Electronic Journal of Differential Equations, Vol. 2006(2006), No. 53, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

NONLINEAR TRANSMISSION PROBLEM WITH A DISSIPATIVE BOUNDARY CONDITION OF MEMORY TYPE

DOHERTY ANDRADE, LUCI HARUE FATORI, JAIME E. MUÑOZ RIVERA

Abstract. We consider a differential equation that models a material consisting of two elastic components. One component is clamped while the other is in a viscoelastic fluid producing a dissipative mechanism on the boundary. So, we have a transmission problem with boundary damping condition of memory type. We prove the existence of a global solution and its uniformly decay to zero as time approaches infinity. More specifically, the solution decays exponentially provided the relaxation function decays exponentially.

1. Introduction

In this paper, we model the oscillation of a solid consisting of two elastic materials. We suppose that a part of the boundary is inside a viscoelastic fluid producing a dissipative mechanism of memory type while the other part of the boundary is clamped. The corresponding mathematical equations which model this situation is called a transmission problem with boundary dissipation.

Boundary dissipation was studied for several authors, see for example, [8, 29, 11, 30, 4, 21, 31, 3] and the references therein, all of them dealing with frictional damping. Models with memory dissipation are physically and mathematically more interesting, physically because our model follows the constitutive equations for materials with memory and Mathematically because the estimates we need to show the exponential decay are more delicate and depends on the relaxation function, see for example [2] and the references therein.

Memory dissipation is produced by the interaction of materials with memory. Such types of dissipation are subtle and their analysis are more delicate than the frictional damping, because introduce another type of technical difficulties. So, we have only a few works in this direction.

In this work we show the existence of solutions of a nonlinear transmission problem with boundary dissipation of memory type. Moreover we will prove that under suitable conditions on the relaxation functions the solution will decay uniformly as time goes to infinity. The transmission problem considered here is

$$\rho_1 u_{tt} - \gamma_1 \Delta u + f(u) = 0, \quad \text{in } \Omega_1 \times]0,T[, \tag{1.1}$$

$$\rho_2 v_{tt} - \gamma_2 \Delta v + g(v) = 0, \quad \text{in } \Omega_2 \times]0,T[, \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35B40, 35L70.
Key words and phrases. Wave equation; asymptotic behavior; memory.

Submitted November 10, 2005. Published April 28, 2006.

with boundary condition

$$u(x,t) + \int_0^t k(t-\tau)\frac{\partial u}{\partial \nu} d\tau = 0 \quad \text{on} \quad \Gamma \tag{1.3}$$

and satisfying the transmission condition

$$u = v, \quad \text{and} \quad \gamma_1 \frac{\partial u}{\partial \nu} = \gamma_2 \frac{\partial v}{\partial \nu} \quad \text{on} \quad \Gamma_1. \tag{1.4}$$

Additionally we assume that v satisfies Dirichlet boundary condition over Γ_2,

$$v(x,t) = 0, \quad \text{on } \Gamma_2 \times]0,T[, \tag{1.5}$$

and verifies the initial conditions

$$\begin{gathered} u(x,0) = u_0(x), \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega_1 \\ v(x,0) = v_0(x), \quad \text{and} \quad v_t(x,0) = v_1(x) \quad \text{in } \Omega_2. \end{gathered}$$

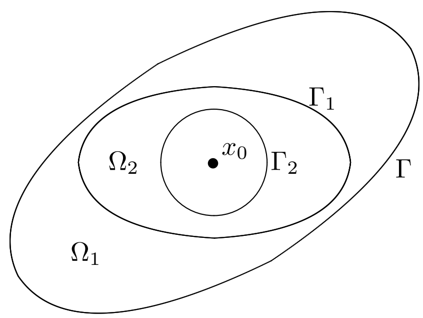

Figure 1. The configuration

The transmission problem (1.1)-(1.2) can be consider as a semilinear wave equation with discontinuous coefficients and discontinuous and non linear terms; that is, denoting

$$U = \begin{cases} u(x), & \text{if } x \in \Omega_1 \\ v(x), & \text{if } x \in \Omega_2, \end{cases} \quad \rho(x) = \begin{cases} \rho_1, & \text{if } x \in \Omega_1 \\ \rho_2, & \text{if } x \in \Omega_2, \end{cases}$$

$$a(x) = \begin{cases} \gamma_1, & \text{if } x \in \Omega_1 \\ \gamma_2, & \text{if } x \in \Omega_2, \end{cases} \quad F(x) = \begin{cases} f(x), & \text{if } x \in \Omega_1 \\ g(x), & \text{if } x \in \Omega_2. \end{cases}$$

Note that (1.1)-(1.2) is equivalent to

$$\rho(x)U_{tt} - a(x)\Delta U + F(U) = 0, \quad \text{in } \Omega \times (0,T)$$

where $\Omega = \Omega_1 \times \Omega_2$.

2. Existence of solutions

Lemma 2.1. *For each function $\alpha \in C^1$ and each $\varphi \in W^{1,2}(0,T)$, we have*

$$\int_0^t \alpha(t-\tau)\varphi(\tau)d\tau\varphi_t = -\frac{1}{2}\alpha(t)|\varphi(t)|^2 + \frac{1}{2}\alpha'\Box\varphi - \frac{1}{2}\frac{d}{dt}\big\{\alpha\Box\varphi - \Big(\int_0^t \alpha\Big)|\varphi|^2\big\}. \tag{2.1}$$

Let a be a function that satisfies

$$k(0)a + k' * a = -\frac{k'}{k(0)}. \tag{2.2}$$

By $*$ we denote the convolution product; that is, $k * g(\cdot,t) = \int_0^t k(t-\tau)g(\cdot,\tau)\,d\tau$. The function a is called the resolvent kernel of k. Using the Volterra's resolvent, we have

$$\frac{\partial u}{\partial \nu} = -\frac{1}{k(0)}u_t - a * u_t$$

after performing an integration by parts, the above identity is equivalent to

$$\frac{\partial u}{\partial \nu} = -\frac{1}{k(0)}u_t - a(0)u - a' * u + a(t)u_0. \tag{2.3}$$

We assume the following hypotheses on a:

$$a(t) > 0, \quad a'(t) < 0, \quad a''(t) > 0, \quad \forall t \geq 0 \tag{2.4}$$

$$-c_0 a'(t) \leq a''(t) \leq -c_1 a'(t), \quad \forall t \geq 0, \tag{2.5}$$

where c_i are positive constants. To facilitate our calculation we introduce the following notation

$$(\alpha \Box f)(t) = \int_0^t \alpha(t-\tau)\,|f(t) - f(\tau)|^2\,d\tau, \tag{2.6}$$

$$(\alpha \Diamond f)(t) = \int_0^t g(t-\tau)\,[f(t) - f(\tau)]\,d\tau. \tag{2.7}$$

It follows that

$$(\alpha * f)(t) = \Big(\int_0^t \alpha(s)ds\Big) f(t) - (\alpha \Diamond f)(t). \tag{2.8}$$

From hypothesis (2.2), we know that the behavior of a is similar to the behavior of k. We can find the following Lemma in [28].

