Electronic Journal of Differential Equations, Vol. 2006(2006), No. 54, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

GROWTH OF SOLUTIONS OF COMPLEX DIFFERENTIAL EQUATIONS WITH COEFFICIENTS OF FINITE ITERATED ORDER

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Abstract. In this paper, we investigate the growth of solutions to the differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_0(z)f = F(z),$$

where the coefficients are of finite iterated order.

1. Introduction

It is well known that all solutions of the complex differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_0(z)f = 0, \tag{1.1}$$

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_0(z)f = F(z) \tag{1.2}$$

are entire functions, provided that the coefficients $A_0(z), A_1(z), \dots, A_{k-1}(z), F(z)$ are entire functions with $A_0(z) \not\equiv 0$. A natural question arises: What conditions on $A_0(z), A_1(z), \dots, A_{k-1}(z), F(z)$ will guarantee that every solution $f \not\equiv 0$ has infinite order? Also: For solutions of infinite order, how to express the growth of them explicitly, it is a very important problem. Partial results have been available since a paper of Frei [4]. For high order differential equations, the following results have been obtained.

Theorem 1.1 ([3, Theorem 2.1]). *Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be entire functions with $A_0(z) \not\equiv 0$, such that for some real constants $\alpha, \beta, \mu, \theta_1, \theta_2$, with $0 \le \beta < \alpha, \mu > 0, \theta_1 < \theta_2$, we have*

$$|A_0(z)| \ge e^{\alpha |z|^\mu}, \tag{1.3}$$

$$|A_j(z)| \le e^{\beta |z|^\mu}, \quad j = 1, \dots, k-1, \tag{1.4}$$

as $z \to \infty$ with $\theta_1 \le \arg z \le \theta_2$. Then every solution $f \not\equiv 0$ of (1.1) has infinite order.

Theorem 1.2 ([1, Theorem 1]). *Let H be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| : z \in H\} > 0$, and let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be entire functions and*

2000 *Mathematics Subject Classification.* 30D35, 34M10.
Key words and phrases. Differential equations; growth of solutions; iterated order.

Submitted January 17, 2006. Published April 28, 2006.
Supported by grant 10371009 from the Natural Science Foundation of China.

satisfy (1.3) *and* (1.4) *as* $z \to \infty$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of* (1.1) *satisfies* $\sigma(f) = \infty$ *and* $\sigma_2(f) \geq \mu$.

Theorem 1.3 ([1, Theorem 2]). *Let* H *be a set of complex numbers satisfying* $\overline{\text{dens}}\{|z| : z \in H\} > 0$, *and let* $A_0(z), A_1(z), \dots, A_{k-1}(z)$ *be entire functions with* $\max\{\sigma(A_j) : j = 1, \dots, k-1\} \leq \sigma(A_0) = \sigma < +\infty$ *such that for some constants* $0 \leq \beta < \alpha$ *and for any* $\varepsilon > 0$, *we have*

$$|A_0(z)| \geq e^{\alpha |z|^{\sigma - \varepsilon}}, \tag{1.5}$$

$$|A_j(z)| \leq e^{\beta |z|^{\sigma - \varepsilon}}, \quad j = 1, \dots, k-1, \tag{1.6}$$

as $z \to \infty$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of* (1.1) *satisfies* $\sigma(f) = \infty$ *and* $\sigma_2(f) = \sigma(A_0)$.

Theorem 1.4 ([2, Theorem 1.1]). *Let* $H, A_0(z), A_1(z), \dots, A_{k-1}(z)$ *satisfy the hypotheses of Theorem 1.3, and let* $F \not\equiv 0$ *be an entire function with* $\sigma(F) < +\infty$. *Then every solution* $f(z)$ *of* (1.2) *satisfies* $\overline{\lambda}_2(f) = \sigma_2(f) = \sigma$, *with at most one exceptional solution* f_0 *satisfying* $\sigma_2(f_0) < \sigma$.

