key idea is to adapt the argument that was used in Subsection 6.3 to transform (6.21)-(6.24) into (6.27), (6.29).

6.5. **Local existence and uniqueness Theorem.** The main result in this Subsection is as follows.

Theorem 6.2. *Given $\lambda(\cdot)$ in $C[\zeta_0,\zeta_0+1]$ we define a function $Z(\zeta;\zeta_0)$, for $\zeta\in[\zeta_0,\zeta_0+1]$ as the unique solution of* (6.20), (6.21) *with $\bar\zeta=\zeta_0$. Suppose that $\lambda(\zeta)$, $Z(\zeta;\zeta_0)$ satisfy the compatibility conditions* (6.31), (6.39), (6.41) *with $g(\zeta;\zeta_0)$ as in* (6.34), *$\psi(\zeta;\zeta_0)$ is defined by means of* (6.37) *and $K_1(\zeta_0)$, $K_2(\zeta_0)$ are as in* (6.42), (6.43). *Then, for any $\zeta_0>0$ large enough, there exists $\delta>0$ depending only on $\|\lambda(\cdot)\|_{L^\infty[\zeta_0,\zeta_0+1]}$ such that the problem* (6.20)-(6.22) *has a unique solution $\lambda(\cdot)\in C[\zeta_0,\zeta_0+1+\delta]$.*

Proof. We wish to transform the problem in a perturbed version of (6.27) where it is possible to apply classical fixed point arguments. We remark that for an arbitrary function $\lambda(\cdot)\in C[\zeta_0,\zeta_0+1+\delta]$ a function $Z(\zeta;\bar\zeta)$ solving (6.20), (6.21) does not satisfy in general (6.22). It is then convenient to define, by technical reasons, two functions $Z_-(\zeta;\bar\zeta)$, $Z_+(\zeta;\bar\zeta)$ as follows. For any $\bar\zeta\in[\zeta_0,\zeta_0+\delta]$ we denote as $Z_-(\zeta;\bar\zeta)$, $Z_+(\zeta;\bar\zeta)$ the unique solutions of (6.20), (6.21) and (6.20), (6.22) respectively. We also define

$$
Z(\zeta;\bar\zeta)\equiv\begin{cases}Z_-(\zeta;\bar\zeta)\,, & \zeta\in[\zeta_0,\zeta_0+\frac12)\\ Z_+(\zeta;\bar\zeta)\,, & \zeta\in[\zeta_0+\frac12,\zeta_0+1]\end{cases} \tag{6.44}
$$

Note that for an arbitrary function $\lambda(\cdot) \in C[\zeta_0, \zeta_0 + 1 + \delta]$ the function $Z(\zeta; \bar{\zeta})$ is discontinuous at the point $\zeta = \zeta_0 + \frac{1}{2}$. Actually the functions $\lambda(\cdot)$ solving (6.20)-(6.22) are precisely those functions for which:

$$Z_-(\zeta_0 + \frac{1}{2}; \bar{\zeta}) = Z_+(\zeta_0 + \frac{1}{2}; \bar{\zeta}) \,.$$

We define a function $\Phi(\bar{\zeta})$, $\bar{\zeta} \in [\zeta_0, \zeta_0 + \delta]$ by means of (6.30). Due to (6.44) we have

$$\begin{aligned} \Phi(\bar{\zeta}) &\equiv \int_{\bar{\zeta}}^{\zeta_0+\frac{1}{2}} \frac{\sin^2(\pi(\zeta - \bar{\zeta} - Z_-(\zeta, \bar{\zeta})))}{\pi^2} [\lambda(\zeta) + R(Z_-(\zeta, \bar{\zeta}), \zeta, \bar{\zeta})] d\zeta \\ &\quad + \int_{\zeta_0+\frac{1}{2}}^{\bar{\zeta}+1} \frac{\sin^2(\pi(\zeta - \bar{\zeta} - Z_+(\zeta, \bar{\zeta})))}{\pi^2} [\lambda(\zeta) + R(Z_+(\zeta, \bar{\zeta}), \zeta, \bar{\zeta})] d\zeta \\ &= 0 \,, \quad \bar{\zeta} \geq \zeta_0 \end{aligned} \tag{6.45}$$

Arguing as in the proof of (6.38) we obtain

$$\begin{aligned} &\frac{\partial Z_-}{\partial \bar{\zeta}}(\zeta, \bar{\zeta}) + 1 \\ &= e^{-\psi_-(\zeta, \bar{\zeta})} \Big(1 + \int_{\bar{\zeta}}^{\zeta} e^{\psi_-(\eta, \bar{\zeta})} [\frac{\sin^2(\pi(\eta - \bar{\zeta} - Z_-(\eta, \bar{\zeta})))}{\pi^2} g(\eta, \bar{\zeta})] d\eta \Big), \end{aligned} \tag{6.46}$$

$$\begin{aligned} &\frac{\partial Z_+}{\partial \bar{\zeta}}(\zeta, \bar{\zeta}) + 1 \\ &= e^{-\psi_+(\zeta, \bar{\zeta})} \Big(1 - \int_{\zeta}^{\bar{\zeta}+1} e^{\psi_+(\eta, \bar{\zeta})} [\frac{\sin^2(\pi(\eta - \bar{\zeta} - Z_+(\eta, \bar{\zeta})))}{\pi^2} g(\eta, \bar{\zeta})] d\eta \Big), \end{aligned} \tag{6.47}$$

$$\psi_-(\zeta, \bar{\zeta}) \equiv \frac{1}{\pi} \int_{\bar{\zeta}}^{\zeta} \sin(2\pi(\eta - \bar{\zeta} - Z_-(\eta, \bar{\zeta})))[\lambda(\eta) + R(Z_-(\eta, \bar{\zeta}), \eta, \bar{\zeta})] d\eta, \tag{6.48}$$

$$\begin{aligned} &\psi_+(\zeta, \bar{\zeta}) \\ &\equiv -\frac{1}{\pi} \int_{\zeta}^{\bar{\zeta}+1} \sin(2\pi(\eta - \bar{\zeta} - Z_+(\eta, \bar{\zeta})))[\lambda(\eta) + R(Z_+(\eta, \bar{\zeta}), \eta, \bar{\zeta})] d\eta, \end{aligned} \tag{6.49}$$

where $g(\eta, \bar{\zeta})$ is defined by means of (6.34) with the values of $Z(\zeta; \bar{\zeta})$ required in each region of integration. We rewrite (6.46), (6.47) as

$$\frac{\partial Z}{\partial \bar{\zeta}}(\zeta, \bar{\zeta}) + 1 = H_0(\zeta, \bar{\zeta}) + H_1(\zeta, \bar{\zeta}), \tag{6.50}$$

where

$$H_0(\zeta, \bar{\zeta}) \equiv \begin{cases} e^{-\psi_-(\zeta, \bar{\zeta})}, & \zeta \in [\zeta_0, \zeta_0 + \frac{1}{2}) \\ e^{-\psi_+(\zeta, \bar{\zeta})}, & \zeta \in [\zeta_0 + \frac{1}{2}, \zeta_0 + 1] \,. \end{cases} \tag{6.51}$$

Note that

$$\begin{aligned} H_0(\bar{\zeta}, \bar{\zeta}) = H_0(\bar{\zeta} + 1, \bar{\zeta}) = 1 \,, \\ H_1(\bar{\zeta}, \bar{\zeta}) = H_1(\bar{\zeta} + 1, \bar{\zeta}) = 0 \,. \end{aligned} \tag{6.52}$$

Differentiating (6.45) and using (6.46), (6.47), we obtain

$$\frac{d\Phi(\bar{\zeta})}{d\bar{\zeta}} = -e^{-\psi_-(\zeta_0 + \frac{1}{2}, \bar{\zeta})} + e^{-\psi_+(\zeta_0 + \frac{1}{2}, \bar{\zeta})} - G_1(\bar{\zeta}, \zeta_0), \tag{6.53}$$

$$
\begin{aligned}
&G_1(\bar\zeta,\zeta_0)\\
&\equiv \frac{e^{-\psi_-(\zeta_0+\frac12,\bar\zeta)}}{\pi^2}\int_{\bar\zeta}^{\zeta_0+\frac12} e^{\psi_-(\eta,\bar\zeta)}\sin^2(\pi(\eta-\bar\zeta-Z_-(\eta,\bar\zeta)))g(\eta,\bar\zeta)d\eta\\
&\quad+\frac{e^{-\psi_+(\zeta_0+\frac12,\bar\zeta)}}{\pi^2}\int_{\zeta_0+\frac12}^{\bar\zeta+1} e^{\psi_+(\eta,\bar\zeta)}\sin^2(\pi(\eta-\bar\zeta-Z_+(\eta,\bar\zeta)))g(\eta,\bar\zeta)d\eta,
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2\Phi(\bar\zeta)}{d\bar\zeta^2} &= e^{-\psi_-(\zeta_0+\frac12,\bar\zeta)}\frac{\partial\psi_-(\zeta_0+\frac12,\bar\zeta)}{\partial\bar\zeta}\\
&\quad - e^{-\psi_+(\zeta_0+\frac12,\bar\zeta)}\frac{\partial\psi_+(\zeta_0+\frac12,\bar\zeta)}{\partial\bar\zeta} - \frac{\partial G_1(\bar\zeta,\zeta_0)}{\partial\bar\zeta}\,.
\end{aligned}\tag{6.54}
$$

Using (6.53) to eliminate $\exp\big(-\psi_+(\zeta_0+\frac12,\bar\zeta)\big)$ in (6.54), we obtain

$$
\begin{aligned}
&\frac{d^2\Phi(\bar\zeta)}{d\bar\zeta^2}+\frac{\partial\psi_+(\zeta_0+\frac12,\bar\zeta)}{\partial\bar\zeta}\frac{d\Phi(\bar\zeta)}{d\bar\zeta}\\
&= e^{-\psi_-(\zeta_0+\frac12,\bar\zeta)}\frac{\partial(\psi_-(\zeta_0+\frac12,\bar\zeta)-\psi_+(\zeta_0+\frac12,\bar\zeta))}{\partial\bar\zeta}\\
&\quad - \frac{\partial\psi_+(\zeta_0+\frac12,\bar\zeta)}{\partial\bar\zeta}G_1(\bar\zeta,\zeta_0) - \frac{\partial G_1(\bar\zeta,\zeta_0)}{\partial\bar\zeta}\,.
\end{aligned}\tag{6.55}
$$

Differentiating (6.55) with respect to $\bar\zeta$, we obtain

$$
\begin{aligned}
&\frac{d}{d\bar\zeta}\Big(\frac{d^2\Phi(\bar\zeta)}{d\bar\zeta^2}+\frac{\partial\psi_+(\zeta_0+\frac12,\bar\zeta)}{\partial\bar\zeta}\frac{d\Phi(\bar\zeta)}{d\bar\zeta}\Big)\\
&= e^{-\psi_-(\zeta_0+\frac12,\bar\zeta)}\frac{\partial^2(\psi_-(\zeta_0+\frac12,\bar\zeta)-\psi_+(\zeta_0+\frac12,\bar\zeta))}{\partial\bar\zeta^2}+U(\bar\zeta,\zeta_0)\,,
\end{aligned}\tag{6.56}
$$

where

$$
\begin{aligned}
U(\bar\zeta,\zeta_0) &\equiv -\frac{\partial\psi_-(\zeta_0+\frac12,\bar\zeta)}{\partial\bar\zeta}\frac{\partial(\psi_-(\zeta_0+\frac12,\bar\zeta)-\psi_+(\zeta_0+\frac12,\bar\zeta))}{\partial\bar\zeta}e^{-\psi_-(\zeta_0+\frac12,\bar\zeta)}\\
&\quad - \frac{\partial}{\partial\bar\zeta}\Big(\frac{\partial\psi_+(\zeta_0+\frac12,\bar\zeta)}{\partial\bar\zeta}G_1(\bar\zeta,\zeta_0)+\frac{\partial G_1(\bar\zeta,\zeta_0)}{\partial\bar\zeta}\Big)\,.
\end{aligned}\tag{6.57}
$$

Equation (6.56) will play a role analogous to (6.26) in the analysis of the linearized problem considered in Subsection 6.3. To formulate the analogous of the problem (6.27) we need to compute the term $\frac{\partial^2(\psi_-(\zeta_0+\frac12,\bar\zeta)-\psi_+(\zeta_0+\frac12,\bar\zeta))}{\partial\bar\zeta^2}$. Using (6.44), (6.48), (6.49) we have

$$
\begin{aligned}
&\psi_-(\zeta_0+\frac12,\bar\zeta)-\psi_+(\zeta_0+\frac12,\bar\zeta)\\
&=\frac1\pi\int_{\bar\zeta}^{\bar\zeta+1}\sin(2\pi(\eta-\bar\zeta-Z(\eta,\bar\zeta)))[\lambda(\eta)+g(\eta,\bar\zeta)]d\eta;
\end{aligned}\tag{6.58}
$$

whence, using (6.50),

$$
\begin{aligned}
&\frac{\partial(\psi_-(\zeta_0+\frac12,\bar\zeta)-\psi_+(\zeta_0+\frac12,\bar\zeta))}{\partial\bar\zeta}\\
&=-2\int_{\bar\zeta}^{\bar\zeta+1}\cos(2\pi(\eta-\bar\zeta-Z(\eta,\bar\zeta)))(H_0(\zeta,\bar\zeta)+H_1(\zeta,\bar\zeta))[\lambda(\eta)+g(\eta,\bar\zeta)]d\eta
\end{aligned}
$$

$$+\frac{1}{\pi}\int_{\bar\zeta}^{\bar\zeta+1}\sin(2\pi(\eta-\bar\zeta-Z(\eta,\bar\zeta)))\frac{\partial g(\eta,\bar\zeta)}{\partial\bar\zeta}d\eta\,.$$