Lemma 2.2. *If b and α satisfy $b + \alpha = -b * \alpha$, then*

(i) *Suppose that $|\alpha(t)| \leq c_\alpha \mathrm{e}^{-\gamma t}$, for all $t > 0$, for some $\gamma > 0$, and $c_\alpha > 0$, then for any $0 < \varepsilon < \gamma$ and $c_\alpha < \gamma - \varepsilon$, we have*

$$|b(t)| \leq \frac{c_\alpha(\gamma - \varepsilon)}{\gamma - \varepsilon - c_\alpha}\mathrm{e}^{-\varepsilon t}, \quad \forall t > 0.$$

(ii) *If α satisfies $|\alpha(t)| \leq c_\alpha(1+t)^{-p}$, for some $p > 1$, $c_\alpha > 0$ and*

$$\frac{1}{c_\alpha} > c_p := \sup_{0 \leq t < \infty} \int_0^t (1+t)^p (1+t-\tau)^{-p}(1+\tau)^{-p} d\tau,$$

then

$$|b(t)| \leq \frac{c_\alpha}{1 - c_\alpha c_p}(1+t)^{-p}, \forall\, t > 0.$$

Let us introduce the following two vector spaces

$$W = \{w \in H^1(\Omega_2) : w(x) = 0 \text{ on } \Gamma_2\},$$

$$V = \{(u,v) \in H^1(\Omega_1) \times W : u = v \text{ on } \Gamma_1\}.$$

Let us consider $f, g \in C^1(\mathbb{R})$ satisfying

$$|f(s)| \leq C_1|s|^\rho + C_2 \quad \text{and} \quad |g(s)| \leq C_1|s|^\rho + C_2, \tag{2.9}$$

$$|f'(s)| \le C_1|s|^{\rho-1} + C_2 \quad \text{and} \quad |g'(s)| \le C_1|s|^{\rho-1} + C_2, \tag{2.10}$$

where C_1 and C_2 are positive constants. When the space dimension is $n \le 2$, we use $1 \le \rho < \infty$, and when $n \ge 3$, we use $1 \le \rho \le \frac{n}{n-2}$. We also assume that for $s \in \mathbb{R}$,

$$F(s) = \int_0^s f(\sigma)\,d\sigma \ge 0 \quad \text{and} \quad G(s) = \int_0^s g(\sigma)\,d\sigma \ge 0. \tag{2.11}$$

Let us introduce the definition of weak solution to system (1.1)–(1.5).

Definition 2.3. We say that the couple (u, v) is a weak solution of (1.1)–(1.5) when

$$(u, v) \in L^\infty(0, T; V), \quad (u_t, v_t) \in L^\infty(0, T; L^2(\Omega_1) \times L^2(\Omega_2)),$$

and satisfies

$$\begin{aligned}
&\int_0^T \int_{\Omega_1} [\rho_1 u\phi_{tt} + \gamma_1 \nabla u \nabla \phi + f(u)\phi]\,dx\,dt \\
&+ \int_0^T \int_{\Omega_2} [\rho_2 v\psi_{tt} + \gamma_2 \nabla v \nabla \psi + g(v)\psi]\,dx\,dt \\
&= \int_{\Omega_1} u_1\phi(0)dx - \int_{\Omega_1} u_0\phi_t(0)dx + \int_{\Omega_2} v_1\psi(0)dx - \int_{\Omega_2} v_0\psi_t(0)dx \\
&\quad - \int_\Gamma \Big(\frac{1}{k(0)}u_t + a(0)u + a' * u - a(t)u_0\Big)\,\phi\,d\Gamma,
\end{aligned}$$

for any $(\phi, \psi) \in C^2(0, T; V)$ such that

$$\phi(T) = \phi_t(T) = \psi(T) = \psi_t(T) = 0.$$

To show the existence of strong solutions we need a regularity result for the elliptic system associated with the problem (1.1)–(1.5). For the reader's convenience we recall the following result whose proof can be found in the book by O. A. Ladyzhenskaya and N. N. Ural'tseva [10, Theorem 16.2].

Lemma 2.4. *For any given functions $F \in L^2(\Omega_1)$, $G \in L^2(\Omega_2)$, $g \in H^{1/2}(\Gamma)$, $\gamma_1, \gamma_2 \in \mathbb{R}^+$, then there exists only one solution (u, v), with $u \in H^2(\Omega_1)$ and $v \in H^2(\Omega_2)$, to the system*

$$\begin{gathered}
-\gamma_1 \Delta u = F \quad \text{in} \quad \Omega_1, \\
-\gamma_2 \Delta v = G \quad \text{in} \quad \Omega_2, \\
v(x) = 0 \quad \text{on} \quad \Gamma_2 \\
\frac{\partial u}{\partial \nu} = g, \ \text{on } \Gamma, \\
u(x) = v(x) \quad \text{on } \Gamma_1 \\
\gamma_1 \frac{\partial u}{\partial \nu} = \gamma_2 \frac{\partial v}{\partial \nu} \quad \text{on } \Gamma_1\,.
\end{gathered}$$

The existence result is summarized in the following theorem.

Theorem 2.5. *Suppose that f and g are C^1-functions satisfying* (2.9)–(2.11) *and let us take initial data such that*

$$(u_0, v_0) \in V, \quad (u_1, v_1) \in L^2(\Omega_1) \times L^2(\Omega_2), \quad u_0 = 0 \text{on } \Gamma.$$

Then, there exists a solution (u,v) *of system* (1.1)–(1.5)*, such that*

$$(u,v) \in C(0,T;V) \cap C^1(0,T;L^2(\Omega_1) \times L^2(\Omega_2)).$$

In addition, if the second-order regularity holds, that is, $(u_0,v_0) \in H^2(\Omega_1) \times H^2(\Omega_2)$ *and* $(u_1,v_1) \in V$*, and*

$$u_2 := \frac{\gamma_1}{\sigma_1}\Delta u_0 - f(u_0) \in L^2(\Omega_1)$$
$$v_2 := \frac{\gamma_2}{\sigma_2}\Delta u_0 - g(v_0) \in L^2(\Omega_2),$$

satisfying the compatibility conditions

$$\frac{\partial u_0}{\partial \nu} = -\frac{1}{k(0)}u_1 - au_0 \quad on\,\Gamma$$
$$u_0 = v_0 \quad and \quad \gamma_1\frac{\partial u}{\partial \nu} = \gamma_2\frac{\partial v}{\partial \nu}, \quad on\ \Gamma_1$$

then there exists a strong solution satisfying (u,v) *in the space*

$$C(0,T;H^2(\Omega_1) \times H^2(\Omega_2)) \cap C^1(0,T;V) \cap C^2(0,T;L^2(\Omega_1) \times L^2(\Omega_2)).$$

Proof. To show the existence of solutions we use the Galerkin method. Let (φ_i,ω_i), $i = 1,\dots,\infty$ be a basis of V and let us write

$$(u^m(t),v^m(t)) = \sum_{i=1}^m h_i(t)(\varphi_i,\omega_i),$$

where u^m and v^m satisfy

$$\begin{aligned}
&\int_{\Omega_1}\{\rho_1 u^m_{tt}\varphi_i + \gamma_1\nabla u^m\nabla\varphi_i + f(u^m)\varphi_i\}dx \\
&+ \int_{\Omega_2}\{\rho_2 v^m_{tt}\omega_i + \gamma_2\nabla v^m\nabla\omega_i + g(v^m)\omega_i\}dx \\
&= -\int_\Gamma\Big(\frac{1}{k(0)}u^m_t + a(0)u^m + a' * u^m - a(t)u^m_0\Big)\phi_i\,d\Gamma, \quad i = 1,2,\dots,m.
\end{aligned} \tag{2.12}$$

This is a m-dimensional system of ODEs in $h_i(t)$ and has a local solution in t. With the estimates obtained below, we can extend u^m and v^m to the whole interval $[0,T]$.