2. Notation and results

In this section, we prove some results concerning the above questions when the coefficients of (1.1) and (1.2) are of finite iterated order. For $r \in [0, \infty)$, we define $\exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r)$ $(i \in \mathbb{N})$. For r sufficiently large, we define $\log_1 r = \log r$, $\log_{i+1} r = \log(\log_i r)$ $(i \in \mathbb{N})$. To express the rate of growth of entire function of infinite order, we introduce the notion of iterated order [8].

Definition 2.1. The iterated i-order of an entire function f is defined by

$$\sigma_i(f) = \limsup_{r \to \infty} \frac{\log_{i+1} M(r, f)}{\log r} = \limsup_{r \to \infty} \frac{\log_i T(r, f)}{\log r} \quad (i \in \mathbb{N}). \tag{2.1}$$

Definition 2.2. The finiteness degree of the order of an entire function f is defined by

$$i(f) = \begin{cases} 0 & \text{if } f \text{ is a polynomial}, \\ \min\{j \in \mathbb{N} : \sigma_j(f) < \infty\} & \text{if } f \text{ is transcendental with} \\ & \quad \sigma_j(f) < \infty \text{ for some } j \in \mathbb{N}, \\ \infty & \text{if } \sigma_j(f) = \infty \ \forall j \in \mathbb{N}. \end{cases} \tag{2.2}$$

Definition 2.3. The iterated convergence exponent of the sequence of zeros of an entire function f is defined by

$$\lambda_i(f) = \limsup_{r \to \infty} \frac{\log_i n(r, 1/f)}{\log r} \quad (i \in \mathbb{N}). \tag{2.3}$$

The linear measure of a set $E \subset [0, +\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)\, dt$. The logarithmic measure of a set $E \subset [1, +\infty)$ is defined by $lm(E) = \int_1^{+\infty} \chi_E(t)/t\, dt$, where $\chi_E(t)$ is the characteristic function of E. The upper and lower densities of E are

$$\overline{\text{dens}} E = \limsup_{r \to \infty} \frac{m(E \cap [0, r])}{r}, \quad \underline{\text{dens}} E = \liminf_{r \to \infty} \frac{m(E \cap [0, r])}{r}. \tag{2.4}$$

In this paper, we improve the results of Belaïdi [1, 2, 3], and we obtain the following results:

Theorem 2.4. *Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be entire functions with $A_0(z) \not\equiv 0$ such that for real constants $\alpha, \beta, \mu, \theta_1, \theta_2$ and positive integer p with $0 \le \beta < \alpha, \mu > 0, \theta_1 < \theta_2, 1 \le p < \infty$, we have*

$$|A_0(z)| \ge \exp_p\{\alpha|z|^\mu\}, \tag{2.5}$$

$$|A_j(z)| \le \exp_p\{\beta|z|^\mu\}, \quad j = 1, \dots, k-1, \tag{2.6}$$

as $z \to \infty$ with $\theta_1 \le argz \le \theta_2$. Then $\sigma_{p+1}(f) \ge \mu$ holds for all non-trivial solutions of (1.1).

Theorem 2.5. *Let H be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| : z \in H\} > 0$, and let $A_0(z)$, $A_1(z), \dots, A_{k-1}(z)$ be entire functions and satisfy* (2.5) *and* (2.6) *as $z \to \infty$ for $z \in H$, where $0 \le \beta < \alpha, \mu > 0$, $1 \le p < \infty$. Then every solution $f \not\equiv 0$ of* (1.1) *satisfies $\sigma_{p+1}(f) \ge \mu$.*

Theorem 2.6. *Let H be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| : z \in H\} > 0$, and let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be entire functions of iterated order with $\max\{\sigma_p(A_j) : j = 1, \dots, k-1\} \le \sigma_p(A_0) = \sigma < +\infty$, $1 \le p < \infty$ such that for some constants $0 \le \beta < \alpha$ and for any given $\varepsilon > 0$, we have*

$$|A_0(z)| \ge \exp_p\{\alpha|z|^{\sigma-\varepsilon}\} \tag{2.7}$$

$$|A_j(z)| \le \exp_p\{\beta|z|^{\sigma-\varepsilon}\}, \quad j = 1, \dots, k-1, \tag{2.8}$$

as $z \to \infty$ for $z \in H$. Then every solution $f \not\equiv 0$ of (1.1) *satisfies $\sigma_{p+1}(f) = \sigma_p(A_0) = \sigma$.*