Differentiating this formula and using (6.52), we obtain

$$\frac{\partial^2(\psi_-(\zeta_0+\frac12,\bar\zeta)-\psi_+(\zeta_0+\frac12,\bar\zeta))}{\partial\bar\zeta^2}=-2[\lambda(\bar\zeta+1)-\lambda(\bar\zeta)]+V(\bar\zeta,\zeta_0)\,, \tag{6.59}$$

$$\begin{aligned}V(\bar\zeta,\zeta_0)\equiv -4\pi\int_{\bar\zeta}^{\bar\zeta+1}&\sin(2\pi(\eta-\bar\zeta-Z(\eta,\bar\zeta)))(H_0(\zeta,\bar\zeta)\\&+H_1(\zeta,\bar\zeta))^2[\lambda(\eta)+g(\eta,\bar\zeta)]d\eta-2\int_{\bar\zeta}^{\bar\zeta+1}\cos(2\pi(\eta-\bar\zeta-Z(\eta,\bar\zeta)))\\&\times\frac{\partial((H_0(\zeta,\bar\zeta)+H_1(\zeta,\bar\zeta))[\lambda(\eta)+g(\eta,\bar\zeta)])}{\partial\bar\zeta}d\eta\\&+\frac{1}{\pi}\int_{\bar\zeta}^{\bar\zeta+1}\frac{\partial}{\partial\bar\zeta}\Big(\sin(2\pi(\eta-\bar\zeta-Z(\eta,\bar\zeta)))\frac{\partial g(\eta,\bar\zeta)}{\partial\bar\zeta}\Big)d\eta\,.\end{aligned} \tag{6.60}$$

Suppose that $\lambda(\bar\zeta)$ satisfies

$$[\lambda(\bar\zeta+1)-\lambda(\bar\zeta)]=\frac{V(\bar\zeta,\zeta_0)}{2}+\frac{e^{\psi_-(\zeta_0+\frac12,\bar\zeta)}U(\bar\zeta,\zeta_0)}{2}\,. \tag{6.61}$$

Then (6.56) implies

$$\frac{d}{d\bar\zeta}\Big(\frac{d^2\Phi(\bar\zeta)}{d\bar\zeta^2}+\frac{\partial\psi_+(\zeta_0+\frac12,\bar\zeta)}{\partial\bar\zeta}\frac{d\Phi(\bar\zeta)}{d\bar\zeta}\Big)=0. \tag{6.62}$$

Note that the compatibility conditions (6.31), (6.39), (6.41) yield

$$\Phi(\zeta_0)=\Phi'(\zeta_0)=\Phi''(\zeta_0)=0\,.$$

Therefore, (6.62) and, under suitable regularity assumptions for $\frac{\partial\psi_+(\zeta_0+\frac12,\bar\zeta)}{\partial\bar\zeta}$, classical uniqueness theory for ODEs would imply $\Phi(\bar\zeta)=0$ for the values of $\bar\zeta$ for which (6.61) holds. Reciprocally, if $\lambda(\cdot)$ solves (6.20)-(6.22) in an interval $(\zeta_0,\zeta_0+\delta)$, $\Phi(\bar\zeta)=0$ in such interval, whence (6.61) would follow.

We have then reduced the problem of proving Theorem 6.2 to the solution of the equation (6.61), under suitable regularity conditions for $\lambda(\cdot)$. Let us then precise the suitable framework in which it is possible to solve (6.61) as well as (6.62). Let us choose an arbitrary function $\lambda(\zeta)$ for $\zeta\in[\zeta_0,\zeta_0+1]$, $\lambda(\cdot)\in C[\zeta_0,\zeta_0+1]$. Suppose that $|\lambda(\zeta)|\le\varepsilon_0$ for $\zeta\in[\zeta_0,\zeta_0+1]$. We define a Banach space $X_\delta=C[\zeta_0+1,\zeta_0+1+\delta]$ endowed with the L^∞ norm

$$\|\lambda\|_{X_\delta}\equiv\sup_{\zeta\in[\zeta_0+1,\zeta_0+1+\delta]}|\lambda(\zeta)|\,.$$

We rewrite (6.61) as the fixed point problem:

$$\lambda(\bar\zeta+1)=T[\lambda](\bar\zeta)\,,\quad \bar\zeta\in[\zeta_0,\zeta_0+\delta], \tag{6.63}$$

where

$$T[\lambda](\bar\zeta)\equiv\lambda(\bar\zeta)+\frac{V(\bar\zeta,\zeta_0)+e^{\psi_-(\zeta_0+\frac12,\bar\zeta)}U(\bar\zeta,\zeta_0)}{2} \tag{6.64}$$

with $U(\bar{\zeta},\zeta_0)$, $V(\bar{\zeta},\zeta_0)$ as in (6.57), (6.60) and $\psi_-(\zeta,\bar{\zeta})$ is as in (6.48). Note that $\|V(\cdot,\zeta_0)\|_{L^\infty[\zeta_0,\zeta_0+\delta]} \le C(\varepsilon_0+\eta_L)$ where $\eta_L \to 0$ as $L\to\infty$. A similar estimate might be obtained for $\|U(\cdot,\zeta_0)\|_{L^\infty[\zeta_0,\zeta_0+\delta]}$. To derive such estimate some care is needed with the term $\frac{\partial}{\partial\bar{\zeta}}(\frac{\partial\psi_+(\zeta_0+\frac{1}{2},\bar{\zeta})}{\partial\bar{\zeta}}G_1(\bar{\zeta},\zeta_0)+\frac{\partial G_1(\bar{\zeta},\zeta_0)}{\partial\bar{\zeta}})$ in (6.57), and in particular the term $\frac{\partial^2\psi_+(\zeta_0+\frac{1}{2},\bar{\zeta})}{\partial\bar{\zeta}^2}G_1(\bar{\zeta},\zeta_0)$. The term $\frac{\partial^2\psi_+(\zeta_0+\frac{1}{2},\bar{\zeta})}{\partial\bar{\zeta}^2}$ contains a term proportional to $\lambda(\bar{\zeta})$. However, $G_1(\bar{\zeta},\zeta_0)$ might be estimated by a small constant if $\delta>0$ is chosen small enough. Therefore the operator T transforms the ball $\|\lambda(\cdot)\|_{X_\delta}\le 1$ in a ball $\|T[\lambda](\cdot)\|_{X_\delta}\le\nu$ where ν is small if L and ζ_0 are large and $\delta>0$ is small. Moreover, similar bounds show that

$$\|T[\lambda_1](\cdot)-T[\lambda_2](\cdot)\|_{X_\delta}\le\theta\|\lambda_1(\cdot)-\lambda_2(\cdot)\|_{X_\delta}$$

where θ is small if L and ζ_0 are large enough and $\delta>0$ is small. A standard contractive fixed point argument then shows that (6.63) (or equivalently (6.61)) has a unique solution for this range of values of L,ζ_0, δ. Moreover, for these functions $\lambda\in X_\delta$, $\frac{\partial^2\psi_+(\zeta_0+\frac{1}{2},\bar{\zeta})}{\partial\bar{\zeta}^2}$ is a continuous function, whence (6.62) implies that $\Phi(\bar{\zeta})=0$ for $\bar{\zeta}\in[\zeta_0,\zeta_0+\delta]$ and the Theorem follows. □

6.6. **On the existence of functions $\lambda(\cdot)$ satisfying the compatibility conditions for ζ_0 large.** Note that a key assumption in Theorem 6.2 is the existence of a function $\lambda(\cdot)$ for which the compatibility conditions (6.31), (6.39), (6.41) hold for ζ_0 large. The existence of such functions $\lambda(\cdot)$ is not obvious at all. The purpose of this Section, is to show that there is indeed a large class of functions $\lambda(\cdot)$ satisfying $|\lambda(\bar{\zeta})|\le\varepsilon_0$ as well as (6.31), (6.39), (6.41).Several of the formulae derived in this Subsection will be useful later proving that the function $\lambda(\bar{\zeta})$ is globally defined for $\bar{\zeta}\in[\zeta_0,\infty)$.

As a first step we rewrite the compatibility conditions obtained in Subsection 6.4 in a more convenient manner. Note that we can rewrite the compatibility condition (6.31) for $\lambda(\zeta)$ in $\zeta\in(\zeta_0,\zeta_0+1)$ as

$$\int_{\zeta_0}^{\zeta_0+1}\big[\frac{\sin^2(\pi(\zeta-\zeta_0-Z(\zeta,\zeta_0)))\lambda(\zeta)}{\pi^2}+H(\zeta,\zeta_0)\big]d\zeta=0 \tag{6.65}$$

with $Z(\zeta,\zeta_0)$ defined by means of (6.20), (6.21) with $\bar{\zeta}=\zeta_0$, and where from now on:

$$H(\zeta,\zeta_0)\equiv\frac{\sin^2(\pi(\zeta-\zeta_0-Z(\zeta,\zeta_0)))R(Z(\zeta,\zeta_0),\zeta,\zeta_0)}{\pi^2}\,. \tag{6.66}$$

Differentiating $\Phi(\bar{\zeta})$ in (6.30), and choosing $\bar{\zeta}=\zeta_0$ we obtain:

$$\begin{aligned}&-\frac{1}{\pi}\int_{\zeta_0}^{\zeta_0+1}\sin(2\pi(\zeta-\zeta_0-Z(\zeta,\zeta_0)))(1+\frac{\partial Z(\zeta,\zeta_0)}{\partial\zeta_0})\lambda(\zeta)d\zeta\\&+\frac{d}{d\zeta_0}(\int_{\zeta_0}^{\zeta_0+1}H(\zeta,\zeta_0)d\zeta)=0\end{aligned} \tag{6.67}$$

We can simplify the above expression after computing $\frac{\partial Z(\zeta,\zeta_0)}{\partial\zeta_0}$. Differentiating (6.20), with respect to $\bar{\zeta}$, we obtain

$$(\frac{\partial Z}{\partial\bar{\zeta}})_\zeta=-\frac{\sin(2\pi(\zeta-\bar{\zeta}-Z(\zeta,\bar{\zeta})))\lambda(\zeta)}{\pi}(1+\frac{\partial Z}{\partial\bar{\zeta}})+\frac{\partial H(\zeta,\bar{\zeta})}{\partial\bar{\zeta}}\,. \tag{6.68}$$

Integrating (6.68), (6.35), we obtain

$$\begin{aligned}\frac{\partial Z}{\partial \bar\zeta}(\zeta,\bar\zeta) &= -\int_{\bar\zeta}^{\zeta} e^{-\bar\psi(\zeta,\bar\zeta)+\bar\psi(\eta,\bar\zeta)}\frac{\partial\bar\psi(\eta,\bar\zeta)}{\partial\zeta}d\eta \\ &\quad+\int_{\bar\zeta}^{\zeta} e^{-\bar\psi(\zeta,\bar\zeta)+\bar\psi(\eta,\bar\zeta)}\frac{\partial H(\eta,\bar\zeta)}{\partial\bar\zeta}d\eta \\ &= (e^{-\bar\psi(\zeta,\bar\zeta)}-1)+\int_{\bar\zeta}^{\zeta} e^{-\bar\psi(\zeta,\bar\zeta)+\bar\psi(\eta,\bar\zeta)}\frac{\partial H(\eta,\bar\zeta)}{\partial\bar\zeta}d\eta,\end{aligned}\tag{6.69}$$

where

$$\bar\psi(\zeta,\bar\zeta)\equiv\frac{1}{\pi}\int_{\bar\zeta}^{\zeta}\sin(2\pi(\eta-\bar\zeta-Z(\eta,\bar\zeta)))\lambda(\eta)d\eta\,.\tag{6.70}$$

We can rewrite (6.36) as

$$\frac{\partial Z}{\partial\bar\zeta}(\zeta,\bar\zeta)+1=e^{-\bar\psi(\zeta,\bar\zeta)}\Big(1+\int_{\bar\zeta}^{\zeta}e^{\bar\psi(\eta,\bar\zeta)}\frac{\partial H(\eta,\bar\zeta)}{\partial\bar\zeta}d\eta\Big)\,.\tag{6.71}$$