Weak Solutions. Multiplying the above equation by $h_i'(t)$ and summing up from $i = 1$ to m, we have

$$\frac{d}{dt}E^m(t) = -\frac{1}{2k(0)}\int_\Gamma |u^m_t|^2 d\Gamma + \frac{1}{2}a'(t)\int_\Gamma |u^m|^2 d\Gamma - \frac{1}{2}\int_\Gamma a''\Box u^m d\Gamma,$$

where

$$\begin{aligned}
E^m(t) = \frac{1}{2}\int_{\Omega_1}\{\rho_1|u^m_t|^2 + \gamma_1|\nabla u^m|^2 + 2F(u^m)\}dx + a(t)\int_\Gamma |u|^2 d\Gamma - \int_\Gamma a'\Box u d\Gamma \\
+ \frac{1}{2}\int_{\Omega_2}\{\rho_2|v^m_t|^2 + \gamma_2|\nabla v^m|^2 + 2G(v^m)\}dx.
\end{aligned}$$

Then we deduce that

$$(u^m,v^m) \quad \text{is bounded in} \quad L^\infty(0,T;H^1(\Omega_1) \times H^1(\Omega_2)), \tag{2.13}$$
$$(u^m_t,v^m_t) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega_1) \times L^2(\Omega_2)), \tag{2.14}$$

which imply that

$$(u^m, v^m) \rightharpoonup (u, v) \quad \text{weakly} \star \text{ in } L^\infty(0,T; H^1(\Omega_1) \times H^1(\Omega_2)),$$
$$(u_t^m, v_t^m) \rightharpoonup (u_t, v_t) \quad \text{weakly} \star \text{ in } L^\infty(0,T; L^2(\Omega_1) \times L^2(\Omega_2)).$$

Application of the Lions-Aubin's Lemma [13, Theorem 5.1], we have

$$(u^m, v^m) \to (u, v) \quad \text{strongly in } L^2(0,T; L^2(\Omega_1) \times L^2(\Omega_2)),$$

and consequently

$$u^m \to u \quad \text{a.e. in } \Omega_1 \quad \text{and} \quad f(u^m) \to f(u) \quad \text{a.e. in } \Omega_1,$$
$$v^m \to v \quad \text{a.e. in } \Omega_2 \quad \text{and} \quad g(v^m) \to g(v) \quad \text{a.e. in } \Omega_2.$$

From the growth condition (2.9), we have

$$f(u^m) \quad \text{is bounded in } L^\infty(0,T; L^2(\Omega_1)),$$
$$g(v^m) \quad \text{is bounded in } L^\infty(0,T; L^2(\Omega_2));$$

therefore,

$$f(u^m) \rightharpoonup f(u) \quad \text{weakly in } L^2(0,T; L^2(\Omega_1)),$$
$$g(v^m) \rightharpoonup g(v) \quad \text{weakly in } L^2(0,T; L^2(\Omega_2)).$$

The rest of the proof of the existence of weak solution is a matter of routine.

Strong Solutions. To show the regularity we take a basis of such that (u_0, v_0) and (u_1, v_1) are in $B = \{(\phi_i, w_i), i \in \mathbb{N}\}$. Therefore,

$$u_0^m = u_0, \quad v_0^m = v_0, \quad u_1^m = u_1, \quad v_1^m = v_1, \quad \forall m.$$

Differentiate the approximate equation and multiply by $h_i''(t)$. Using a similar argument as before, we obtain

$$\frac{d}{dt} E_2^m(t) \le \int_{\Omega_1} |f'(u^m)| u_t^m u_{tt}^m \, dx + \int_{\Omega_2} |g'(v^m)| v_t^m v_{tt}^m \, dx, \tag{2.15}$$

where

$$\begin{aligned} E_2^m(t) &= \frac{1}{2} \int_{\Omega_1} \rho_1 |u_{tt}^m|^2 + \gamma_1 |\nabla u_t^m|^2 \, dx + \frac{1}{2} \int_{\Omega_2} \rho_2 |v_{tt}^m|^2 + \gamma_2 |\nabla v_t^m|^2 \, dx \\ &\quad + \frac{1}{2} a(t) \int_\Gamma |u_t^m|^2 \, d\Gamma + \frac{1}{2} \int_\Gamma a' \Box u_t^m \, d\Gamma. \end{aligned} \tag{2.16}$$

Note that $E_2^m(0)$ is bounded, in fact is constant, because of our choice of the basis. Let us estimate the right hand side of (2.16). From (2.10) we have

$$\begin{aligned} &\int_{\Omega_1} |f'(u^m) u_t^m u_{tt}^m| \, dx \\ &\le \frac{C_1}{2} \int_{\Omega_1} |u^m|^{2(\rho-1)} |u_t^m|^2 \, dx + \frac{C_2}{2} \int_{\Omega_1} |u_t^m|^2 \, dx + \frac{C_1 + C_2}{2} \int_{\Omega_1} |u_{tt}^m|^2 \, dx. \end{aligned}$$

But since $(\rho - 1) \le 2/(n-2)$ and $\frac{1}{r} + \frac{1}{s} = 1$ with $r = n/2$ and $s = n/(n-2)$, we see that

$$\int_{\Omega_1} |u^m|^{2(\rho-1)} |u_t^m|^2 \, dx \le \Big(\int_{\Omega_1} |u^m|^{2^*} dx \Big)^{1/r} \Big(\int_{\Omega_1} |u_t^m|^{2^*} dx \Big)^{1/s},$$

where $2^* = 2n/(n-2)$. Then from Sobolev imbeddings and (2.13) there exists a constant $C > 0$ such that

$$\int_{\Omega_1} |u^m|^{2(\rho-1)} |u_t^m|^2 \, dx \le C + C \int_{\Omega_1} |\nabla u_t^m|^2 \, dx.$$

It follows that

$$\int_{\Omega_1} |f'(u^m) u_t^m u_{tt}^m| \, dx \le C + C \int_{\Omega_1} \{|u_{tt}^m|^2 + |\nabla u_t^m|^2\} \, dx,$$

and similarly

$$\int_{\Omega_2} |g'(v^m) v_t^m v_{tt}^m| \, dx \le C + C \int_{\Omega_2} \{|v_{tt}^m|^2 + |\nabla v_t^m|^2\} \, dx.$$

Hence, from (2.15) and the Gronwall inequality we conclude that

$$\begin{gathered}
(u_t^m, v_t^m) \quad \text{is bounded in} \quad L^\infty(0,T; H^1(\Omega_1) \times H^1(\Omega_2)), \\
(u_{tt}^m, v_{tt}^m) \quad \text{is bounded in} \quad L^\infty(0,T; L^2(\Omega_1) \times L^2(\Omega_2)),
\end{gathered}$$

which imply that

$$\begin{gathered}
(u_t^m, v_t^m) \rightharpoonup (u_t, v_t) \quad \text{weakly} * \text{ in} \quad L^\infty(0,T; H^1(\Omega_1) \times H^1(\Omega_2)), \\
(u_{tt}^m, v_{tt}^m) \rightharpoonup (u_{tt}, v_{tt}) \quad \text{weakly} * \text{ in} \quad L^\infty(0,T; L^2(\Omega_1) \times L^2(\Omega_2)).
\end{gathered}$$

Therefore, (u, v) satisfies (1.1)-(1.5). Moreover

$$\frac{\partial u}{\partial \nu} = -\frac{1}{k(0)} u_t - a(0)u - a' * u + a(t) u_0.$$

Integrating by parts,

$$\frac{\partial u}{\partial \nu} = -\frac{1}{k(0)} u_t - a * u_t.$$

Since u_t is bounded in $H^1(\Omega_1)$, $\frac{\partial u}{\partial \nu} \in H^{\frac{1}{2}}(\Gamma)$. So we have

$$\begin{gathered}
-\gamma_1 \Delta u = u_{tt} - f(u) \in L^2(\Omega_1), \\
-\gamma_2 \Delta v = v_{tt} - g(v) \in L^2(\Omega_2) \\
u = v \quad \text{and} \quad \gamma_1 \frac{\partial u}{\partial \nu} = \gamma_2 \frac{\partial v}{\partial \nu} \quad \text{in } \Gamma_1 \\
v = 0 \quad \text{in } \Gamma_2, \quad \frac{\partial u}{\partial \nu} \in H^{1/2}(\Gamma).
\end{gathered}$$

Then using Lemma 2.4 we have the required regularity to (u, v). □

3. Asymptotic behavior

In this section we prove that the solution decay exponentially as time approaches infinity. First, we need some preliminaries results.