Theorem 2.7. *Let $H, A_0(z)$, $A_1(z), \dots, A_{k-1}(z)$ satisfy the hypotheses of Theorem 2.6, and let $F \not\equiv 0$ be an entire function of iterated order with $i(F) = q$.*

- (i) *If $q < p+1$ or $q = p+1, \sigma_{p+1}(F) < \sigma_p(A_0)$, then every solution $f(z)$ of* (1.2) *satisfies $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma$, with at most one exceptional solution f_0 satisfying $i(f) < p+1$ or $\sigma_{p+1}(f_0) < \sigma$.*
- (ii) *If $q > p+1$ or $q = p+1, \sigma_p(A_0) < \sigma_{p+1}(F) < +\infty$, then every solution $f(z)$ of* (1.2) *satisfies $i(f) = q$ and $\sigma_q(f) = \sigma_q(F)$.*

3. Preliminaries for proving the main results

To prove the above theorems, we need the following lemmas:

Lemma 3.1 ([5])**.** *Let $f(z)$ be a nontrivial entire function, and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then there exist a constant $c > 0$ and a set $E_1 \subset [0, \infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have*

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| \le c[T(\alpha r, f) r^\varepsilon \log T(\alpha r, f)]^k \quad (k \in \mathbb{N}). \tag{3.1}$$

Lemma 3.2 (Wiman-Valiron [6, 9])**.** *Let $f(z)$ be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then for all $|z|$ outside a set E_2 of r of finite logarithmic measure, we have*

$$\frac{f^{(k)}(z)}{f(z)} = \Big(\frac{\nu_f(r)}{z}\Big)^k (1 + o(1)) \quad (k \in \mathbb{N}, r \notin E_2). \tag{3.2}$$

where $\nu_f(r)$ is the central index of f.

Lemma 3.3 ([7]). *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be an entire function, $\mu(r)$ be the maximum term, i.e. $\mu(r)=\max\{|a_n|r^n; n=0,1,\dots\}$, and let $\nu_f(r)$ be the central index of f. Then*

(i) *For $|a_0|\neq 0$,*

$$\log\mu(r)=\log|a_0|+\int_0^r\frac{\nu_f(t)}{t}dt, \tag{3.3}$$

(ii) *For $r<R$,*

$$M(r,f)<\mu(r)\{\nu_f(R)+\frac{R}{R-r}\}. \tag{3.4}$$

Lemma 3.4. *Let $f(z)$ be an entire function with $\sigma_{p+1}(f)=\sigma$, and let $\nu_f(r)$ be the central index of f, then*

$$\limsup_{r\to\infty}\frac{\log_{p+1}\nu_f(r)}{\log r}=\sigma. \tag{3.5}$$

Proof. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$, without loss of generality, we can assume that $|a_0|\neq 0$. From (3.3), we have

$$\log\mu(2r)=\log|a_0|+\int_0^{2r}\frac{\nu_f(t)}{t}dt\geq\log|a_0|+\nu_f(r)\log 2. \tag{3.6}$$

Using the Cauchy inequality, it is easy to see that $\mu(2r)\leq M(2r,f)$. Hence

$$\nu_f(r)\log 2\leq\log M(2r,f)+c_1,$$

where $c_1>0$ is a constant. By (2.1) and (3),

$$\limsup_{r\to\infty}\frac{\log_{p+1}\nu_f(r)}{\log r}\leq\limsup_{r\to\infty}\frac{\log_{p+2}M(r,f)}{\log r}=\sigma. \tag{3.7}$$

On the other hand, from (3.4), we have

$$M(r,f)<\mu(r)\{\nu_f(2r)+2\}=|a_{\nu_f(r)}|r^{\nu_f(r)}\{\nu_f(2r)+2\}, \tag{3.8}$$