Substituting (6.71) with $\bar\zeta=\zeta_0$ into (6.67) and using also (6.70), we obtain

$$\begin{aligned}&-\int_{\zeta_0}^{\zeta_0+1}\frac{\partial\bar\psi(\zeta,\zeta_0)}{\partial\zeta}e^{-\bar\psi(\zeta,\zeta_0)}d\zeta\\ &-\int_{\zeta_0}^{\zeta_0+1}\frac{\partial\bar\psi(\zeta,\zeta_0)}{\partial\zeta}e^{-\bar\psi(\zeta,\zeta_0)}\int_{\zeta_0}^{\zeta}e^{\bar\psi(\eta,\bar\zeta)}\frac{\partial H(\eta,\zeta_0)}{\partial\bar\zeta}d\eta d\zeta+\frac{d}{d\zeta_0}\Big(\int_{\zeta_0}^{\zeta_0+1}H(\zeta,\zeta_0)d\zeta\Big)\\ &=0\,.\end{aligned}$$

After some integrations by parts we arrive to the following compatibility condition

$$(e^{\bar\psi(\zeta_0+1,\zeta_0)}-1)+\int_{\zeta_0}^{\zeta_0+1}e^{\bar\psi(\zeta,\zeta_0)}\frac{\partial H(\zeta,\zeta_0)}{\partial\zeta_0}d\zeta=0\,.\tag{6.72}$$

Actually (6.72) holds for any $\bar\zeta\ge\zeta_0$ if ζ_0 is replaced by $\bar\zeta$. We obtain an additional compatibility condition that must be satisfied by $\lambda(\cdot)$ differentiating the resulting equation with respect to $\bar\zeta$ and particularizing the value $\bar\zeta=\zeta_0$. Equivalently we can just differentiate with respect to ζ_0 in (6.72) to obtain

$$e^{\bar\psi(\zeta_0+1,\zeta_0)}\frac{d\bar\psi(\zeta_0+1,\zeta_0)}{d\zeta_0}+\frac{d}{d\zeta_0}\Big(\int_{\zeta_0}^{\zeta_0+1}e^{\bar\psi(\zeta,\zeta_0)}\frac{\partial H(\zeta,\zeta_0)}{\partial\zeta_0}d\zeta\Big)=0\,.\tag{6.73}$$

On the other hand, using (6.70), as well as (6.71) we arrive at

$$\begin{aligned}&\frac{d\bar\psi(\zeta_0+1,\zeta_0)}{d\zeta_0}\\ &=-2\int_{\zeta_0}^{\zeta_0+1}\cos(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0)))e^{-\bar\psi(\eta,\zeta_0)}\lambda(\eta)d\eta-\\ &-2\int_{\zeta_0}^{\zeta_0+1}\cos(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0)))\lambda(\eta)e^{-\bar\psi(\eta,\zeta_0)}\int_{\bar\zeta}^{\eta}e^{\bar\psi(\xi,\zeta_0)}\frac{\partial H(\xi,\zeta_0)}{\partial\zeta_0}d\xi d\eta\,.\end{aligned}$$

Substituting this formula into (6.39), we obtain the compatibility condition

$$
\begin{aligned}
& e^{\bar{\psi}(\zeta_0+1,\zeta_0)} \int_{\zeta_0}^{\zeta_0+1} \cos(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0))) e^{-\bar{\psi}(\eta,\zeta_0)} \lambda(\eta) d\eta \\
& + e^{\bar{\psi}(\zeta_0+1,\zeta_0)} \int_{\zeta_0}^{\zeta_0+1} \cos(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0))) \lambda(\eta) e^{-\bar{\psi}(\eta,\zeta_0)} \\
& \times \Big[\int_{\zeta_0}^{\eta} e^{\bar{\psi}(\xi,\zeta_0)} \frac{\partial H(\xi,\zeta_0)}{\partial \zeta_0} d\xi \Big] d\eta \\
& = \frac{1}{2} \frac{d}{d\zeta_0} \Big(\int_{\zeta_0}^{\zeta_0+1} e^{\bar{\psi}(\zeta,\zeta_0)} \frac{\partial H(\zeta,\zeta_0)}{\partial \zeta_0} d\zeta \Big)
\end{aligned}
\tag{6.74}
$$

Equations (6.65), (6.72) and (6.74) are just the reformulation of the compatibility conditions for $\lambda(\cdot)$ that we wanted to obtain.

As indicated above, the main goal of this Subsection is to show that the compatibility conditions (6.65), (6.72) and (6.74) are satisfied by a large class of functions $\lambda(\cdot)$ for any ζ_0 large enough. A first technical problem is the following. Note that for ζ_0 large enough, the function $R(Z,\zeta,\zeta_0)$ is not necessarily small. Indeed, $R(Z,\zeta,\zeta_0)$ is defined in (6.15), (6.23) and since h_1, h_2 are just bounded functions but $(f'(\zeta))^2 \to \infty$ as $\zeta \to \infty$ it follows that $R(Z,\zeta,\zeta_0)$ becomes large for some large values of Z. In particular, due to this growth it is not obvious at all if $Z(\zeta,\zeta_0)$ is a bounded function for $\zeta_0 \to \infty$ even if $|\lambda|$ is assumed to be small. Moreover, for the same reason it is not "a priori" obvious if the function $H(\zeta,\zeta_0)$ defined in (6.66) is small for $\zeta_0 \to \infty$. The following Lemma shows that this is actually the case:

Lemma 6.3. *Suppose that $Z(\zeta;\bar{\zeta})$ solves (6.20)-(6.22). Let us assume also that $|\lambda(\zeta)| \le \varepsilon_0$ for $\zeta \in [\bar{\zeta},\bar{\zeta}+1]$. There exist $C>0$ and $\hat{\zeta}$ large such that for any $\bar{\zeta} \ge \hat{\zeta}$ such that*

$$
|Z(\zeta;\bar{\zeta})| \le C\varepsilon_0 \min\{|\zeta-\bar{\zeta}|, |\zeta-\bar{\zeta}-1|\}, \quad \text{for } \zeta \in [\bar{\zeta},\bar{\zeta}+1]. \tag{6.75}
$$

Moreover, for $\bar{\zeta} \ge \hat{\zeta}$, with $\hat{\zeta}$ large, the following inequality holds:

$$
|H(\zeta,\bar{\zeta})| \le \frac{C}{(L(\tau))^3}, \quad \text{for } \zeta \in [\bar{\zeta},\bar{\zeta}+1]. \tag{6.76}
$$

Proof. We use a classical continuity argument. By assumption (6.75) holds for $\zeta = \bar{\zeta}$. As long as

$$
|Z(\zeta;\bar{\zeta})| \le \frac{1}{4} \min\big\{|\zeta-\bar{\zeta}|, |\zeta-\bar{\zeta}-1|\big\} \tag{6.77}
$$

we have that (6.76) holds true, due to (6.23) as well as the fact that $f'(\zeta)$ is approximately constant in intervals of the form $[\bar{\zeta},\bar{\zeta}+\frac{C}{f'(\zeta)}]$. Integrating the equation (6.20) we recover (6.77) whence Lemma 6.3 follows. □

We need to rewrite the compatibility conditions (6.72), (6.74) in a more convenient form. We have

$$
(e^{\bar{\psi}(\zeta_0+1,\zeta_0)} - 1) + J(\zeta_0+1,\zeta_0) = 0, \tag{6.78}
$$

$$\begin{aligned}
&\int_{\zeta_0}^{\zeta_0+1}\cos(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0)))e^{-\bar{\psi}(\eta,\zeta_0)}\lambda(\eta)d\eta+\\
&\int_{\zeta_0}^{\zeta_0+1}\cos(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0)))\lambda(\eta)e^{-\bar{\psi}(\eta,\zeta_0)}J(\eta,\zeta_0)d\eta\\
&=\frac{e^{-\bar{\psi}(\zeta_0+1,\zeta_0)}}{2}\frac{d}{d\zeta_0}(J(\zeta_0+1,\zeta_0)),
\end{aligned} \tag{6.79}$$

where

$$\begin{aligned}
J(\zeta,\zeta_0)&\equiv\frac{d}{d\zeta_0}(\int_{\zeta_0}^{\zeta}e^{\bar{\psi}(\eta,\zeta_0)}H(\eta,\zeta_0)d\eta)-\int_{\zeta_0}^{\zeta}e^{\bar{\psi}(\eta,\zeta_0)}\frac{\partial\bar{\psi}(\eta,\zeta_0)}{\partial\zeta_0}H(\eta,\zeta_0)d\eta\\
&\equiv\frac{d}{d\zeta_0}(F_1(\zeta,\zeta_0))-F_2(\zeta,\zeta_0)
\end{aligned} \tag{6.80}$$

A crucial step in all the forthcoming arguments is to derive better estimates for $Z(\zeta,\zeta_0)$ and its derivatives as those in Lemma 6.3.

Lemma 6.4. *Under the assumptions of Lemma 6.3 the following estimates hold*

$$|\frac{\partial^k Z(\zeta,\bar{\zeta})}{\partial\bar{\zeta}^k}|\le C\big[\varepsilon_0|\zeta-\bar{\zeta}|^{3-k}+\frac{1}{(f'(\bar{\zeta}))^{\beta}}\big]\,,\ \textit{for } \bar{\zeta}\le\zeta\le\bar{\zeta}+\frac{1}{2} \tag{6.81}$$

$$|\frac{\partial^k Z(\zeta,\bar{\zeta})}{\partial\bar{\zeta}^k}|\le C\big[\varepsilon_0|\zeta-\bar{\zeta}-1|^{3-k}+\frac{1}{(f'(\bar{\zeta}+1))^{\beta}}\big]\,,\ \textit{for } \bar{\zeta}+\frac{1}{2}\le\zeta\le\bar{\zeta}+1 \tag{6.82}$$

where $k=0,1,2,3$ and $\bar{\zeta}$ is large enough, and where $\beta\in(0,1)$ might be chosen arbitrarily close to one.

Proof. Let us sketch the main argument in the proof of this result. The equation (6.20) might be rewritten in the form:

$$\frac{\partial Z}{\partial\zeta}=H(\zeta-\bar{\zeta}-Z)\lambda(\zeta)+W(\zeta,\bar{\zeta},Z) \tag{6.83}$$

where $H(x)=\frac{\sin^2(\pi x)}{\pi^2}$. Due to the (6.20), (6.23) we have that $W(\zeta,\bar{\zeta},Z)$ is a smooth function that has approximately the form

$$W(\zeta,\bar{\zeta},Z)=\Phi(f'(\zeta)(\zeta-\bar{\zeta}-Z)),$$

with $\Phi(x)=0$ for $0\le x\le L$ and $|\Phi(x)|\le\frac{C}{x}$ globally on x. Due to Lemma 6.3 it follows that Z is small and to the leading order can be neglected. With this assumption it would be possible to approximate Z solution of (6.21), (6.83) as

$$Z(\zeta,\bar{\zeta})\approx Z_1(\zeta,\bar{\zeta})+Z_2(\zeta,\bar{\zeta})\equiv\int_{\bar{\zeta}}^{\zeta}H(\eta-\bar{\zeta})\lambda(\eta)d\eta+\int_{\bar{\zeta}}^{\zeta}\Phi(f'(\eta)(\eta-\bar{\zeta}))d\eta\,. \tag{6.84}$$

Note that the function $Z_2(\zeta,\bar{\zeta})$ is small away from a boundary layer close to $\zeta\approx\bar{\zeta}$. Therefore, the results in Appendix A imply that in that region $Z_2(\zeta,\bar{\zeta})$ might be approximated as

$$Z_2(\zeta,\bar{\zeta})\approx\int_{\bar{\zeta}}^{\zeta}\Phi(f'(\bar{\zeta})(\eta-\bar{\zeta}))d\eta=\frac{1}{f'(\bar{\zeta})}\int_0^{f'(\bar{\zeta})(\zeta-\bar{\zeta})}\Phi(x)dx\,;$$

i.e. $Z_2(\zeta,\bar{\zeta})$ is roughly of order $\frac{1}{f'(\bar{\zeta})}$ in that boundary layer. On the other hand $Z_1(\zeta,\bar{\zeta})$ is roughly of order $\frac{\lambda(\bar{\zeta})}{3}(\zeta-\bar{\zeta})^3$ for $\zeta\approx\bar{\zeta}$. Then, $Z_2(\zeta,\bar{\zeta})$ is the leading term for $\zeta \approx \bar{\zeta}$ and $Z_1(\zeta,\bar{\zeta})$ becomes the leading one for $|\zeta-\bar{\zeta}|$ of order one. Note that $Z_2(\zeta,\bar{\zeta})$ is smooth due to the smoothness of f. In particular its derivatives with respect to $\bar{\zeta}$ might be estimated as $1/(f'(\bar{\zeta}))^{\beta}$ for $\zeta \approx \bar{\zeta}$. On the contrary only three derivatives of $Z_1(\zeta,\bar{\zeta})$ are bounded if the only assumption made on λ is boundedness. Therefore the decomposition (6.84) would imply the estimate (6.81). A similar argument in the region $\zeta \approx \bar{\zeta}+1$ would imply (6.82).