Lemma 3.1. *Suppose that the initial data satisfies the second order regularity as in Theorem 2.5, then*

$$\frac{d}{dt} E(t) = -\frac{1}{k(0)} \int_\Gamma |u_t|^2 \, d\Gamma + \frac{a'(t)}{2} \int_\Gamma |u|^2 \, d\Gamma - \frac{1}{2} \int_\Gamma a'' \Box u \, d\Gamma,$$

where

$$
\begin{aligned}
E(t) = \frac{1}{2}\int_{\Omega_1} \rho_1|u_t|^2 + \gamma_1|\nabla u|^2 + 2F(u)dx + \gamma_1\int_\Gamma a(t)|u|^2 \; - a'\square u \, d\Gamma \\
+ \frac{1}{2}\int_{\Omega_2} \rho_2|v_t|^2 + \gamma_2|\nabla v|^2 + 2G(v)dx.
\end{aligned} \tag{3.1}
$$

Proof. Multiply by u_t equation (1.1) and by v_t equation (1.2), summing up and using identity (2.3) and Lemma 2.1 we get the result. $\square$

Let f and g be such that

$$0 \le F(s) := \int_0^s f(t)dt \le \frac{1}{m+1} sf(s), \tag{3.2}$$

$$0 \le G(s) := \int_0^s g(t)dt \le \frac{1}{l+1} sg(s), \tag{3.3}$$

$$F(s) \le G(s) \tag{3.4}$$

where $l, m > 1$. Note that odd polynomials satisfy (3.2)-(3.3). Let

$$\delta < \min\{\frac{l-1}{l+1}n, \frac{m-1}{m+1}n, 1\} \tag{3.5}$$

and

$$J_0(t) = \int_{\Omega_1} \rho_1 u_t q \cdot \nabla u \, dx + \int_{\Omega_2} \rho_2 v_t q \cdot \nabla v \, dx.$$

Lemma 3.2. *Under the hypothesis of Lemma 3.1, consider $q(x) = x - x_0 \in C^1(\overline{\Omega})$, $\gamma_1 > \gamma_2$ and $\rho_1 > \rho_2$. Then any strong solution of* (1.1)–(1.5) *satisfies*

$$
\begin{aligned}
\frac{d}{dt}J_0(t) \le \gamma_1 \int_\Gamma \frac{\partial u}{\partial \nu} q \cdot \nabla u \, dx - \frac{\gamma_1}{2}\int_\Gamma q\cdot\nu|\nabla u|^2 \, dx + \frac{\rho_1}{2}\int_\Gamma q\cdot\nu|u_t|^2 \, d\Gamma \\
- \frac{n}{2}\int_{\Omega_1} \rho_1|u_t|^2 - \gamma_1|u|^2 dx + n\int_{\Omega_1} F(u)dx - \gamma_1\int_{\Omega_1} |\nabla u|^2 \, dx \\
- \frac{n}{2}\int_{\Omega_2} \rho_2|v_t|^2 - \gamma_2|\nabla v|^2 \, dx + n\int_{\Omega_2} G(v) \, dx - \gamma_2\int_{\Omega_2} |\nabla v|^2 dx.
\end{aligned}
$$

Proof. Using equation (1.1),

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_1} \rho_1 u_t q_k \frac{\partial u}{\partial x_k} dx \\
&= \int_{\Omega_1} \rho_1 u_{tt} q_k \frac{\partial u}{\partial x_k} dx + \int_{\Omega_1} \rho_1 u_t q_k \frac{\partial u_t}{\partial x_k} dx \\
&= \int_{\Omega_1} \gamma_1 \Delta u q_k \frac{\partial u}{\partial x_k} dx - \int_{\Omega_1} f(u) q_k \frac{\partial u}{\partial x_k} dx + \frac{\rho_1}{2}\int_{\Omega_1} q_k \frac{\partial |u_t|^2}{\partial x_k} dx \\
&= \gamma_1 \int_{\partial\Omega_1} \frac{\partial u}{\partial \nu} q_k \frac{\partial u}{\partial x_k} d\Gamma - \frac{\gamma_1}{2}\int_{\partial\Omega_1} q_k\nu_k|\nabla u|^2 d\Gamma + \frac{\gamma_1}{2}\int_{\Omega_1} \frac{\partial q_k}{\partial x_k}|\nabla u|^2 dx \\
&\quad - \int_{\partial\Omega_1} F(u) q_k \nu_k d\Gamma + \int_{\Omega_1} F(u)\frac{\partial q_k}{\partial x_k} dx + \frac{\rho_1}{2}\int_{\partial\Omega_1} q_k\nu_k|u_t|^2 d\Gamma \\
&\quad - \frac{\rho_1}{2}\int_{\Omega_1} \frac{\partial q_k}{\partial x_k}|u_t|^2 dx - \gamma_1\int_{\Omega_1} \nabla u \cdot \nabla q_k \frac{\partial u}{\partial x_k} dx.
\end{aligned}
$$

So we have,

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega_1}\rho_1 u_t q_k\frac{\partial u}{\partial x_k}dx &= \gamma_1\int_{\partial\Omega_1}\frac{\partial u}{\partial\nu}q_k\frac{\partial u}{\partial x_k}d\Gamma-\frac{\gamma_1}{2}\int_{\partial\Omega_1}q_k\nu_k|\nabla u|^2d\Gamma\\
&\quad-\int_{\partial\Omega_1}q_k\nu_kF(u)d\Gamma+\frac{\rho_1}{2}\int_{\partial\Omega_1}q_k\nu_k|u_t|^2d\Gamma\\
&\quad-\frac{1}{2}\int_{\Omega_1}\frac{\partial q_k}{\partial x_k}\left\{\rho_1|u_t|^2-\gamma_1|\nabla u|^2\right\}dx\\
&\quad+\int_{\Omega_1}F(u)\frac{\partial q_k}{\partial x_k}dx-\gamma_1\int_{\Omega_1}\nabla u\cdot\nabla q_k\frac{\partial u}{\partial x_k}dx.
\end{aligned}
\tag{3.6}
$$

Similarly using equation (1.2), we obtain

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega_2}\rho_2 v_t q_k\frac{\partial v}{\partial x_k}dx &= -\gamma_2\int_{\partial\Omega_2}\frac{\partial v}{\partial\nu}q_k\frac{\partial v}{\partial x_k}d\Gamma+\frac{\gamma_2}{2}\int_{\partial\Omega_2}q_k\nu_k|\nabla v|^2d\Gamma\\
&\quad+\int_{\Gamma_1}q_k\nu_kG(v)d\Gamma-\frac{\rho_2}{2}\int_{\Gamma_1}q_k\nu_k|v_t|^2d\Gamma\\
&\quad-\frac{1}{2}\int_{\Omega_2}\frac{\partial q_k}{\partial x_k}\left\{\rho_2|v_t|^2-\gamma_2|\nabla v|^2\right\}dx\\
&\quad+\int_{\Omega_2}\frac{\partial q_k}{\partial x_k}G(v)dx-\gamma_2\int_{\Omega_2}\nabla v\cdot\nabla q_k\frac{\partial v}{\partial x_k}dx.
\end{aligned}
\tag{3.7}
$$