Since $\{|a_n|\}$ is a bounded sequence, we have

$$\log_{p+2}M(r,f)\leq\log_{p+1}\nu_f(2r)\big[1+\frac{\log_{p+2}\nu_f(2r)}{\log_{p+1}\nu_f(2r)}\big]+\log_{p+2}r+c_2, \tag{3.9}$$

where $c_2>0$ is a constant. Hence

$$\sigma=\limsup_{r\to\infty}\frac{\log_{p+2}M(r,f)}{\log r}\leq\limsup_{r\to\infty}\frac{\log_{p+1}\nu_f(2r)}{\log 2r}=\limsup_{r\to\infty}\frac{\log_{p+1}\nu_f(r)}{\log r}. \tag{3.10}$$

From (3.7) and (3.10), we obtain the conclusion (3.5). □

Lemma 3.5 ([8]). *Let $f(z)$ be an entire function with $i(f)=p+1$, then*

$$\sigma_{p+1}(f)=\sigma_{p+1}(f'). \tag{3.11}$$

Lemma 3.6. *Let $A_0(z),\dots,A_{k-1}(z)$ be entire functions, with , $F\not\equiv 0$ and let $f(z)$ be a solution of (1.2) satisfying one of the following conditions:*

(i) $\max\{i(F)=q, i(A_j)(j=0,\dots,k-1)\}<i(f)=p+1$ *($1\leq p<\infty$),*
(ii) $\max\{\sigma_p(F),\sigma_p(A_j)(j=0,\dots,k-1)\}<\sigma_{p+1}(f)=\sigma$.

Then $\overline{\lambda}_{p+1}(f)=\lambda_{p+1}(f)=\sigma_{p+1}(f)=\sigma$.

Proof. From (1.2), we have

$$\frac{1}{f}=\frac{1}{F}\Big(\frac{f^{(k)}}{f}+A_{k-1}\frac{f^{(k-1)}}{f}+\dots+A_0\Big), \tag{3.12}$$

it is easy to see that if f has a zero at z_0 of order $\alpha(>k)$, then F must have a zero at z_0 of order $\alpha-k$, hence

$$n(r,\frac{1}{f})\le k\overline{n}(r,\frac{1}{f})+n(r,\frac{1}{F}), \tag{3.13}$$

$$N(r,\frac{1}{f})\le k\overline{N}(r,\frac{1}{f})+N(r,\frac{1}{F}). \tag{3.14}$$

By (3.12), we have

$$m(r,\frac{1}{f})\le m(r,\frac{1}{F})+\sum_{j=0}^{k-1}m(r,A_j)+O\left(\log T(r,f)+\log r\right)(r\notin E_3), \tag{3.15}$$

where E_3 is a subset of r of finite linear measure. By (3.14) and (3.15), for $r\notin E_3$, we get

$$T(r,f)=T(r,\frac{1}{f})+O(1)\le k\overline{N}(r,\frac{1}{f})+T(r,F)+\sum_{j=0}^{k-1}T(r,A_j)+O\{\log(rT(r,f))\}. \tag{3.16}$$

For sufficiently large r, we have

$$O\{\log r+\log T(r,f)\}\le\frac{1}{2}T(r,f), \tag{3.17}$$

$$T(r,A_0)+\dots+T(r,A_{k-1})\le k\exp_{p-1}\{r^{\sigma+\varepsilon}\}, \tag{3.18}$$

$$T(r,F)\le\exp_{p-1}\{r^{\sigma(F)+\varepsilon}\}. \tag{3.19}$$

Thus, by (3.16)-(3.19), for $r\notin E_3$, we have

$$T(r,f)\le 2k\overline{N}(r,\frac{1}{f})+2k\exp_{p-1}\{r^{\sigma+\varepsilon}\}+2\exp_{p-1}\{r^{\sigma(F)+\varepsilon}\}. \tag{3.20}$$

Hence for any f with $\sigma_{p+1}(f)=\sigma$, by (3.20), we have $\sigma_{p+1}(f)\le\overline{\lambda}_{p+1}(f)$. Therefore, $\overline{\lambda}_{p+1}(f)=\lambda_{p+1}(f)=\sigma_{p+1}(f)=\sigma$. □