To illustrate how to make the argument above rigorous we derive (6.81) for $k=1$. Note that (6.81), (6.82) for $k=0$ are just a consequence of (6.75) and the formula

$$Z(\zeta,\bar{\zeta}) = \int_{\bar{\zeta}}^{\zeta} H(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta}))\lambda(\eta)d\eta + \int_{\bar{\zeta}}^{\zeta} W(\eta,\bar{\zeta},Z(\eta,\bar{\zeta}))d\eta\,.$$

The integral terms can then be estimated basically as the terms $Z_1(\zeta,\bar{\zeta})$, $Z_2(\zeta,\bar{\zeta})$ above. To prove (6.81) we differentiate (6.83) with respect to $\bar{\zeta}$, whence

$$\begin{aligned}\frac{\partial}{\partial\zeta}\Big(\frac{\partial Z}{\partial\bar{\zeta}}\Big) &= \big[-H_Z(\zeta-\bar{\zeta}-Z)\lambda(\zeta) + \frac{\partial W(\zeta,\bar{\zeta},Z)}{\partial Z}\big]\frac{\partial Z}{\partial\bar{\zeta}}\\ &\quad + \big[-H_{\bar{\zeta}}(\zeta-\bar{\zeta}-Z)\lambda(\zeta) + W_{\bar{\zeta}}(\zeta,\bar{\zeta},Z)\big]\,.\end{aligned}$$

Differentiating (6.21) we obtain $\frac{\partial Z}{\partial\zeta}(\bar{\zeta},\bar{\zeta})=0$. Then

$$\begin{aligned}\frac{\partial Z}{\partial\bar{\zeta}}(\zeta,\bar{\zeta}) &= \int_{\bar{\zeta}}^{\zeta} e^{\int_{\bar{\zeta}}^{\eta}\big[-H_Z(\xi-\bar{\zeta}-Z)\lambda(\xi)+\frac{\partial W(\xi,\bar{\zeta},Z)}{\partial Z}\big]d\xi}\\ &\quad\times\Big[-H_{\bar{\zeta}}(\eta-\bar{\zeta}-Z)\lambda(\eta)+W_{\bar{\zeta}}(\eta,\bar{\zeta},Z)\Big]d\eta\,.\end{aligned}\tag{6.85}$$

The exponential factor containing H_Z might be estimated by means of a constant using (6.75). We then need to estimate the term $\exp\big(\int_{\bar{\zeta}}^{\eta}\frac{\partial W(\xi,\bar{\zeta},Z)}{\partial Z}d\xi\big)$. To this end we remark that this term can be estimated as $e^{\int_{\bar{\zeta}}^{\eta} f'(\zeta)|\Phi'(f'(\zeta)(\zeta-\bar{\zeta}-Z))|d\xi}$ and this can be estimated also by means of a constant (actually close to one if L, $\bar{\zeta}$ are large). After estimating the exponential factors in this manner the terms left in (6.85) can be estimated as the derivatives of the functions $Z_1(\zeta,\bar{\zeta})$, $Z_2(\zeta,\bar{\zeta})$ above. This yields (6.81) with $k=1$.

Higher order derivatives can be estimated in an analogous manner. The main difference arises for $k=3$, because in that case $\frac{\partial^3 Z}{\partial\zeta^3}(\bar{\zeta},\bar{\zeta}) = -2\lambda(\bar{\zeta})$. In particular this yields a global term $C\varepsilon_0$. Moreover, due to this higher derivatives cannot be estimated in this manner unless additional regularity for $\lambda(\cdot)$ is assumed. $\square$

To show that there exist functions $\lambda(\cdot)$ satisfying (6.65), (6.78), (6.80) for ζ_0 large we need to obtain estimates for $J(\zeta,\zeta_0)$and some of its derivatives, or equivalently $F_1(\zeta,\zeta_0)$, $F_2(\zeta,\zeta_0)$ and their derivatives. To this end, we write

$$\begin{aligned}F_1(\zeta,\zeta_0) &= F_{1,1}(\zeta,\zeta_0)+F_{1,2}(\zeta,\zeta_0)+F_{1,3}(\zeta,\zeta_0)\,,\\ F_2(\zeta,\zeta_0) &= F_{2,1}(\zeta,\zeta_0)+F_{2,2}(\zeta,\zeta_0)+F_{2,3}(\zeta,\zeta_0),\end{aligned}$$

where

$$F_{1,1}(\zeta,\zeta_0) \equiv \int_{\zeta_0}^{\min\{\zeta,\zeta_0+\frac{1}{(f(\zeta_0))^{\alpha}}\}} e^{\bar{\psi}(\eta,\zeta_0)}H(\eta,\zeta_0)d\eta\,,$$

$$F_{1,2}(\zeta,\zeta_0) \equiv \int_{\min\{\zeta,\zeta_0+\frac{1}{(f(\zeta_0))^{\alpha}}\}}^{\min\{\zeta,\zeta_0+1-\frac{1}{(f(\zeta_0+1))^{\alpha}}\}} e^{\bar{\psi}(\eta,\zeta_0)}H(\eta,\zeta_0)d\eta\,,$$

$$F_{1,3}(\zeta,\zeta_0) \equiv \int_{\min\{\zeta,\zeta_0+1-\frac{1}{(f(\zeta_0+1))^\alpha}\}}^{\min\{\zeta,\zeta_0+1\}} e^{\bar\psi(\eta,\zeta_0)} H(\eta,\zeta_0)d\eta\,,$$

$$F_{2,1}(\zeta,\zeta_0) \equiv \int_{\zeta_0}^{\min\{\zeta,\zeta_0+\frac{1}{(f(\zeta_0))^\alpha}\}} e^{\bar\psi(\eta,\zeta_0)} \frac{\partial\bar\psi(\eta,\zeta_0)}{\partial\zeta_0} H(\eta,\zeta_0)d\eta\,,$$

$$F_{2,2}(\zeta,\zeta_0) \equiv \int_{\min\{\zeta,\zeta_0+\frac{1}{(f(\zeta_0))^\alpha}\}}^{\min\{\zeta,\zeta_0+1-\frac{1}{(f(\zeta_0+1))^\alpha}\}} e^{\bar\psi(\eta,\zeta_0)} \frac{\partial\bar\psi(\eta,\zeta_0)}{\partial\zeta_0} H(\eta,\zeta_0)d\eta\,,$$

$$F_{2,3}(\zeta,\zeta_0) \equiv \int_{\min\{\zeta,\zeta_0+\frac{1}{(f(\zeta_0))^\alpha}\}}^{\min\{\zeta,\zeta_0+1-\frac{1}{(f(\zeta_0+1))^\alpha}\}} e^{\bar\psi(\eta,\zeta_0)} \frac{\partial\bar\psi(\eta,\zeta_0)}{\partial\zeta_0} H(\eta,\zeta_0)d\eta\,.$$

and where $\alpha \in (0,1)$ might be chosen arbitrarily close to one.

The terms $F_{1,2}(\zeta,\zeta_0)$ and $F_{2,2}(\zeta,\zeta_0)$ might be easily estimated if ζ_0 is large enough.

Lemma 6.5. *Suppose that the assumptions of Lemma 6.3 are satisfied. Then*

$$\Big|\frac{d^k}{d\zeta_0^k}(F_{1,2}(\zeta,\zeta_0))\Big| + \Big|\frac{d^j}{d\zeta_0^j}(F_{2,2}(\zeta,\zeta_0))\Big| \le \frac{C}{(f(\zeta_0))^\beta} \tag{6.86}$$

for ζ_0 large enough, $\zeta_0 \le \zeta \le \zeta_0+1$, where $k = 0,1,2,3$, $j = 0,1,2$, and $\beta > 0$ might be chosen arbitrarily close to one.

Proof. This result is just a consequence of Lemma 6.4, the definitions of $F_{1,2}(\zeta,\zeta_0)$, $F_{2,2}(\zeta,\zeta_0)$ and the definitions of $H(\zeta,\zeta_0)$, $\bar\psi(\zeta,\zeta_0)$ (cf. (6.66), (6.70)). $\square$

In other words, the integration in regions away from the boundaries $\zeta = \zeta_0$, $\zeta = \zeta_0 + 1$ yields a negligible contribution into $F_1(\zeta,\zeta_0)$, $F_2(\zeta,\zeta_0)$. On the other hand in order to estimate the contributions to these functions due to the regions close to the boundaries $\zeta = \zeta_0$, $\zeta = \zeta_0 + 1$ we need some additional bounds for $Z(\zeta,\bar\zeta)$ and its derivatives.

Lemma 6.6. *Under the assumptions of Lemma 6.3 the following estimates hold*

$$|\frac{d^k}{d\zeta_0^k}(F_{1,1}(\zeta,\zeta_0))| + |\frac{d^j}{d\zeta_0^j}(F_{2,1}(\zeta,\zeta_0))| \le \frac{C}{(f(\zeta_0))^\beta} \tag{6.87}$$

$$|\frac{d^k}{d\zeta_0^k}(F_{1,3}(\zeta,\zeta_0))| + |\frac{d^j}{d\zeta_0^j}(F_{2,3}(\zeta,\zeta_0))| \le \frac{C}{(f(\zeta_0+1))^\beta} \tag{6.88}$$

for ζ_0 large enough, $\zeta_0 \le \zeta \le \zeta_0+1$, where $k = 0,1,2$, $j = 0,1$ and $\beta > 0$, might be chosen arbitrarily close to one.

Proof. We rewrite

$$\begin{aligned} F_{1,1}(\zeta,\zeta_0) &\equiv \int_0^{\min\{\zeta,\zeta_0+\frac{1}{(f(\zeta_0))^\alpha}\}-\zeta_0} H(\xi+\zeta_0,\zeta_0)d\xi \\ &\quad + \int_{\zeta_0}^{\min\{\zeta,\zeta_0+\frac{1}{(f(\zeta_0))^\alpha}\}} [e^{\bar\psi(\eta,\zeta_0)} - 1]H(\eta,\zeta_0)d\eta \\ &\equiv F_{1,1,1}(\zeta,\zeta_0) + F_{1,1,2}(\zeta,\zeta_0)\,. \end{aligned}$$

Since

$$H(\xi+\zeta_0,\zeta_0) \equiv \frac{\sin^2(\pi(\xi - Z(\zeta_0+\xi,\zeta_0)))R(Z(\zeta_0+\xi,\zeta_0),\zeta_0+\xi,\zeta_0)}{\pi^2},$$

using Lemma 6.4 and (6.20), we obtain

$$\Big|\frac{\partial^k (H(\xi+\zeta_0,\zeta_0))}{\partial \zeta_0^k}\Big| \le C\Big(\frac{1}{|\xi|(f(\zeta_0))^{\beta}}\Big)^k$$

for $k=0,1,2$, $0\le\xi\le\frac12$. Then, using (6.23) we arrive at

$$\Big|\frac{\partial^k (F_{1,1,1}(\zeta,\zeta_0))}{\partial \zeta_0^k}\Big| \le C\frac{(f(\zeta_0))^{(1-\beta)k}}{(f(\zeta_0))^{\alpha}}$$

for $k=0,1,2$, $0\le\xi\le\frac12$. On the other hand since $|e^{\bar\psi(\eta,\zeta_0)}-1|\le C\varepsilon_0|\zeta-\zeta_0|^2$ and using the fact that each derivative of H yields a contribution of order $f'(\zeta_0)$ we obtain

$$\Big|\frac{\partial^k (F_{1,1,2}(\zeta,\zeta_0))}{\partial \zeta_0^k}\Big| \le \frac{C}{(f(\zeta_0))^{\alpha}}$$

for $k=0,1,2$, $0\le\xi\le\frac12$. This yields

$$\Big|\frac{d^k}{d\zeta_0^k}(F_{1,1}(\zeta,\zeta_0))\Big| \le \frac{C}{(f(\zeta_0))^{\beta}} \tag{6.89}$$

for a new value of β close to one.

On the other hand, in order to estimate $F_{2,1}(\zeta,\zeta_0)$ we use the fact that $|\bar\psi(\zeta,\bar\zeta)|\le C\varepsilon_0|\zeta-\zeta_0|$, whence, since each derivative of H yields a new multiplicative factor $f'(\zeta_0)$ we obtain (6.87), using also (6.89). The proof of (6.88) is similar. $\square$

We can now prove the main result of this Subsection that shows that it is possible to choose functions $\lambda(\cdot)$ satisfying the compatibility conditions (6.65), (6.78), (6.79) for ζ_0 large enough in infinite different manners.