Using that $\nabla u=\frac{\partial u}{\partial\nu}\nu+\nabla_\tau u$ and $v=0$ on Γ_2 we have from (3.6) and (3.7) that

$$
\begin{aligned}
&\frac{d}{dt}J_0(t)\\
&=\gamma_1\int_\Gamma\frac{\partial u}{\partial\nu}q\cdot\nabla u\,d\Gamma-\frac{\gamma_1}{2}\int_\Gamma q\cdot\nu|\nabla u|^2\,d\Gamma+\frac{\rho_1}{2}\int_{\Gamma_1}q\cdot\nu|u_t|^2\,d\Gamma\\
&\quad+\frac{\gamma_1}{2}\int_{\Gamma_1}\frac{\partial u}{\partial\nu}q\cdot\nabla_\tau u\,d\Gamma-\frac{\gamma_1}{2}\int_{\Gamma_1}q\cdot\nu|\nabla_\tau u|^2\,d\Gamma+\frac{\rho_1}{2}\int_\Gamma q\cdot\nu|u_t|^2\,d\Gamma\\
&\quad-\int_{\partial\Omega_1}q\cdot\nu F(u)\,d\Gamma-\frac{\gamma_2}{2}\int_{\Gamma_1}q\cdot\nu|\frac{\partial v}{\partial\nu}|^2\,d\Gamma\\
&\quad-\gamma_2\int_{\Gamma_1}\frac{\partial v}{\partial\nu}q\cdot\nabla_\tau v\,dx+\frac{\gamma_2}{2}\int_{\Gamma_1}q\cdot\nu|\nabla_\tau v|^2\,d\Gamma\\
&\quad+\int_{\Gamma_1}q\cdot\nu G(v)\,d\Gamma+\frac{\gamma_1}{2}\int_{\Gamma_1}q\cdot\nu|\frac{\partial u}{\partial\nu}|^2\,d\Gamma\\
&\quad-\frac{\rho_2}{2}\int_{\Gamma_1}|v_t|^2q\cdot\nu\,d\Gamma+\frac{\gamma_2}{2}\int_{\Gamma_2}q\cdot\nu|\frac{\partial v}{\partial\nu}|^2d\Gamma-\frac{1}{2}\int_{\Omega_1}\frac{\partial q_k}{\partial x_k}(\rho_1|u_t|^2-\gamma_1|\nabla u|^2)\,dx\\
&\quad+\int_{\Omega_1}F(u)\frac{\partial q_k}{\partial x_k}dx-\gamma_1\int_{\Omega_1}\nabla u\cdot\nabla q_k\frac{\partial u}{\partial x_k}dx-\frac{1}{2}\int_{\Omega_2}\frac{\partial q_k}{\partial x_k}(\gamma_2|v_t|^2-\rho_2|\nabla v|^2)dx\\
&\quad+\int_{\Omega_2}G(v)\frac{\partial q_k}{\partial x_k}dx-\gamma_2\int_{\Omega_2}\nabla v\cdot\nabla q_k\frac{\partial v}{\partial x_k}\,dx.
\end{aligned}
$$

Since $u=v$ in Γ_1 then $\nabla_\tau u=\nabla_\tau v$ in Γ_1; therefore,

$$
\begin{aligned}
&\frac{d}{dt}J_0(t)\\
&=\gamma_1\int_\Gamma\frac{\partial u}{\partial\nu}q\cdot\nabla u\,d\Gamma-\frac{\gamma_1}{2}\int_\Gamma q\cdot\nu|\nabla u|^2\,d\Gamma+\frac{\rho_1}{2}\int_\Gamma q\cdot\nu|u_t|^2\,d\Gamma
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\gamma_1}{2}\big(\frac{\gamma_1-\gamma_2}{\gamma_2}\big)\int_{\Gamma_1}|\frac{\partial u}{\partial \nu}|^2 q\cdot\nu\, d\Gamma - \big(\frac{\gamma_1-\gamma_2}{2}\big)\int_{\Gamma_1} q\cdot\nu|\nabla_\tau u|^2\, d\Gamma\\
&+ \big(\frac{\rho_1-\rho_2}{2}\big)\int_{\Gamma_1} q\cdot\nu|u_t|^2\, d\Gamma - \int_\Gamma q\cdot\nu F(u)\, d\Gamma\\
&- \int_{\Gamma_1} q\cdot\nu[F(u)-G(u)]\, d\Gamma + \frac{\gamma_2}{2}\int_{\Gamma_2} q\cdot\nu|\frac{\partial v}{\partial\nu}|^2 d\Gamma\\
&- \frac{n}{2}\int_{\Omega_1}\rho_1|u_t|^2 - \gamma_1|\nabla u|^2\, dx + n\int_{\Omega_1}F(u)dx\\
&- \gamma_1\int_{\Omega_1}|\nabla u|^2\, dx - \frac{n}{2}\int_{\Omega_2}\rho_2|v_t|^2 - \gamma_2|\nabla v|^2 dx + n\int_{\Omega_2}G(v)dx - \gamma_2\int_{\Omega_2}|\nabla v|^2 dx.
\end{aligned}
$$

Using that $(x-x_0)\cdot\nu > 0$ in Γ then we conclude our proof. □

Lemma 3.3. *Under the hypothesis in Lemma 3.1,*

$$
\begin{aligned}
&\frac{d}{dt}\{\int_{\Omega_1}\rho_1 u u_t dx + \int_{\Omega_2}\rho_2 v_t v dx\}\\
&= \int_{\Omega_1}\rho_1|u_t|^2 - \gamma_1|\nabla u|^2 dx + \gamma_1\int_\Gamma \frac{\partial u}{\partial\nu} u d\Gamma\\
&\quad - \int_{\Omega_1} f(u)u dx + \int_{\Omega_2}\rho_2|v_t|^2 - \gamma_2|\nabla v|^2 dx - \int_{\Omega_2} g(v)v dx.
\end{aligned}
$$

Proof. Multiply (1.1) by u and (1.2) by v and summing up the product the our result follows. □

Let us define the functional

$$
\Phi(t) = J_0(t) + \big(\frac{n-\delta}{2}\big)\big[\int_{\Omega_1}\rho_1 u u_t dx + \int_{\Omega_2}\rho_2 v_t v dx\big]
$$

where we consider $q(x) = x - x_0$ as before.

Lemma 3.4. *Under the hypotheses of Lemmas 3.1 and 3.2, there exists a positive constant δ_0 such that*

$$
\frac{d}{dt}\Phi(t) \le C\int_\Gamma\big|\frac{\partial u}{\partial\nu}\big|^2 d\Gamma + \big(\frac{n-\delta}{2}\big)\gamma_1\int_\Gamma u\frac{\partial u}{\partial\nu} d\Gamma - \delta_0 E_0(t) + \frac{\rho_1}{2}\int_\Gamma q\cdot\nu|u_t|^2 d\Gamma,
$$

where

$$
E_0(t) = \frac{1}{2}\int_{\Omega_1}\rho_1|u_t|^2 + \gamma_1|\nabla u|^2 + F(u)dx + \frac{1}{2}\int_{\Omega_2}\rho_2|v_t|^2 + \gamma_2|\nabla v|^2 + G(v)dx.
$$

Proof. From Lemma 3.2 and Lemma 3.3 we have,

$$
\begin{aligned}
&\frac{d}{dt}\Phi(t)\\
&\le C\int_\Gamma\big|\frac{\partial u}{\partial\nu}\big|^2 d\Gamma - \frac{\delta}{2}\Big[\int_{\Omega_1}\rho_1|u_t|^2 + \gamma_1|\nabla u|^2 dx + \int_{\Omega_2}\rho_2|v_t|^2 + \gamma_2|\nabla v|^2 dx\Big]\\
&\quad + n\int_{\Omega_1}F(u)dx + n\int_{\Omega_2}G(v)dx - \big(\frac{n-\delta}{2}\big)\int_{\Omega_1} f(u)u dx\\
&\quad - \big(\frac{n-\delta}{2}\big)\int_{\Omega_2} g(v)v dx + \big(\frac{n-\delta}{2}\big)\gamma_1\int_\Gamma\frac{\partial u}{\partial\nu} u d\Gamma + \frac{\rho_1}{2}\int_\Gamma q\cdot\nu|u_t|^2 d\Gamma.
\end{aligned}
$$