4. Proofs of theorems

Proof of Theorem 2.4. Let f be a solution of (1.1), and rewritten (1.1) as

$$A_0=-\Big(\frac{f^{(k)}}{f}+A_{k-1}\frac{f^{(k-1)}}{f}+\dots+A_1\frac{f'}{f}\Big). \tag{4.1}$$

By Lemma 3.1, there exist a constant $c>0$ and a set $E_1\subset[0,\infty)$ having finite linear measure such that $|z|=r\notin E_1$ for all $z=re^{i\theta}$. Then we have

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big|\le c[rT(2r,f)]^{2k},\quad j=1,\dots,k-1. \tag{4.2}$$

By (4.1), (4.2) and the hypothesis of Theorem 2.4, we get

$$\exp_p\{\alpha|z|^{\mu}\}\le|A_0(z)|\le k\exp_p\{\beta|z|^{\mu}\}c[rT(2r,f)]^{2k} \tag{4.3}$$

as $z\to\infty$ with $|z|=r\notin E_1,\theta_1\le\arg z=\theta\le\theta_2$. By (4.3) and (2.1), we have $\sigma_{p+1}(f)\ge\mu$. □

Proof of Theorem 2.5. From (1.1), it follows that

$$|A_0(z)| \le \Big|\frac{f^{(k)}(z)}{f(z)}\Big| + |A_{k-1}(z)|\Big|\frac{f^{(k-1)}(z)}{f(z)}\Big| + \dots + |A_1(z)|\Big|\frac{f'(z)}{f(z)}\Big|. \tag{4.4}$$

By the hypotheses of Theorem 2.5, there exists a set H with $\overline{\text{dens}}\{|z| : z \in H\} > 0$ such that for all z satisfying $z \in H$, we have

$$|A_0(z)| \ge \exp_p\{\alpha |z|^\mu\}, \tag{4.5}$$

$$|A_j(z)| \le \exp_p\{\beta |z|^\mu\}, \quad j = 1, \dots, k-1, \tag{4.6}$$

as $z \to \infty$. Hence from (4.2), (4.4)-(4.6), it follows that for all z satisfying $z \in H$ and $z \notin E_1$, we have

$$\exp_p\{\alpha |z|^\mu\} \le k \exp_p\{\beta |z|^\mu\} c[rT(2r, f)]^{2k} \tag{4.7}$$

as $z \to \infty$. Thus, there exists a set $H_1 = H \setminus E_1$ with $\overline{\text{dens}}\{|z| : z \in H_1\} > 0$ such that

$$\exp_p\{(\alpha - \beta)|z|^\mu\} \le kc[rT(2r, f)]^{2k} \tag{4.8}$$

as $z \to \infty$. Therefore, by (4.8) and Definition 2.1, we obtain $\sigma_{p+1}(f) \ge \mu$. □

Proof of Theorem 2.6. By Theorem 2.5, we have $\sigma_{p+1}(f) \ge \sigma - \varepsilon$, since ε is arbitrary, we get $\sigma_{p+1}(f) \ge \sigma_p(A_0) = \sigma$. On the other hand, by Lemma 3.2, there exists a set $E_2 \subset [1, \infty)$ having finite logarithmic measure such that (3.2) holds for all z satisfying $|z| = r \notin [0, 1] \bigcup E_2$ and $|f(z)| = M(r, f)$. By Definition 2.1, for any given $\varepsilon > 0$ and for sufficiently large r, we have

$$|A_j(z)| \le \exp_p\{r^{\sigma+\varepsilon}\}, \quad j = 0, 1, \dots, k-1. \tag{4.9}$$

Substituting (3.2) and (4.9) in (1.1), for all z satisfying $|z| = r \notin [0, 1] \bigcup E_2$ and $|f(z)| = M(r, f)$, we have

$$\Big(\frac{\nu_f(r)}{|z|}\Big)^k |1 + o(1)| \le k\Big(\frac{\nu_f(r)}{|z|}\Big)^{k-1} |1 + o(1)| \exp_p\{r^{\sigma+\varepsilon}\}. \tag{4.10}$$

By (4.10), we get

$$\limsup_{r\to\infty} \frac{\log_{p+1} \nu_f(r)}{\log r} \le \sigma + \varepsilon. \tag{4.11}$$