Proposition 6.7. *Suppose that $\lambda(\cdot)$ has the form*

$$\lambda(\zeta)=\alpha_0+\alpha_1\cos(2\pi(\zeta-\zeta_0))+\beta_1\sin(2\pi(\zeta-\zeta_0))+\tilde\lambda(\zeta-\zeta_0) \tag{6.90}$$

where:

$$\int_{\zeta_0}^{\zeta_0+1} e^{2\pi\ell(\zeta-\zeta_0)}\tilde\lambda(\zeta-\zeta_0)d\zeta=0\,,\quad \ell=0,\pm1,$$
$$|\tilde\lambda(\zeta-\zeta_0)|\le\varepsilon_0\,,\quad \zeta\in[\zeta_0,\zeta_0+1]\,.$$

Suppose that $L>0$ is large enough. Then, for any ζ_0 large enough there exist constants α_0, α_1, β_1 such that the function $\lambda(\cdot)$ in (6.90) satisfies the compatibility conditions (6.65), (6.78), (6.79), as well as an estimate of the form

$$|\lambda(\zeta)|\le C\varepsilon_0\,,\quad \zeta\in[\zeta_0,\zeta_0+1]\,.$$

Proof. Formally linearizing the compatibility conditions (6.65), (6.78), (6.79), we obtain

$$\int_{\zeta_0}^{\zeta_0+1}\sin^2(\pi(\zeta-\zeta_0))\lambda(\zeta)d\zeta=-\int_{\zeta_0}^{\zeta_0+1}H(\zeta,\zeta_0)d\zeta\equiv f_1(\zeta_0)\,,$$
$$\int_{\zeta_0}^{\zeta_0+1}\sin(2\pi(\eta-\zeta_0))\lambda(\eta)d\eta=-J(\zeta_0+1,\zeta_0)\equiv f_2(\zeta_0)\,,$$

$$
\begin{aligned}
&\int_{\zeta_0}^{\zeta_0+1} \cos(2\pi(\eta-\zeta_0))\lambda(\eta)d\eta \\
&= -\int_{\zeta_0}^{\zeta_0+1} \cos(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0)))\lambda(\eta)e^{-\bar{\psi}(\eta,\zeta_0)}J(\eta,\zeta_0)d\eta \\
&\quad - \frac{e^{-\bar{\psi}(\zeta_0+1,\zeta_0)}}{2}\frac{d}{d\zeta_0}(J(\zeta_0+1,\zeta_0)) \equiv f_3(\zeta_0)\,.
\end{aligned}
$$

Using (6.90) it follows that these equations can be rewritten as

$$
\frac{\alpha_0}{2} = f_1(\zeta_0)\,,\quad \frac{\alpha_1}{2} = f_2(\zeta_0)\,,\quad \frac{\beta_1}{2} = f_3(\zeta_0)\,. \tag{6.91}
$$

Note that, due to the definition of $J(\zeta,\zeta_0)$ as well as Lemmas 6.5, 6.6 the functions $f_\ell(\zeta_0)$, $\ell = 1,2,3$ can be made arbitrarily small if $L$is large and ζ_0 is large enough, assuming that $|\lambda(\zeta)| \le 2\varepsilon_0$ for $\zeta \in [\zeta_0,\zeta_0+1]$. In particular the left-hand sides of (6.91) is larger than the right-hand sides if $\sqrt{(\alpha_0)^2+(\alpha_1)^2+(\beta_1)^2} = 4\varepsilon_0$. Therefore, Proposition 6.7 follows just using standard Degree Theory (cf. [16]) ☐

7. Analysis of the transition problem: Global well posedness

7.1. **Reducing the Transition Problem to a delay equation.** In the previous Sections we have proved that the Transition Problem (6.20)-(6.22) might be solved, in infinite different ways, in intervals $[\zeta_0,\zeta_0+1+\delta]$ with $\delta > 0$ small. To conclude the proof of Theorem 2.1 it only remains to show that the solution of (6.20)-(6.22) can be extended for arbitrarily large values of ζ.

Note that as long as $\lambda(\cdot)$ solves (6.20)-(6.22) the compatibility conditions (6.65), (6.78), (6.79) are satisfied with $\bar{\zeta}$ replacing ζ_0. We need to rewrite (6.61) in a more convenient manner. To this end we first rewrite (6.79) as

$$
\int_{\bar{\zeta}}^{\bar{\zeta}+1} \cos(2\pi(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta})))e^{-\bar{\psi}(\eta,\bar{\zeta})}\lambda(\eta)d\eta = Q(\bar{\zeta}), \tag{7.1}
$$

where

$$
\begin{aligned}
Q(\zeta_0) \equiv& \frac{e^{-\bar{\psi}(\zeta_0+1,\zeta_0)}}{2}\frac{d}{d\zeta_0}(J(\zeta_0+1,\zeta_0)) \\
&- \int_{\zeta_0}^{\zeta_0+1} \cos(2\pi(\eta-\zeta_0-Z(\eta,\zeta_0)))\lambda(\eta)e^{-\bar{\psi}(\eta,\zeta_0)}J(\eta,\zeta_0)d\eta\,.
\end{aligned}
$$

Differentiating (7.1), we obtain

$$
\begin{aligned}
&e^{-\bar{\psi}(\bar{\zeta}+1,\bar{\zeta})}\lambda(\bar{\zeta}+1) - \lambda(\bar{\zeta}) \\
&+ 2\pi\int_{\bar{\zeta}}^{\bar{\zeta}+1} \sin(2\pi(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta})))(1+\frac{\partial Z(\eta,\bar{\zeta})}{\partial\bar{\zeta}})e^{-\bar{\psi}(\eta,\bar{\zeta})}\lambda(\eta)d\eta \\
&- \int_{\bar{\zeta}}^{\bar{\zeta}+1} \cos(2\pi(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta})))e^{-\bar{\psi}(\eta,\bar{\zeta})}\frac{\partial\bar{\psi}(\eta,\bar{\zeta})}{\partial\bar{\zeta}}\lambda(\eta)d\eta \\
&= \frac{dQ(\bar{\zeta})}{d\bar{\zeta}}\,.
\end{aligned}
$$

Using (6.71), this formula becomes

$$e^{-\bar{\psi}(\bar{\zeta}+1,\bar{\zeta})}\lambda(\bar{\zeta}+1)-\lambda(\bar{\zeta})+\int_{\bar{\zeta}}^{\bar{\zeta}+1}K(\eta,\bar{\zeta})\lambda(\eta)d\eta=W_1(\bar{\zeta})\,,$$

where

$$\begin{aligned} W_1(\bar{\zeta}) \equiv & -2\pi\int_{\bar{\zeta}}^{\bar{\zeta}+1}\sin(2\pi(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta})))\Big(\int_{\bar{\zeta}}^{\eta}e^{\bar{\psi}(\xi,\bar{\zeta})}\frac{\partial H(\xi,\bar{\zeta})}{\partial\bar{\zeta}}d\xi\Big)e^{-2\bar{\psi}(\eta,\bar{\zeta})}\lambda(\eta)d\eta \\ & +\frac{dQ(\bar{\zeta})}{d\bar{\zeta}}, \end{aligned}$$

$$\begin{aligned} K(\eta,\bar{\zeta}) \equiv & \Big[2\pi\sin(2\pi(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta})))e^{-2\bar{\psi}(\eta,\bar{\zeta})} \\ & +\cos(2\pi(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta})))e^{-\bar{\psi}(\eta,\bar{\zeta})}\frac{\partial\bar{\psi}(\eta,\bar{\zeta})}{\partial\bar{\zeta}}\Big] \end{aligned}$$

On the other hand, using (6.78), we obtain

$$\lambda(\bar{\zeta}+1)-\lambda(\bar{\zeta})+\int_{\bar{\zeta}}^{\bar{\zeta}+1}K(\eta,\bar{\zeta})\lambda(\eta)d\eta=W_1(\bar{\zeta})+W_2(\bar{\zeta}), \tag{7.2}$$

with

$$W_2(\bar{\zeta})\equiv -e^{-\bar{\psi}(\bar{\zeta}+1,\bar{\zeta})}J(\bar{\zeta}+1,\bar{\zeta})$$

Equation (7.2) is well suited to prove global existence for the solutions of the problem (6.20)-(6.22). It turns out that the right hand side of (7.2) converges to zero as $\bar{\zeta}\to\infty$. On the other hand, the solution obtained for the linearized transition problem in Subsection 6.3 shows that for $R=0$ the function λ would be periodic with period one. Since the right-hand side of (7.2) vanishes for $R=0$, the integral term on the left-hand side should vanish too. To see this more clearly we rewrite the integral term as

$$\begin{aligned} & \int_{\bar{\zeta}}^{\bar{\zeta}+1}K(\eta,\bar{\zeta})\lambda(\eta)d\eta \\ & =\int_{\bar{\zeta}}^{\bar{\zeta}+1}\Big[2\pi\sin(2\pi(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta})))e^{-2\bar{\psi}(\eta,\bar{\zeta})} \\ & \quad +\cos(2\pi(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta})))e^{-\bar{\psi}(\eta,\bar{\zeta})}\frac{\partial\bar{\psi}(\eta,\bar{\zeta})}{\partial\bar{\zeta}}\Big]\lambda(\eta)d\eta \\ & =2\pi^2\int_{\bar{\zeta}}^{\bar{\zeta}+1}\frac{\partial\bar{\psi}(\eta,\bar{\zeta})}{\partial\zeta}e^{-2\bar{\psi}(\eta,\bar{\zeta})}d\eta-\frac{1}{\pi}\int_{\bar{\zeta}}^{\bar{\zeta}+1}d\eta\lambda(\eta)\cos(2\pi(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta}))) \\ & \quad\times e^{-\bar{\psi}(\eta,\bar{\zeta})}\int_{\bar{\zeta}}^{\eta}\cos(2\pi(\xi-\bar{\zeta}-Z(\xi,\bar{\zeta})))\big(1+\frac{\partial Z}{\partial\bar{\zeta}}(\xi,\bar{\zeta})\big)\lambda(\xi)d\xi\,. \end{aligned}$$

Using (6.71), we can write

$$\begin{aligned} \int_{\bar{\zeta}}^{\bar{\zeta}+1}K(\eta,\bar{\zeta})\lambda(\eta)d\eta = & \pi^2(1-e^{-2\bar{\psi}(\bar{\zeta}+1,\bar{\zeta})}) \\ & -\frac{1}{2\pi}\Big(\int_{\bar{\zeta}}^{\bar{\zeta}+1}\cos(2\pi(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta})))e^{-\bar{\psi}(\eta,\bar{\zeta})}\lambda(\eta)d\eta\Big)^2 \\ & -W_3(\bar{\zeta}), \end{aligned}$$

where

$$
\begin{aligned}
&W_3(\bar\zeta)\\
&= -\frac{1}{\pi}\int_{\bar\zeta}^{\bar\zeta+1} d\eta\lambda(\eta)\cos(2\pi(\eta-\bar\zeta-Z(\eta,\bar\zeta)))e^{-\bar\psi(\eta,\bar\zeta)}\\
&\quad\times\int_{\bar\zeta}^{\eta}\cos(2\pi(\xi-\bar\zeta-Z(\xi,\bar\zeta)))e^{-\bar\psi(\xi,\bar\zeta)}\Big(\int_{\bar\zeta}^{\xi} e^{\bar\psi(\upsilon,\bar\zeta)}\frac{\partial H(\upsilon,\bar\zeta)}{\partial\bar\zeta}d\upsilon\Big)\lambda(\xi)d\xi\,.
\end{aligned}
$$

Using (6.78) and (6.79), we obtain

$$
\int_{\bar\zeta}^{\bar\zeta+1} K(\eta,\bar\zeta)\lambda(\eta)d\eta = -W_3(\bar\zeta)-W_4(\bar\zeta)-W_5(\bar\zeta),
$$

where

$$
W_4(\bar\zeta) = \pi^2(e^{-\bar\psi(\bar\zeta+1,\bar\zeta)}+1)e^{-\bar\psi(\bar\zeta+1,\bar\zeta)}J(\bar\zeta+1,\bar\zeta),
$$
$$
W_5(\bar\zeta) = \frac{1}{2\pi}(Q(\bar\zeta))^2\,.
$$

Therefore, we can finally write (7.2) as

$$
\lambda(\bar\zeta+1)-\lambda(\bar\zeta) = W(\bar\zeta)\equiv\sum_{i=1}^{5}W_i(\bar\zeta)\,. \tag{7.3}
$$

It turns out that the function $W(\bar\zeta)$ approaches zero as $\bar\zeta\to\infty$ fast enough. Using this fact it is possible to show that $\lambda(\bar\zeta)$ is globally bounded as $\bar\zeta\to\infty$. In the rest of this Section we will make precise this argument, and we will determine in which sense the different terms in $W(\bar\zeta)$ are small.

7.2. **Reformulating the delay equation as an integral equation.** The estimates so far derived yield bounds for the terms $W_i(\bar\zeta)$ with $i=2,\dots,5$.

Proposition 7.1. *Suppose that $|\lambda(\bar\zeta)|\le\varepsilon_0$ for any $\zeta_0\le\bar\zeta\le\zeta_0+M$, for some M large. Then:*

$$
\Big|\sum_{i=2}^{5}W_i(\bar\zeta)\Big|\le\frac{C}{(f(\bar\zeta))^\beta}\,,\quad \text{for } \zeta_0\le\bar\zeta\le\zeta_0+M
$$

for some $\beta\in(0,1)$.