Using the hypotheses on F and G, we obtain

$$\begin{aligned} n\int_{\Omega_1} F(u)dx - \frac{n-\delta}{2}\int_{\Omega_1} f(u)udx &\le \Big(\frac{n}{m+1} - \frac{n-\delta}{2}\Big)\int_{\Omega_1} f(u)udx \\ &\le -\alpha\int_{\Omega_1} f(u)udx, \end{aligned}$$

where by our assumption on δ we have that $\alpha > 0$. Similarly

$$n\int_{\Omega_2} G(v)dx - \frac{n-\delta}{2}\int_{\Omega_2} g(v)vdx \le -\beta\int_{\Omega_2} g(v)vdx.$$

From where it follows that

$$\begin{aligned} \frac{d}{dt}\Phi(t) &\le \gamma_1\int_\Gamma |\frac{\partial u}{\partial\nu}|^2 d\Gamma + (\frac{n-\delta}{2})\gamma_1\int_\Gamma u\frac{\partial u}{\partial\nu}d\Gamma \\ &\quad - \frac{\delta}{2}\Big[\int_{\Omega_1}\rho_1|u|^2 + \gamma_1|\nabla u|^2dx + \int_{\Omega_2}\rho_2|v_t|^2 + \gamma_2|\nabla v|^2dx\Big] \\ &\quad - \alpha\int_{\Omega_1} uf(u)dx - \beta\int_{\Omega_2} vg(v)dx + \frac{\rho_1}{2}\int_\Gamma q\cdot\nu|u_t|^2d\Gamma, \end{aligned}$$

which implies that for $\delta_0 = \min\big\{\frac{\delta}{2}, \alpha(m+1), \beta(l+1)\big\}$, we have

$$\frac{d}{dt}\Phi(t) \le \gamma_1\int_\Gamma |\frac{\partial u}{\partial\nu}|^2 d\Gamma + \left(\frac{n-\delta}{2}\right)\gamma_1\int_\Gamma u\frac{\partial u}{\partial\nu}d\Gamma - \delta_0 E_0(t) + \frac{\rho_1}{2}\int_\Gamma q\cdot\nu|u_t|^2d\Gamma.$$

□

Theorem 3.5. *With hypotheses in Lemma 3.2, there exists a positive constants such that any strong solution satisfies*

$$E(t) \le CE(0)\exp(-\delta_1 t),$$

provided (2.4)-(2.5) *holds.*

Proof. Note that from (1.3) and (2.8) we have

$$\frac{\partial u}{\partial\nu} = -\frac{1}{k(0)}u_t - a(t)u - a'\Diamond u$$

from where it follows

$$|\frac{\partial u}{\partial\nu}|^2 \le 2\big\{\frac{1}{k^2(0)}|u_t|^2 + a^2(t)|u|^2 + |a'\Diamond u|^2\big\}.$$

Since

$$|a'\Diamond u|^2 = \Big|\int_0^t a'(t-s)\{u(s)-u(t)\}ds\Big|^2 \le \Big(\int_0^t |a'(t-s)|ds\Big)|a'|\Box u.$$

From this inequality and (2.5) it follows that

$$\left|\frac{\partial u}{\partial\nu}\right|^2 \le k_0\{|u_t|^2 + a(t)|u|^2 + a'\Box u\}. \tag{3.8}$$

On the other hand,

$$
\begin{aligned}
|\int_\Gamma u\frac{\partial u}{\partial\nu}d\Gamma| &\le \Big(\int_\Gamma |u|^2 d\Gamma\Big)^{1/2}\Big(\int_\Gamma |\frac{\partial u}{\partial\nu}|^2 d\Gamma\Big)^{1/2}\\
&\le \delta_1\int_\Gamma |u|^2 d\Gamma + C_{\delta_1}\int_\Gamma \{|u_t|^2 + a(t)|u|^2 + a'\Box u\}d\Gamma \qquad (3.9)\\
&\le \delta_1\int_\Gamma |u|^2 d\Gamma + C\int_\Gamma \{|u_t|^2 + |u|^2 + a'\Box u\}d\Gamma.
\end{aligned}
$$

Since

$$
\int_\Gamma |u|^2\, d\Gamma \le C\int_\Omega |\nabla u|^2 + |\nabla v|^2\, dx,
$$

we have that $\mathcal{L}(t) = NE(t) + \Phi(t)$ satisfies

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \le& -\frac{N\gamma_1}{k(0)}\int_\Gamma |u_t|^2 d\Gamma + \frac{N\gamma_1 a'(t)}{2}\int_\Gamma |u|^2 d\Gamma - \frac{N\gamma_1}{2}\int_\Gamma a''\Box u d\Gamma\\
&+ C\int_\Gamma |\frac{\partial u}{\partial\nu}|^2 d\Gamma + (\frac{n-\delta}{2})\gamma_1\int_\Gamma u\frac{\partial u}{\partial\nu}d\Gamma\\
&- \frac{\delta_0}{2}E_0(t) + \rho_1\int_\Gamma q\cdot\nu|u_t|^2 d\Gamma.
\end{aligned}
$$

Using (4.2) and (4.3) we conclude that

$$
\begin{gathered}
\frac{d}{dt}\mathcal{L}(t) \le -(\frac{N\gamma_1}{k(0)} - C_2)\int_\Gamma |u_t|^2 d\Gamma - (\frac{N\gamma_1}{2} - C_2)\int_\Gamma a''\Box u d\Gamma - \frac{\delta_0}{2}E_0(t)\\
\frac{d}{dt}\mathcal{L}(t) \le -\frac{\delta_0}{2}E(t) \le -c\mathcal{L}(t).
\end{gathered} \qquad (3.10)
$$

from where our conclusion follows. □

We remark that standard density arguments, the above result is also valid for weak solutions.

4. Polynomial rate of decay

Here our attention turns to the uniform rate of decay when k decays polynomially as $(1+t)^{-p}$. In this case we will show that the solution also decays polynomially with the same rate. Let us consider the following hypotheses:

$$
\begin{gathered}
0 < a(t) \le b_0(1+t)^{-p},\\
-b_1 a^{1+\frac{1}{p}}(t) \le a'(t) \le -b_2 a^{1+\frac{1}{p}}(t),\\
b_3[-a'(t)]^{1+\frac{1}{p+1}} \le a''(t) \le b_4[-a'(t)]^{1+\frac{1}{p+1}},
\end{gathered} \qquad (4.1)
$$

where $p > 1$ and $b_i > 0$ for $i = 0,\dots,4$. The following lemmas will play an important role in the sequel.