Since ε is arbitrary, by (4.11) and Lemma 3.4, we obtain $\sigma_{p+1}(f) \le \sigma$. This and the fact that $\sigma_{p+1}(f) \ge \sigma$ yield $\sigma_{p+1}(f) = \sigma$. □

Proof of Theorem 2.7. (i) First, we show that (1.2) can possess at most one exceptional solution f_0 satisfying $\sigma_{p+1}(f_0) \le \sigma$ or $i(f_0) < p + 1$. In fact, if f^* is a second solution with $\sigma_{p+1}(f^*) \le \sigma$ or $i(f^*) < p + 1$, then $\sigma_{p+1}(f_0 - f^*) \le \sigma$ or $i(f_0 - f^*) < p + 1$. But $f_0 - f^*$ is a solution of the corresponding homogeneous equation (1.1) of (1.2), this contradicts Theorem 2.6. We assume that f is a solution with $\sigma_{p+1}(f) \ge \sigma$, and $f_1, f_2, \dots, f_k$ is a solution base of the corresponding homogeneous equation (1.1). Then f can be expressed in the form

$$f(z) = B_1(z)f_1(z) + B_2(z)f_2(z) + \dots + B_k(z)f_k(z), \tag{4.12}$$

where $B_1(z), \dots, B_k(z)$ are determined by

$$\begin{aligned} B_1'(z)f_1(z) + B_2'(z)f_2(z) + \dots + B_k'(z)f_k(z) &= 0, \\ B_1'(z)f_1'(z) + B_2'(z)f_2'(z) + \dots + B_k'(z)f_k'(z) &= 0, \\ &\vdots \\ B_1'(z)f_1^{(k-1)}(z) + B_2'(z)f_2^{(k-1)}(z) + \dots + B_k'(z)f_k^{(k-1)}(z) &= F(z). \end{aligned} \tag{4.13}$$

Since the Wronskian $W(f_1, f_2, \dots, f_k)$ is a differential polynomial in $f_1, f_2, \dots, f_k$ with constant coefficients, it is easy to deduce that $\sigma_{p+1}(W) \le \sigma_{p+1}(f_j) = \sigma_p(A_0) = \sigma$. From (4.13),

$$B_j' = F \cdot G_j(f_1, \dots, f_k) \cdot W(f_1, \dots, f_k)^{-1}, \quad j = 1, \dots, k, \tag{4.14}$$

where $G_j(f_1, \dots, f_k)$ are differential polynomials in $f_1, f_2, \dots, f_k$ with constant coefficients, thus

$$\sigma_{p+1}(G_j) \le \sigma_{p+1}(f_j) = \sigma_p(A_0) = \sigma. \tag{4.15}$$

Since $i(F) < p+1$ or $i(F) = p+1, \sigma_{p+1}(F) < \sigma_p(A_0)$, by Lemma 3.5 and (4.15), for $j = 1, \dots, k$, we have

$$\sigma_{p+1}(B_j) = \sigma_{p+1}(B_j') \le \max\{\sigma_{p+1}(F), \sigma_p(A_0)\} = \sigma_p(A_0) = \sigma\,. \tag{4.16}$$

Then from (4.12) and (4.16), we get

$$\sigma_{p+1}(f) \le \max\{\sigma_{p+1}(f_j), \sigma_{p+1}(B_j)\} = \sigma_p(A_0) = \sigma. \tag{4.17}$$

This and the assumption $\sigma_{p+1}(f) \ge \sigma$ yield $\sigma_{p+1}(f) = \sigma$. If f is a solution of equation (1.2) satisfying $\sigma_{p+1}(f) = \sigma$, by Lemma 3.6, we have

$$\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma.$$

(ii) From the hypotheses of Theorem 2.7 and (4.12)-(4.17), we obtain

$$\sigma_q(f) \le \sigma_q(F). \tag{4.18}$$

From (1.2), a simple consideration of order implies

$$\sigma_q(f) \ge \sigma_q(F).$$

By this inequality and (4.18), $\sigma_q(f) = \sigma_q(F)$ which completes the proof. □

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