Proof. Lemmas 6.5, 6.6 imply that $|J(\bar\zeta+1,\bar\zeta)|\le\frac{C}{(f(\bar\zeta))^\beta}$ for $\zeta_0\le\bar\zeta\le\zeta_0+M$. Therefore $|W_2(\bar\zeta)|+|W_4(\bar\zeta)|\le\frac{C}{(f(\bar\zeta))^\beta}$. On the other hand, Lemmas 6.5, 6.6 imply also that $\sup_{\bar\zeta\le\zeta\le\bar\zeta+1}|J(\zeta,\bar\zeta)|\le\frac{C}{(f(\bar\zeta))^\beta}$, and also $\sup_{\bar\zeta\le\zeta\le\bar\zeta+1}|\frac{\partial J(\zeta,\bar\zeta)}{\partial\bar\zeta}|\le\frac{C}{(f(\bar\zeta))^\beta}$. Moreover, (6.80) implies

$$
\frac{\partial J(\zeta,\bar\zeta)}{\partial\zeta} = \frac{\partial}{\partial\bar\zeta}(e^{\bar\psi(\zeta,\bar\zeta)}H(\zeta,\bar\zeta)) - e^{\bar\psi(\zeta,\bar\zeta)}\frac{\partial\bar\psi(\zeta,\bar\zeta)}{\partial\bar\zeta}H(\zeta,\bar\zeta);
$$

whence due to (6.66) and the definition of $h_1(W,\tau)$, $h_2(W,\tau)$ we have $\frac{\partial J(\bar\zeta+1,\bar\zeta)}{\partial\zeta}=0$. Then $|\frac{dJ(\bar\zeta+1,\bar\zeta)}{d\bar\zeta}|\le\frac{C}{(f(\bar\zeta))^\beta}$, whence $|Q(\bar\zeta)|\le\frac{C}{(f(\bar\zeta))^\beta}$. Therefore, $|W_5(\bar\zeta)|\le\frac{C}{(f(\bar\zeta))^\beta}$. To estimate $W_3(\bar\zeta)$ we need to estimate the term

$$
\int_{\bar\zeta}^{\xi} e^{\bar\psi(\upsilon,\bar\zeta)}\frac{\partial H(\upsilon,\bar\zeta)}{\partial\bar\zeta}d\upsilon = \frac{\partial}{\partial\bar\zeta}(\int_{\bar\zeta}^{\xi} e^{\bar\psi(\upsilon,\bar\zeta)}H(\upsilon,\bar\zeta)d\upsilon) - \int_{\bar\zeta}^{\xi} e^{\bar\psi(\upsilon,\bar\zeta)}\frac{\partial\bar\psi(\upsilon,\bar\zeta)}{\partial\bar\zeta}H(\upsilon,\bar\zeta)d\upsilon
$$

we can argue as in the estimate of $F_{1,1}$ in Lemma 6.5. Then $|\int_{\bar\zeta}^{\xi} e^{\bar\psi(v,\bar\zeta)}\frac{\partial H(v,\bar\zeta)}{\partial\bar\zeta}dv| \leq \frac{C}{(f(\bar\zeta))^\beta}$, whence $|W_3(\bar\zeta)| \leq \frac{C}{(f(\bar\zeta))^\beta}$ and Proposition 7.1 follows. □

The term $W_1(\bar\zeta)$ on the right-hand side of (7.3) contains some terms that cannot be neglected as $\bar\zeta\to\infty$ in the study of the long time asymptotics of (7.3). Using the definitions of $W_1(\bar\zeta)$, $Q(\bar\zeta)$ we obtain

$$
\begin{aligned}
&W_1(\bar\zeta)\\
&= -2\pi\int_{\bar\zeta}^{\bar\zeta+1}\sin(2\pi(\eta-\bar\zeta-Z(\eta,\bar\zeta)))\Big(\int_{\bar\zeta}^{\eta} e^{\bar\psi(\xi,\bar\zeta)}\frac{\partial H(\xi,\bar\zeta)}{\partial\bar\zeta}d\xi\Big)e^{-2\bar\psi(\eta,\bar\zeta)}\lambda(\eta)d\eta\\
&\quad-\frac{d}{d\bar\zeta}\Big(\int_{\bar\zeta}^{\bar\zeta+1}\cos(2\pi(\eta-\bar\zeta-Z(\eta,\bar\zeta)))\lambda(\eta)e^{-\bar\psi(\eta,\bar\zeta)}J(\eta,\bar\zeta)d\eta\Big)\\
&\quad-\frac12\frac{d}{d\bar\zeta}(\bar\psi(\bar\zeta+1,\bar\zeta))e^{-\bar\psi(\bar\zeta+1,\bar\zeta)}\frac{d}{d\bar\zeta}(J(\bar\zeta+1,\bar\zeta))+e^{-\bar\psi(\bar\zeta+1,\bar\zeta)}\frac{d^2}{d\bar\zeta^2}(J(\bar\zeta+1,\bar\zeta))
\end{aligned}
$$

The three first terms on the right-hand side can be bounded as $C/(f(\bar\zeta))^\beta$ arguing as in the proof of Proposition 7.1.

Lemma 7.2. *Under the assumptions of Proposition 7.1,*

$$
W_1(\bar\zeta) = W_6(\bar\zeta) + e^{-\bar\psi(\bar\zeta+1,\bar\zeta)}\frac{d^2}{d\bar\zeta^2}(J(\bar\zeta+1,\bar\zeta)), \tag{7.4}
$$

where

$$
|W_6(\bar\zeta)| \leq \frac{C}{(f(\bar\zeta))^\beta}, \quad \text{for } \zeta_0\leq\bar\zeta\leq\zeta_0+M \tag{7.5}
$$

for some $\beta\in(0,1)$.

To estimate $W_1(\bar\zeta)$ then reduces to deriving approximations for $\frac{d^2}{d\bar\zeta^2}(J(\bar\zeta+1,\bar\zeta))$. This requires basically to obtain rather precise approximations for $Z(\zeta,\bar\zeta)$ in the regions $\zeta\approx\bar\zeta$, $\zeta\approx\bar\zeta+1$. We can further simplify the terms in $W_1(\bar\zeta)$ to be

$$
\begin{aligned}
J(\bar\zeta+1,\bar\zeta) &= \frac{\partial}{\partial\bar\zeta}\Big(\int_{\bar\zeta}^{\bar\zeta+1} e^{\bar\psi(\eta,\bar\zeta)}H(\eta,\bar\zeta)d\eta\Big) - \int_{\bar\zeta}^{\bar\zeta+1} e^{\bar\psi(\eta,\bar\zeta)}\frac{\partial\bar\psi(\eta,\bar\zeta)}{\partial\bar\zeta}H(\eta,\bar\zeta)d\eta\\
&= \int_{\bar\zeta}^{\bar\zeta+1} e^{\bar\psi(\eta,\bar\zeta)}\frac{\partial H(\eta,\bar\zeta)}{\partial\bar\zeta}d\eta\\
&= \frac{\partial}{\partial\bar\zeta}\Big(\int_{\bar\zeta}^{\bar\zeta+1} H(\eta,\bar\zeta)d\eta\Big) + \int_{\bar\zeta}^{\bar\zeta+1}\bar\psi(\eta,\bar\zeta)\frac{\partial H(\eta,\bar\zeta)}{\partial\bar\zeta}d\eta\\
&\quad+\int_{\bar\zeta}^{\bar\zeta+1}(e^{\bar\psi(\eta,\bar\zeta)}-1-\bar\psi(\eta,\bar\zeta))\frac{\partial H(\eta,\bar\zeta)}{\partial\bar\zeta}d\eta\\
&\equiv J_1(\bar\zeta+1,\bar\zeta)+J_2(\bar\zeta+1,\bar\zeta)+J_3(\bar\zeta+1,\bar\zeta)\,.
\end{aligned}
$$

Our main goal now is to obtain suitable approximations for the functions $\frac{d^2}{d\bar\zeta^2}(J_1(\bar\zeta+1,\bar\zeta))$, $\frac{d^2}{d\bar\zeta^2}(J_2(\bar\zeta+1,\bar\zeta))$. We also wish to show that $\frac{d^2}{d\bar\zeta^2}(J_3(\bar\zeta+1,\bar\zeta))$ is small as $\bar\zeta\to\infty$. As indicated above, this requires to derive good approximations for $Z(\zeta,\bar\zeta)$ in the regions $\zeta\approx\bar\zeta$, $\zeta\approx\bar\zeta+1$.

The key result of this section is the following.

Proposition 7.3. *Suppose that the assumptions of Proposition 7.1 hold. There exist functions $K_1(\cdot)$, $K_2(\cdot)$ satisfying*

$$\begin{gathered}|K_1(x)|+|K_2(x)|\le \min\big\{\frac{C}{1+x^2},\frac{1}{(L(\tau))^3}\big\},\\ \int_0^\infty K_1(x)dx=\int_{-\infty}^0 K_2(x)dx=0\end{gathered}\tag{7.6}$$

such that

$$\begin{aligned}&\frac{d^2}{d\bar\zeta^2}(J_1(\bar\zeta+1,\bar\zeta)+J_2(\bar\zeta+1,\bar\zeta))\\ &=f'(\bar\zeta)\int_{\bar\zeta}^{\bar\zeta+\delta}K_1(f'(\bar\zeta)(\eta-\bar\zeta))\lambda(\eta)d\eta\\ &\quad+f'(\bar\zeta+1)\int_{\bar\zeta+1-\delta}^{\bar\zeta+1}K_2(f'(\bar\zeta+1)(\eta-(\bar\zeta+1)))\lambda(\eta)d\eta+W_7(\bar\zeta),\end{aligned}\tag{7.7}$$

where $\delta>0$ is a small fixed number and

$$|W_7(\bar\zeta)|+|\frac{d^2}{d\bar\zeta^2}(J_3(\bar\zeta+1,\bar\zeta))|\le\frac{C}{(f'(\bar\zeta))^\beta},\quad \zeta_0\le\bar\zeta\le\zeta_0+M\tag{7.8}$$

with $\beta\in(0,1)$.

Remark 7.4. Note that Proposition 7.3 states that $\frac{d^2}{d\bar\zeta^2}(J(\bar\zeta+1,\bar\zeta))$ might be approximated as two "Dirac-mass approximating" kernels in the regions $\zeta\approx\bar\zeta$, $\zeta\approx\bar\zeta+1$.

Proof. Let us sketch the main ideas in the proof of Proposition 7.3. We write

$$\begin{gathered}\frac{d^2}{d\bar\zeta^2}(J_1(\bar\zeta+1,\bar\zeta)+J_2(\bar\zeta+1,\bar\zeta))=Y_1(\bar\zeta)+Y_2(\bar\zeta)+Y_3(\bar\zeta),\\ Y_1(\bar\zeta)\equiv\frac{d^3}{d\bar\zeta^3}(\int_{\bar\zeta}^{\bar\zeta+\delta}H(\eta,\bar\zeta)d\eta)+\frac{d^2}{d\bar\zeta^2}(\int_{\bar\zeta}^{\bar\zeta+\delta}\bar\psi(\eta,\bar\zeta)\frac{\partial H(\eta,\bar\zeta)}{\partial\bar\zeta}d\eta),\\ Y_2(\bar\zeta)\equiv\frac{d^3}{d\bar\zeta^3}(\int_{\bar\zeta+\delta}^{\bar\zeta+1-\delta}H(\eta,\bar\zeta)d\eta)+\frac{d^2}{d\bar\zeta^2}(\int_{\bar\zeta+\delta}^{\bar\zeta+1-\delta}\bar\psi(\eta,\bar\zeta)\frac{\partial H(\eta,\bar\zeta)}{\partial\bar\zeta}d\eta),\\ Y_3(\bar\zeta)\equiv\frac{d^3}{d\bar\zeta^3}(\int_{\bar\zeta+1-\delta}^{\bar\zeta+1}H(\eta,\bar\zeta)d\eta)+\frac{d^2}{d\bar\zeta^2}(\int_{\bar\zeta+1-\delta}^{\bar\zeta+1}\bar\psi(\eta,\bar\zeta)\frac{\partial H(\eta,\bar\zeta)}{\partial\bar\zeta}d\eta)\,.\end{gathered}$$

Using Lemma 6.5 we obtain the estimate $|Y_2(\bar\zeta)|\le C/(f'(\bar\zeta))^\beta$.