Lemma 4.1. *Let m and h be integrable functions, $0 \le r < 1$ and $q > 0$. Then, for $t \ge 0$,*

$$
\begin{aligned}
&\int_0^t |m(t-s)h(s)|ds\\
&\le \Big(\int_0^t |m(t-s)|^{1+\frac{1-r}{q}}|h(s)|ds\Big)^{q/(q+1)}\Big(\int_0^t |m(t-s)|^r|h(s)|ds\Big)^{1/(q+1)}.
\end{aligned}
$$

Proof. Let

$$v(s) := |m(t-s)|^{1-\frac{r}{q+1}}|h(s)|^{\frac{q}{q+1}}, \quad w(s) := |m(t-s)|^{\frac{r}{q+1}}|h(s)|^{\frac{1}{q+1}}.$$

Applying Hölder's inequality to $|m(s)h(s)| = v(s)w(s)$ with exponents $\delta = q/(q+1)$ for v and $\delta^* = q+1$ for w our conclusion follows. □

Lemma 4.2. *Let $\phi \in L^\infty(0,T;L^2(\Gamma))$. Then, for $p > 1$, $0 \le r < 1$ and $t \ge 0$, we have*

$$\begin{aligned} &\Big(\int_\Gamma |a'|\Box\phi d\Gamma\Big)^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}} \\ &\le 2\Big(\int_0^t |a'(s)|^r ds \|\phi\|^2_{L^\infty(0,t;L^2(\Gamma))}\Big)^{\frac{1}{(1-r)(p+1)}} \int_\Gamma |a'|^{1+\frac{1}{p+1}}\Box\phi\, d\Gamma, \end{aligned}$$

while for $r = 0$ we get

$$\begin{aligned} &\Big(\int_{\Gamma 1} |a'|\Box\phi d\Gamma\Big)^{\frac{p+2}{p+1}} \\ &\le 2\Big(\int_0^t \|\phi(s,.)\|^2_{L^2(\Gamma)}ds + t\|\phi(s,.)\|^2_{L^2(\Gamma)}\Big)^{p+1} \int_\Gamma |a'|^{1+\frac{1}{p+1}}\Box\phi\, d\Gamma. \end{aligned}$$

Proof. The above inequalities are a immediate consequence of Lemma 4.1 taking

$$m(s) := |a'(s)|, \quad h(s) := \int_\Gamma |\phi(t,x) - \phi_(s,x)|^2 d\Gamma, \quad q := (1-r)(p+1).$$

This concludes our assertion. □

Theorem 4.3. *With the hypotheses in Lemma 3.1 and Lemma 3.2, if the resolvent kernel $a(t)$ satisfies condition (4.1), then there is a positive constant c such that*

$$E(t) \le \frac{c}{(1+t)^{p+1}} E(0).$$

Proof. Note that from (1.3) and (2.8) we have

$$\frac{\partial u}{\partial \nu} = -\frac{1}{k(0)}u_t - a(t)u - a'\diamondsuit u$$

from which it follows

$$|\frac{\partial u}{\partial \nu}|^2 \le 2\big\{\frac{1}{a^2(0)}|u_t|^2 + a^2(t)|u|^2 + |a'\diamondsuit u|^2\big\}.$$

Since

$$|a'\diamondsuit u|^2 = \Big|\int_0^t a'(t-s)\{u(s)-u(t)\}ds\Big|^2 \le \Big(\int_0^t |a'(t-s)|^{1-\frac{1}{p}} ds\Big)[-a']^{1+\frac{1}{p}}\Box u,$$

and (2.5), it follows that

$$\Big|\frac{\partial u}{\partial \nu}\Big|^2 \le k_0\{|u_t|^2 + [-a']^{1+\frac{1}{p}}(t)|u|^2 + [-a']^{1+\frac{1}{p}}\Box u\}. \tag{4.2}$$

On the other hand,

$$\begin{aligned}\Big|\int_\Gamma u\frac{\partial u}{\partial\nu}d\Gamma\Big| &\le \Big(\int_\Gamma |u|^2 d\Gamma\Big)^{1/2}\Big(\int_\Gamma \big|\frac{\partial u}{\partial\nu}\big|^2 d\Gamma\Big)^{1/2}\\ &\le \delta_1\int_\Gamma |u|^2 d\Gamma + \delta_1\int_\Gamma \big\{|u_t|^2 + [-a']^{1+\frac{1}{p}}(t)|u|^2 + [-a']^{1+\frac{1}{p}}\Box u\big\}d\Gamma\\ &\le C\int_\Gamma \big\{|u_t|^2 + |u|^2 + [-a']^{1+\frac{1}{p}}\Box u\big\}d\Gamma.\end{aligned} \tag{4.3}$$

Since

$$\int_\Gamma |u|^2\, d\Gamma \le C\int_\Omega |\nabla u|^2 + |\nabla v|^2\, dx,$$

we have $\mathcal{L}(t) = NE(t) + \Phi(t)$ which satisfies

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\le -\frac{N\gamma_1}{k(0)}\int_\Gamma |u_t|^2 d\Gamma + \frac{N\gamma_1 a'(t)}{2}\int_\Gamma |u|^2 d\Gamma - \frac{N\gamma_1}{2}\int_\Gamma [-a']^{1+\frac{1}{p}}\Box u d\Gamma\\ &\quad + C\int_\Gamma [-a']^{1+\frac{1}{p}}\Box u d\Gamma + C\int_\Gamma \big|\frac{\partial u}{\partial\nu}\big|^2 d\Gamma + \big(\frac{n-\delta}{2}\big)\gamma_1\int_\Gamma u\frac{\partial u}{\partial\nu}d\Gamma\\ &\quad - \frac{\delta_0}{2}E_0(t) + \rho_1\int_\Gamma q\cdot\nu|u_t|^2 d\Gamma.\end{aligned}$$

Using (4.2) and (4.3), we conclude that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\le -\big(\frac{N\gamma_1}{k(0)} - C_2\big)\int_\Gamma |u_t|^2 d\Gamma - \big(\frac{N\gamma_1}{2} - C_2\big)\int_\Gamma [-a']^{1+\frac{1}{p}}\Box u d\Gamma\\ &\quad - \frac{\delta_0}{2}E_0(t),\end{aligned} \tag{4.4}$$

from where we have that for N large enough we get

$$\frac{d}{dt}\mathcal{L}(t) \le -\frac{N\gamma_1}{2k(0)}\int_\Gamma |u_t|^2 d\Gamma - \frac{N\gamma_1}{4}\int_\Gamma [-a']^{1+\frac{1}{p}}\Box u d\Gamma - \frac{\delta_0}{2}E_0(t). \tag{4.5}$$

Let us fix $0 < r < 1$ such that $\frac{1}{p+1} < r < \frac{p}{p+1}$. In this condition from hypothesis (4.1) we have

$$\int_0^\infty [-a']^r \le c\int_0^\infty \frac{1}{(1+t)^{r(p+1)}} < \infty \quad \text{for } i = 1,2,3,4.$$

Using this estimate in Lemma 4.2,

$$\int_\Gamma [-a']^{1+\frac{1}{p+1}}\Box u d\Gamma \ge cE(0)^{-\frac{1}{(1-r)(p+1)}}\Big(\int_\Gamma [-a']\Box u d\Gamma\Big)^{1+\frac{1}{(1-r)(p+1)}}, \tag{4.6}$$

On the other hand, since the energy is bounded we have

$$E(t)^{1+\frac{1}{(1-r)(p+1)}} \le cE(0)^{\frac{1}{(1-r)(p+1)}}E(t). \tag{4.7}$$

Substitutinn (4.6)-(4.7) in (4.5), we arrive to

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\le -cE(0)^{-\frac{1}{(1-r)(p+1)}}E(t)^{1+\frac{1}{(1-r)(p+1)}}\\ &\quad - cE(0)^{-\frac{1}{(1-r)(p+1)}}\Big(\int_\Gamma [-a']\Box u d\Gamma\Big)^{1+\frac{1}{(1-r)(p+1)}}.\end{aligned}$$