We now describe how to approximate $Y_1(\bar\zeta)$. The computation of $Y_3(\bar\zeta)$ is completely similar. Using the form of the function R in (6.23) it would be natural to approximate $Z(\zeta,\bar\zeta)$ in the region $\zeta\approx\bar\zeta$ using the function $\bar Z(\zeta,\bar\zeta)$ solution of

$$\begin{gathered}\bar Z_\zeta=(\zeta-\bar\zeta-\bar Z(\zeta,\bar\zeta))^2\lambda(\zeta)+\Phi_1(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z(\zeta,\bar\zeta)))\\ +\beta(f(\bar\zeta))\Phi_2(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z(\zeta,\bar\zeta))),\\ \bar Z(\bar\zeta^+,\bar\zeta)=0,\end{gathered}\tag{7.9}$$

where $|x\Phi_1(x)| + |\frac{\Phi_2(x)}{x}| \le C$, and $\Phi_1(x)$, $\Phi_2(x)$ vanish if $|x| \le 1/L$. We have used the approximation

$$\begin{aligned} &H(\zeta,\bar\zeta) \\ &\approx \Phi_1(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z(\zeta,\bar\zeta))) + \beta(f(\bar\zeta))\Phi_2(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z(\zeta,\bar\zeta)))\,. \end{aligned} \tag{7.10}$$

Let us assume for the moment that $Z(\zeta,\bar\zeta)$ might be approximated by means of $\bar Z(\zeta,\bar\zeta)$. We can then approximate $\bar Z(\zeta,\bar\zeta)$ as

$$\bar Z(\zeta,\bar\zeta) \approx \bar Z_1(\zeta,\bar\zeta) + \bar Z_2(\zeta,\bar\zeta) + U(\zeta,\bar\zeta), \tag{7.11}$$

where

$$\begin{gathered} \bar Z_{1,\zeta} = \Phi_1(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z_1(\zeta,\bar\zeta))), \\ \bar Z_{2,\zeta} = -f'(\bar\zeta)\Phi_1'(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z_1(\zeta,\bar\zeta)))\bar Z_2(\zeta,\bar\zeta) \\ + \beta(f(\bar\zeta))\Phi_2(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z_1(\zeta,\bar\zeta))), \\ U_\zeta = (\zeta-\bar\zeta-\bar Z_1(\zeta,\bar\zeta)-\bar Z_2(\zeta,\bar\zeta))^2\lambda(\zeta) - P(\zeta,\bar\zeta)U, \end{gathered}$$

where

$$\begin{gathered} \bar Z_1(\bar\zeta^+,\bar\zeta) = \bar Z_2(\bar\zeta^+,\bar\zeta) = U(\bar\zeta^+,\bar\zeta) = 0, \\ P(\zeta,\bar\zeta) \equiv f'(\bar\zeta)\Phi_1'(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z_1(\zeta,\bar\zeta))) \\ + \beta(f(\bar\zeta))f'(\bar\zeta)\Phi_2'(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z_1(\zeta,\bar\zeta)))\,. \end{gathered} \tag{7.12}$$

Note that

$$\bar Z_1(\zeta,\bar\zeta) = \frac{W_1(f'(\bar\zeta)(\zeta-\bar\zeta))}{f'(\bar\zeta)}, \tag{7.13}$$

$$\bar Z_2(\zeta,\bar\zeta) = \frac{\beta(f(\bar\zeta))}{f'(\bar\zeta)}W_2(f'(\bar\zeta)(\zeta-\bar\zeta)), \tag{7.14}$$

where

$$\begin{gathered} W_1'(x) = \Phi_1(x-W_1(x)), \\ W_2'(x) = -\Phi_1'(x-W_1(x))W_2 + \Phi_2(x-W_1(x)), \\ W_1(0) = W_2(0) = 0\,. \end{gathered}$$

On the other hand

$$U(\zeta,\bar\zeta) = \int_{\bar\zeta}^{\zeta} e^{-\int_{\bar\zeta}^s P(\xi,\bar\zeta)d\xi}(s-\bar\zeta-\bar Z_1(s,\bar\zeta)-\bar Z_2(s,\bar\zeta))^2\lambda(s)ds \tag{7.15}$$

Using (7.10) we can approximate $\int_{\bar\zeta}^{\bar\zeta+\delta} H(\eta,\bar\zeta)d\eta$ as

$$\begin{aligned} \int_{\bar\zeta}^{\bar\zeta+\delta} H(\zeta,\bar\zeta)d\zeta \approx \int_{\bar\zeta}^{\bar\zeta+\delta} &[\Phi_1(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z(\zeta,\bar\zeta))) \\ &+ \beta(f(\bar\zeta))\Phi_2(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z(\zeta,\bar\zeta)))]d\zeta\,. \end{aligned}$$

Using (7.11), we obtain the approximation

$$\begin{aligned}
&\int_{\bar\zeta}^{\bar\zeta+\delta} H(\zeta,\bar\zeta)d\zeta \\
&\approx \int_{\bar\zeta}^{\bar\zeta+\delta} \Phi_1(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z_1(\zeta,\bar\zeta)-\bar Z_2(\zeta,\bar\zeta)))d\zeta \\
&\quad+\int_{\bar\zeta}^{\bar\zeta+\delta} \beta(f(\bar\zeta))\Phi_2(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z_1(\zeta,\bar\zeta)-\bar Z_2(\zeta,\bar\zeta)))d\zeta \\
&\quad-\int_{\bar\zeta}^{\bar\zeta+\delta} f'(\bar\zeta)\Phi_1'(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z_1(\zeta,\bar\zeta)-\bar Z_2(\zeta,\bar\zeta)))U(\zeta,\bar\zeta)d\zeta \\
&\quad-\int_{\bar\zeta}^{\bar\zeta+\delta} \beta(f(\bar\zeta))\Phi_2'(f'(\bar\zeta)(\zeta-\bar\zeta-\bar Z_1(\zeta,\bar\zeta)-\bar Z_2(\zeta,\bar\zeta)))U(\zeta,\bar\zeta)d\zeta\,.
\end{aligned} \tag{7.16}$$

The first two terms on the right of (7.16), as well as their derivatives are bounded by $O(\frac{1}{(f'(\bar\zeta))^\beta})$, $\beta>0$, as can be seen using (7.13), (7.14) and the change of variables $\zeta-\bar\zeta=s$. On the other hand, using (7.15) the definition of $P(\zeta,\bar\zeta)$ we can rewrite the last two terms as

$$\begin{aligned}
&-\int_{\bar\zeta}^{\bar\zeta+\delta} d\zeta P(\zeta,\bar\zeta)\int_{\bar\zeta}^{\zeta} e^{-\int_{\bar\zeta}^{\eta}P(\xi,\bar\zeta)d\xi}(\eta-\bar\zeta-\bar Z_1(\eta,\bar\zeta)-\bar Z_2(\eta,\bar\zeta))^2\lambda(\eta)d\eta \\
&\equiv h_1(\bar\zeta)
\end{aligned}$$

We need to compute three derivatives of $h_1(\bar\zeta)$. The contributions due to the extremes of integration vanish, since $P(\bar\zeta,\bar\zeta)=P(\bar\zeta+1,\bar\zeta)=0$. Then

$$\begin{aligned}
\frac{d^3h_1(\bar\zeta)}{d\bar\zeta^3} &= -\int_{\bar\zeta}^{\bar\zeta+\delta} d\eta\lambda(\eta) \\
&\quad\times\frac{d^3}{d\bar\zeta^3}\Big(\int_{\eta}^{\bar\zeta+\delta} P(\zeta,\bar\zeta)e^{-\int_{\bar\zeta}^{\eta}P(\xi,\bar\zeta)d\xi}(\eta-\bar\zeta-\bar Z_1(\eta,\bar\zeta)-\bar Z_2(\eta,\bar\zeta))^2d\zeta\Big)\,.
\end{aligned}$$

The contributions of $P(\zeta,\bar\zeta)$ and its derivatives at $\zeta=\bar\zeta+\delta$ are bounded as $O(1/(f'(\bar\zeta))^\beta)$. Then

$$\begin{aligned}
\frac{d^3h_1(\bar\zeta)}{d\bar\zeta^3} &= -\int_{\bar\zeta}^{\bar\zeta+\delta} d\eta\lambda(\eta)\int_{\eta}^{\bar\zeta+\delta}\frac{d^3}{d\bar\zeta^3}\Big([\int_{\eta}^{\bar\zeta+\delta}P(\zeta,\bar\zeta)d\zeta] \\
&\quad\times e^{-\int_{\bar\zeta}^{\eta}P(\xi,\bar\zeta)d\xi}(\eta-\bar\zeta-\bar Z_1(\eta,\bar\zeta)-\bar Z_2(\eta,\bar\zeta))^2\Big)+O(1/(f'(\bar\zeta))^\beta)
\end{aligned}$$

Using (7.12) and (7.13), it follows that

$$\begin{aligned}
&\int_{\bar\zeta}^{\eta} P(\xi,\bar\zeta)d\xi \\
&=\int_0^{f'(\bar\zeta)(\eta-\bar\zeta)}\Phi_1'(x-W_1(x))dx+\beta(f(\bar\zeta))\int_0^{f'(\bar\zeta)(\eta-\bar\zeta)}\Phi_2'(x-W_1(x))dx\,.
\end{aligned}$$

Using (7.12), (7.13), (7.14), we obtain

$$\int_{\eta}^{\bar\zeta+\delta} P(\zeta,\bar\zeta)d\zeta = f'(\bar\zeta)\int_{\eta}^{\bar\zeta+\delta}\Phi_1'(f'(\bar\zeta)(\zeta-\bar\zeta)-W_1(f'(\bar\zeta)(\zeta-\bar\zeta)))d\zeta$$

$$
\begin{aligned}
&+\beta(f(\bar\zeta))\int_\eta^{\bar\zeta+\delta} f'(\bar\zeta)\Phi_2'\big(f'(\bar\zeta)(\zeta-\bar\zeta)\\
&-W_1(f'(\bar\zeta)(\zeta-\bar\zeta))\big)d\zeta+O(1/(f'(\bar\zeta))^\beta),
\end{aligned}
$$

$$
\begin{aligned}
&(\eta-\bar\zeta-\bar Z_1(\eta,\bar\zeta)-\bar Z_2(\eta,\bar\zeta))^2\\
&=\frac{1}{(f'(\bar\zeta))^2}(f'(\bar\zeta)(\eta-\bar\zeta)-W_1(f'(\bar\zeta)(\eta-\bar\zeta))-\beta(f(\bar\zeta))W_2(f'(\bar\zeta)(\eta-\bar\zeta)))^2
\end{aligned}
$$

Therefore, we arrive to an approximation of the form

$$
\begin{aligned}
&\int_\eta^{\bar\zeta+\delta} P(\zeta,\bar\zeta)d\zeta e^{-\int_{\bar\zeta}^\eta P(\xi,\bar\zeta)d\xi}(\eta-\bar\zeta-\bar Z_1(\eta,\bar\zeta)-\bar Z_2(\eta,\bar\zeta))^2\\
&=\frac{1}{(f'(\bar\zeta))}S(f'(\bar\zeta)(\eta-\bar\zeta),\beta(f(\bar\zeta)))
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d^3h_1(\bar\zeta)}{d\bar\zeta^3}&=-\int_{\bar\zeta}^{\bar\zeta+\delta} d\eta\lambda(\eta)\int_\eta^{\bar\zeta+\delta}\frac{d^3}{d\bar\zeta^3}\Big(\frac{1}{(f'(\bar\zeta))}\\
&\quad\times\tilde S(f'(\bar\zeta)(\eta-\bar\zeta),\beta(f(\bar\zeta)))\Big)d\zeta+O\big(1/(f'(\bar\zeta))^\beta\big)\,.
\end{aligned}
$$

We now remark that if $\delta=1/(f'(\bar\zeta))^\alpha$ with $\alpha<1$ all the contributions due to terms like Φ_2, W_2 and analogous ones yield relative corrections of order $1/(f'(\bar\zeta))^\beta$, with $\beta>0$, perhaps small if α is close to zero, but in any case strictly positive. In particular this implies that if the size of the final terms obtained is of order one the correction due to the presence of $\beta(f(\bar\zeta))$ would be negligible. We then write

$$
\begin{aligned}
\frac{d^3h_1(\bar\zeta)}{d\bar\zeta^3}&=-\int_{\bar\zeta}^{\bar\zeta+\delta} d\eta\lambda(\eta)\Big(\int_\eta^{\bar\zeta+\delta}\frac{d^3}{d\bar\zeta^3}\Big(\frac{1}{(f'(\bar\zeta))}S_1(f'(\bar\zeta)(\zeta-\bar\zeta))\Big)d\zeta\Big)\\
&\quad\times\Big(1+O(1/(f'(\bar\zeta))^\beta)\Big)+O(1/f'(\bar\zeta))^\beta),
\end{aligned}
\tag{7.17}
$$

where

$$
S_1(x)=\Big[\int_x^\infty \Phi_1'(\xi-W_1(\xi))d\xi\Big]e^{-\int_0^x\Phi_1'(\xi-W_1(\xi))d\xi}(x-W_1(x))^2
$$

Since $\log(f^{k)}(\zeta))\sim\log(f(\zeta))$ as $\zeta\to\infty$ (cf. Appendix A), it turns out that the derivatives of $f(\bar\zeta)$ in (7.17) would yield smaller contributions that the derivatives of the term $\bar\zeta$ that, roughly, multiplies the different terms by $f'(\bar\zeta)$. Then

$$
\begin{aligned}
\frac{d^3h_1(\bar\zeta)}{d\bar\zeta^3}&=(f'(\bar\zeta))^2\int_{\bar\zeta}^{\bar\zeta+\delta} d\eta\lambda(\eta)\Big(\int_\eta^{\bar\zeta+\delta} S_1'''(f'(\bar\zeta)(\zeta-\bar\zeta))d\zeta\Big)\\
&\quad\times\Big(1+O(1/(f'(\bar\zeta))^\beta)\Big)+O(1/(f'(\bar\zeta))^\beta),
\end{aligned}
$$

whence

$$
\begin{aligned}
\frac{d^3h_1(\bar\zeta)}{d\bar\zeta^3}&=-f'(\bar\zeta)\Big[\int_{\bar\zeta}^{\bar\zeta+\delta} d\eta\lambda(\eta)S_1''(f'(\bar\zeta)(\eta-\bar\zeta))d\zeta\Big]\\
&\quad\times\Big(1+O(1/(f'(\bar\zeta))^\beta)\Big)+O(1/(f'(\bar\zeta))^\beta)
\end{aligned}
$$

and this formula yields the sough-for structure (7.7). It remains to estimate in a similar manner the term

$$\frac{d^2}{d\bar{\zeta}^2}\Big(\int_{\bar{\zeta}}^{\bar{\zeta}+\delta} \bar{\psi}(\eta,\bar{\zeta})\frac{\partial H(\eta,\bar{\zeta})}{\partial\bar{\zeta}}d\eta\Big)$$

Arguing as in the previous case it would follow that this term might be approximated as

$$\frac{d^2}{d\bar{\zeta}^2}\Big(\int_{\bar{\zeta}}^{\bar{\zeta}+\delta} \bar{\psi}(\eta,\bar{\zeta})\frac{\partial H(\eta,\bar{\zeta})}{\partial\bar{\zeta}}d\eta\Big) = \frac{d^2 h_2(\bar{\zeta})}{d\bar{\zeta}^2} + O(1/(f'(\bar{\zeta}))^{\beta}),$$

where

$$h_2(\bar{\zeta}) = 2f'(\bar{\zeta})\int_{\bar{\zeta}}^{\bar{\zeta}+\delta} \lambda(\eta)(\eta-\bar{\zeta}-Z_1(\eta,\bar{\zeta}))\Big[\int_{\eta}^{\bar{\zeta}+\delta} \Psi(f'(\bar{\zeta})(\zeta-\bar{\zeta}))d\zeta\Big]d\eta,$$
$$\Psi(x) \equiv \frac{d(\Phi_1(x-W_1(x)))}{dx};$$

whence

$$h_2(\bar{\zeta}) = \frac{1}{f'(\bar{\zeta})}\int_{\bar{\zeta}}^{\bar{\zeta}+\delta} \lambda(\eta)S_2(f'(\bar{\zeta})(\eta-\bar{\zeta}))d\eta,$$

where

$$S_2(x) = 2(x-W_1(x))\int_x^{\infty} \Psi(\xi)d\xi\,.$$

Then

$$\frac{d^2 h_2(\bar{\zeta})}{d\bar{\zeta}^2} = f'(\bar{\zeta})\Big[\int_{\bar{\zeta}}^{\bar{\zeta}+\delta} \lambda(\eta)S_2''(f'(\bar{\zeta})(\eta-\bar{\zeta}))d\eta\Big]\Big(1+O(1/(f'(\bar{\zeta}))^{\beta})\Big)$$

and this yields also the structure in (7.7). The bounds for $\frac{d^2}{d\bar{\zeta}^2}(J_3(\bar{\zeta}+1,\bar{\zeta}))$ might be obtained in a similar manner, using the fact that the presence of an additional term $(\eta-\bar{\zeta})^2$ yields smallness. The approximations near the value $\zeta=\bar{\zeta}+1$ can be derived in a similar manner, whence Proposition 7.3 follows. □

7.3. **Bounds for the solutions of the integral equation.**

Proof of Theorem 2.1. In summary, using Propositions 7.1 and 7.3 and Lemma 7.2, we can rewrite (7.3) as

$$\begin{aligned}\lambda(\bar{\zeta}+1)-\lambda(\bar{\zeta}) = e^{-\bar{\psi}(\bar{\zeta}+1,\bar{\zeta})}\Big[&f'(\bar{\zeta})\int_{\bar{\zeta}}^{\bar{\zeta}+\delta} K_1(f'(\bar{\zeta})(\eta-\bar{\zeta}))\lambda(\eta)d\eta \\ &+ f'(\bar{\zeta}+1)\int_{\bar{\zeta}+1-\delta}^{\bar{\zeta}+1} K_2(f'(\bar{\zeta}+1)(\eta-(\bar{\zeta}+1)))\lambda(\eta)d\eta\Big] \\ &+ O(1/(f'(\bar{\zeta}))^{\beta})\end{aligned}$$

On the other hand, using the compatibility condition (6.72) we can replace the exponential factor $e^{-\bar{\psi}(\bar{\zeta}+1,\bar{\zeta})}$ by one, introducing in turn a corrective term

$$\begin{aligned}\lambda(\bar{\zeta}+1)-\lambda(\bar{\zeta}) = \Big[f'(\bar{\zeta})\int_{\bar{\zeta}}^{\bar{\zeta}+\delta} K_1(f'(\bar{\zeta})(\eta-\bar{\zeta}))\lambda(\eta)d\eta \\ + f'(\bar{\zeta}+1)\int_{\bar{\zeta}+1-\delta}^{\bar{\zeta}+1} K_2(f'(\bar{\zeta}+1)(\eta-(\bar{\zeta}+1)))\lambda(\eta)d\eta\Big] \\ + O(1/(f'(\bar{\zeta}))^{\beta})\end{aligned} \tag{7.18}$$

Equation (7.18) is satisfied as long as $|\lambda(\bar{\zeta}+1)| \leq \varepsilon_0$. The local existence Theorem (cf. Theorem 6.2) implies that it is then possible extend the solution in a larger interval. It remains to show that the function λ solution of (7.18) is globally defined in time and that the estimate $|\lambda(\bar{\zeta}+1)| \leq \varepsilon_0$ remains being valid for arbitrarily large values of $\bar{\zeta}$. To this end, we define

$$\psi(\bar{\tau}) = \lambda(\bar{\zeta}+1), \quad \bar{\tau} = f(\bar{\zeta}+1)\,.$$

Using the fact that $S(f(\bar{\zeta}+1)) = f(\bar{\zeta})$ (cf. (4.5)), as well as the asymptotics of f (cf. Appendix A), (7.18) becomes

$$\begin{aligned}\psi(\bar{\tau})-\psi(S(\bar{\tau})) = \int_{f(f^{-1}(\bar{\tau})-\delta)}^{\bar{\tau}} K_2(\bar{\tau}-s)\psi(s)ds \\ + \int_{S(\bar{\tau})}^{f(f^{-1}(S(\bar{\tau}))+\delta)} K_1(s-S(\bar{\tau}))\psi(s)ds + O(\frac{1}{(S(\bar{\tau}))^{\beta}})\end{aligned}$$

where the value of β might change from one formula to the other, but it is always a positive number.

Using (7.8) we can then obtain the inequality

$$\sup_{\tau_n\leq\tau\leq\tau_{n+1}} |\psi(\tau)| \leq \Big(1+\frac{C}{(L(\tau_{n-1}))^3}\Big)\sup_{\tau_{n-1}\leq\tau\leq\tau_n} |\psi(\tau)| + \frac{C}{(\tau_{n-1})^{\beta}}$$

where $\tau_n = S^{-1}(\tau_{n-1})$, and τ_0 is the initial time for $\bar{\tau}$. Using (5.6) it then follows that

$$\sup_{\tau_n\leq\tau\leq\tau_{n+1}} |\psi(\tau)| \leq \big(1+\frac{C}{n-1}\big)\sup_{\tau_{n-1}\leq\tau\leq\tau_n} |\psi(\tau)| + \frac{C}{(\tau_{n-1})^{\beta}}$$

whence, due to the very fast growth of τ_n we obtain, upon iteration

$$\sup_{\tau_{n-1}\leq\tau\leq\tau_n} |\psi(\tau)| \leq C\epsilon_0, \tag{7.19}$$

where ϵ_0 might be arbitrarily small if τ_0 is large enough and $\sup_{\tau_0\leq\tau\leq\tau_1}|\psi(\tau)|$ is small.

Formula (7.19) implies that $\psi(\tau)$ remains small for arbitrarily long times. In particular the assumptions required in Lemma 6.3 and in the subsequent arguments are satisfied. Moreover, (7.19) implies that the solution of the problem (6.20)-(6.22), or equivalently (5.22), (5.51), (5.52) can be extended to arbitrarily long times. Theorem 2.1 is then a consequence of Theorem 5.9. □

8. Appendix: Some properties of the function $f(\zeta)$

In this Appendix we collect several properties of the function $f(\zeta)$ that have been used repeatedly in Section 7. The function $f(\zeta)$ is defined by means of (4.5), (4.6) as well as the compatibility conditions (4.8)-(4.11). Although the properties described in this Appendix could be generalized to more general functions S, we will assume by definiteness that (3.31) holds, as well as similar asymptotic formulae for the derivatives of S. Under these assumptions the function f has the following properties

$$f(\zeta) \gg \exp(\exp(\exp(\dots\exp(\zeta)))) \quad \text{as } \zeta \to \infty \tag{8.1}$$

for any finite number of iterated exponentials.

$$f^{(k-1)}(\zeta) \ll f^{(k)}(\zeta) \ll (f(\zeta))^{1+\varepsilon} \quad \text{as } \zeta \to \infty \tag{8.2}$$

for any $\epsilon > 0$, and any $k = 1, 2, 3$.

$$f(\zeta+\delta) \gg f(\zeta) \quad \text{as } \zeta \to \infty \tag{8.3}$$

for any $\delta > 0$.

$$f(\zeta + \frac{C}{f(\zeta)}) \sim f(\zeta) \quad \text{as } \zeta \to \infty \tag{8.4}$$

for any $C > 0$. Also there holds

$$\log(f'(\zeta)) \sim \log(f(\zeta)) \quad \text{as } \zeta \to \infty \tag{8.5}$$

$$\log(\beta(\tau)) \sim -2\log(\tau) \quad \text{as } \tau \to \infty \tag{8.6}$$

Property (8.1) follows from iterating (4.5) more than k times. Property (8.3) can be proved in a similar manner. Indeed, iterating (4.5) by means of an exponential function it follows that,since $f(\zeta) \to \infty$, that $f(\zeta+\delta) - f(\zeta) \to \infty$, Then

$$\begin{aligned}
f(\zeta+\delta) &= S^{-1}(f(\zeta+\delta-1)) \\
&= S^{-1}([f(\zeta+\delta-1) - f(\zeta-1)] + f(\zeta-1)) \\
&\gg S^{-1}(f(\zeta-1)) \\
&= f(\zeta)
\end{aligned}$$

as $\zeta \to \infty$. Property (8.2) follows from differentiating (4.5), which yields

$$f'(\zeta) = S'(f(\zeta+1))f'(\zeta+1) \sim \frac{a}{f(\zeta+1)} f'(\zeta+1)\,. \tag{8.7}$$

Iterating (8.7) to estimate $f'(\zeta+1)$ it follows from (8.1) that $f'(\zeta) \to \infty$. Combining this with (8.7) we obtain the first inequality in (8.2) with $k = 1$. On the other hand, combining (4.5) and (8.7) we obtain

$$\frac{f'(\zeta+1)}{f(\zeta+1)} = \Big[\frac{1}{S'(f(\zeta+1))} \frac{f(\zeta)}{S^{-1}(f(\zeta))}\Big] \frac{f'(\zeta)}{f(\zeta)}\,. \tag{8.8}$$

The term between brackets is bounded by $Cf(\zeta)$. Iterating (8.8) we obtain

$$\frac{f'(\zeta_0+n)}{f(\zeta_0+n)} = \prod_{\ell=0}^{n-1} \Big[\frac{1}{S'(f(\zeta_0+1+\ell))} \frac{f(\zeta_0+\ell)}{S^{-1}(f(\zeta_0+\ell))}\Big] \frac{f'(\zeta_0)}{f(\zeta_0)}\,. \tag{8.9}$$

The product in (8.9) can be bounded by

$$\prod_{\ell=0}^{n-1} [Cf(\zeta_0+\ell)]\,. \tag{8.10}$$

Taking the logarithm and using that $f(\zeta+1)/f(\zeta) \geq 2$ for ζ large enough we obtain, after adding a geometric series, an upper estimate for the product in (8.10), of the form

$$\exp(B\log(f(\zeta_0+n-1))) = (f(\zeta_0+n-1))^B$$

for some $B > 0$; whence (8.9) yields

$$f'(\zeta) \leq Cf(\zeta)(f(\zeta-1))^B \tag{8.11}$$

and since (4.5) implies that $f(\zeta-1) \leq C\log(f(\zeta))$ we obtain (8.2) for $k=1$.

The proof of (8.2) for $k=2,3$ is similar. To show (8.4) we iterate (4.5) to obtain

$$f\big(\zeta+\frac{C}{f(\zeta)}\big) = S^{-1}\Big(S^{-1}\Big(\dots S^{-1}\Big(f\big(\zeta+\frac{C}{f(\zeta)}-n\big)\Big)\Big)\Big)$$

where the number of iterations n is such that $\zeta+\frac{C}{f(\zeta)}-n \in [\zeta_0,\zeta_0+1]$. Since $f(\zeta)$ is huge we can approximate the terms $S^{-1}(f(\zeta+\frac{C}{f(\zeta)}-n))$ as $S^{-1}(f(\zeta-)n)+\frac{C(S^{-1})'(f(\zeta-n))}{f(\zeta)}$. Using this approximation in $n-1$ iterations, as well as (8.7) we obtain the approximation

$$f\big(\zeta+\frac{C}{f(\zeta)}\big) = S^{-1}\Big(f(\zeta-1)+\frac{Cf'(\zeta-1)}{f(\zeta)}\Big)$$

and using (8.11), (8.4) follows. A more rigorous proof would just replace the approximation using derivatives by upper bounds. Similar approximations might be derived for the derivatives.

Finally, (8.5) follows from (8.3), (8.7) and (8.6) is a consequence of (4.15), (8.1)-(8.4). The detailed derivation of (8.6) as well as more detailed asymptotics for $\beta(\tau)$ have been given in ([14]).

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