Since there exists positive constants satisfying

$$c_0E(t) \le \mathcal{L}(t) \le c_1E(t), \tag{4.8}$$

we obtain
$$\frac{d}{dt}\mathcal{L}(t) \le -\frac{c}{\mathcal{L}(0)^{\frac{1}{(1-r)(p+1)}}}\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}}. \tag{4.9}$$
Therefore, using a Gronwall's type argument we conclude that
$$\mathcal{L}(t) \le \frac{c}{(1+t)^{(1-r)(p+1)}}\mathcal{L}(0). \tag{4.10}$$
Since $(1-r)(p+1) > 1$ we get, for $t \ge 0$, the following bounds
$$t\|u(t,.)\|^2_{L^2(\Gamma)} \le ct\mathcal{L}(t) \le \infty,$$
$$\int_0^t \|u(s,.)\|^2_{L^2(\Gamma)} \le c\int_0^\infty \mathcal{L}(t) \le \infty.$$
Using the above estimates in Lemma 4.2 with $r = 0$, we get
$$\int_\Gamma [-a']^{1+\frac{1}{p+1}}\Box u d\Gamma \ge \frac{c}{E(0)^{\frac{1}{p+1}}}\Big(\int_\Gamma [-a']\Box u d\Gamma\Big)^{1+\frac{1}{p+1}}.$$
Using these inequalities and the same arguments as in the derivation of (4.9), we have
$$\frac{d}{dt}\mathcal{L}(t) \le -\frac{c}{\mathcal{L}(0)^{\frac{1}{p+1}}}\mathcal{L}(t)^{1+\frac{1}{p+1}}.$$
From where we obtain $\mathcal{L}(t) \le \frac{c}{(1+t)^{p+1}}\mathcal{L}(0)$. Then inequality (4.8) implies $E(t) \le \frac{c}{(1+t)^{p+1}}E(0)$, which completes the proof. $\Box$

We remark that by standard density arguments, the above result is also valid for weak solutions.

References

[1] D. Andrade, J. E. Muñoz Rivera, *A Boundary Condition with memory in elasticity.* Appl. Math Letters, v.13, 115-121, 2000.

[2] D. Andrade, J. E. Muñoz Rivera, *Exponential Decay of Nonlinear Wave Equation with a Viscoelastic Boundary Condition.* Mathematical Methods in The Applied Sciences, v.23, 41-61, 2000.

[3] M.M. Cavalcanti, V.N.D. Cavalcanti, J.S. Prates, J.A. Soriano; *Existence and uniform decay of solutions of a degenerate equation with nonlinear boundary damping and boundary memory source term.* Nonlinear Analysis Vol. 38(1), 281- 294, 1999.

[4] F. Conrad, B. Rao; *Decay of solutions of the wave equation in a star-shaped domain with non linear boundary feedback.* Asymptotic Analysis Vol. 7(1), 159- 177, 1993.

[5] C. M. Dafermos; *An abstract Volterra equation with application to linear viscoelasticity.* J. Differential Equations 7, 554-589, 1970.

[6] G. Dassios & F. Zafiropoulos; *Equipartition of energy in linearized 3-d viscoelasticity*, Quart. Appl. Math. 48,715-730, 1990.

[7] J. M. Greenberg & Li Tatsien; *The effect of the boundary damping for the quasilinear wave equation.* Journal of Differential Equations 52, 66-75, 1984.

[8] Greenberg J.M. and Li Tatsien; *The effect of the boundary damping for the quasilinear wave equation.* Journal of Differential Equations Vol. 52(1), 66- 75, 1984.

[9] A. Haraux & E. Zuazua; *Decay estimates for some semilinear damped hyperbolic problems.* Archive for Rational Mechanics and Analysis, Vol. 100(2), 191- 206, 1988.

[10] O. A. Ladyzhenskaya and N. N. Ural'tseva; *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[11] I. Lasiecka; *Global uniform decay rates for the solution to the wave equation with nonlinear boundary conditions*, Applicable Analysis 47, 191-212, 1992.

[12] J. L. Lions; Controlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribus, Tome 1, Masson, Paris, 1988.

[13] J. L. Lions; Quelques méthodes de rèsolution des problèmes aux limites non linèaires, Dunod Gaulthier-Villars, Paris, 1969.

[14] K. Liu & Z. Liu; *Exponential decay of the energy of the Euler Bernoulli beam with locally distributed Kelvin-Void*. SIAM Control and Optimization 36, 1086-1098, 1998.

[15] Weijiu Liu and G. Williams; *The exponential stability of the problem of transmission of the wave equation*. Bull. Austral. Math. Soc. Vol. 57(1), 305- 327, 1998.

[16] M. Nakao; *Decay of solutions of the wave equation with a local nonlinear dissipation*. Mathematisched Annalen 305, 403-417, 1996.

[17] M. Nakao; *Decay of solutions of the wave equation with a local degenerate dissipation*. Israel Journal of Mathematics 95, 25-42, 1996.

[18] M. Nakao; *On the decay of solutions of the wave equation with a local time-dependent nonlinear dissipation*. Advances in Mathematical Science and Applications 7, 317- 331, 1997.

[19] M. Nakao; *Decay of solutions of the wave equation with a local nonlinear dissipation*. Mathematische Annalen Vol. 305(1), 403- 417, 1996.

[20] K. Ono; *A stretched string equation with a boundary dissipation*. Kyushu J. Maths. 28, 265-281, 1994.

[21] K. Ono; *A stretched string equation with a boundary dissipation*. Kyushu J. of Math. Vol. 28(2), 265- 281, 1994.

[22] J. E. Muñoz Rivera; *Asymptotic behaviour in linear viscoelasticity*. Quart. Appl. Math. 52, 629-648, 1994.

[23] J. E. Muñoz Rivera; *Global smooth solution for the Cauchy problem in nonlinear viscoelasticity*. Diff. Integral Equations 7, 257-273, 1994.

[24] J. E. Muñoz Rivera & M. L. Oliveira; *Stability in inhomogeneous and anisotropic thermoelasticity*. Bollettino U.M.I. 7 (11A), 115-127, 1997.

[25] J. E. Muñoz Rivera and Alfonso Peres Salvatierra; *Decay of the energy to partially viscoelastic materials*. Mathematical models and methods for smart materials, Ser. Adv. Math. Appl. Sci., 62, 297–311, 2002.

[26] J. E. Muñoz Rivera and Higidio Portillo Oquendo; *The transmission problem of viscoelastic waves*. Acta Aplicandae Mathematicae Vol. 60(1), 1-21, 2000.

[27] J. E. Muñoz Rivera and M. To Fu; *Exponential stability of a transmission problem*. To appear.

[28] J. E. Muñoz Rivera and R. Racke; *Magneto-thermo-elasticity—large-time behavior for linear systems*. Adv. Differential Equations, 6 (3), 359–384, 2001.

[29] Shen Weixi and Zheng Songmu; *Global smooth solution to the system of one dimensional Thermoelasticity with dissipation boundary condition*. Chin. Ann. of Math. Vol. 7B(3), 303-317, 1986.

[30] M. Tucsnak; *Boundary stabilization for the stretched string equation*. Differential and Integral Equation Vol. 6(4), 925- 935, 1993.

[31] A. Wyler; *Stability of wave equations with dissipative boundary conditions in a bounded domain*. Differential and Integral Equations Vol. 7(2), 345- 366, 1994.

[32] Zhang Xu; *Explicit observability inequalities for the wave equation with lower order terms by means of Carleman inequalities*. SIAM Journal of Control and Optimization Vol. 39(3), 812- 834, 2000.

[33] E. Zuazua; *Exponential decay for the semilinear wave equation with locally distribuited damping*. Communication in PDE Vol. 15(1), 205- 235, 1990.

Doherty Andrade

Maringá State University, Departament of Mathematics, 87020-900 - Maringá, Brazil

E-mail address: doherty@uem.br

Luci H. Fatori

Londrina State University, Departament of Mathematics, 86051-990 - Londrina, Brazil

E-mail address: lucifatori@uel.br

Jaime E. Munõz Rivera

Laboratório Nacional de Computação Científica, Av. Getúlio Vargas, 333, 25651-070 - Petrópolis, Brazil

E-mail address: rivera@lncc.br