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A QUASISTATIC BILATERAL CONTACT PROBLEM WITH FRICTION FOR NONLINEAR ELASTIC MATERIALS

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Abstract. We consider a mathematical model describing the bilateral contact between a deformable body and a foundation. We use a nonlinear elastic constitutive law. The contact takes into account the effects of friction, which are modelled with the regularized friction law. We derive a variational formulation of the problem and establish the existence of a weak solution under a smallness assumption of the friction coefficient. The proof is based on arguments of compactness, lower semicontinuity and time discretization.

1. Introduction

In this paper we study the existence of a solution for a quasistatic bilateral contact problem with friction for nonlinear elastic materials. For linear elastic materials the quasistatic contact problem using a normal compliance law has been solved in [1] by considering incremental problems and in [9] by an other method using a regularization relative to time. The quasistatic contact problem with local or nonlocal friction has been solved respectively in [10] and in [4] by using a time-discretization. The same method was also used in [13] to solve the quasistatic problem with unilateral contact involving nonlocal friction law for nonlinear elastic materials. In [2] the quasistatic contact problem with Coulomb friction was solved by the aid of an established shifting technique used to obtain increased regularity at the contact surface and by the aid of auxiliary problems involving regularized friction terms and a so-called normal compliance penalization technique. Signorini 's problem with friction for nonlinear elastic materials or viscoplastic materials has been solved in [5] by using the fixed point's method. In viscoelasticity, the quasistatic contact problem with normal compliance and friction has been solved in [11] for nonlinear viscoelastic materials by the same fixed point arguments. In the book [8] the authors resolve the quasistatic contact problems in viscoelasticity and viscoplasticity. Carrying out the variational analysis, the authors systymatically use results on elliptic and evolutionary variational inequalities, convex analysis, nonlinear equations with monotone operators, and fixed points of operators.

Here we propose a variational formulation using a regularization of the normal stress. We model the friction by Tresca's law, by nonlocal law as in [13] and

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by a modified version of Coulomb's law which has been derived in [12] to take into account the wear of the contacting surface. The variational formulation is written in the form of a single variational inequality. By means of Euler's implicit scheme as in [4, 10, 13], the bilateral contact problem leads us to solve a well-posed variatonal inequality at each time step. Finally by using lower semicontinuity and compactness arguments we prove that the limit of the discrete solution is a solution to the continuous problem.

2. Problem statement and variational formulation

We consider a nonlinear elastic body which is in frictional contact with a rigid foundation. Time dependent volume forces and surface traction act on it, and as result there is evolution of its mechanical state. Our interest is in modelling this evolution. We assume that the forces and traction vary slowly with time and therefore the accelerations in the system are negligible. Also, we assume that there is no loss of contact between the body and the foundation.

The physical setting is as follows. Let $\Omega \subset \mathbf{R}^d$; $(d = 2, 3)$, be the domain initially occupied by the nonlinear elastic body. Ω is supposed to be open, bounded, with a sufficiently regular boundary Γ. Γ is partitioned into three parts $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open sets and $\operatorname{meas} \Gamma_1 > 0$. The body is acted upon by volume forces of density ϕ_1 on Ω and surface traction of density ϕ_2 on Γ_2. On Γ_3 the body is in bilateral contact with a rigid foundation.

The classical formulation of the mechanical problem is written as follows.

Problem P1. Find a displacement field $u : \Omega \times [0, T] \to \mathbf{R}^d$ such that

$$\operatorname{div} \sigma + \phi_1 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\sigma = F(\varepsilon(u)) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.3}$$

$$\sigma\nu = \phi_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.4}$$

$$\left.\begin{aligned} u_\nu = 0, \quad |\sigma_\tau| \le \mu p(|R\sigma_\nu(u)|) \\ |\sigma_\tau| < \mu p(|R\sigma_\nu(u)|) \Longrightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| = \mu p(|R\sigma_\nu(u)|) \Longrightarrow \exists \lambda \ge 0 : \sigma_\tau = -\lambda \dot{u}_\tau \end{aligned}\right\} \text{on } \Gamma_3 \times (0, T), \tag{2.5}$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{2.6}$$

Equality (2.1) represents the equilibrium equation. Equality (2.2) represents the elastic constitutive law of the material in which F is a given function and $\varepsilon(u)$ denotes the small strain tensor; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal on Γ and $\sigma\nu$ represents the Cauchy stress vector. Conditions (2.5) represents the bilateral contact boundary conditions and the associate friction law in which σ_τ denotes the tangential stress, $\dot{u}_\tau$ denotes the tangential velocity on the boundary and μ is the coefficient of friction. Finally (2.6) represent the initial condition. In (2.5) and below, a dot above a variable represents its derivative with respect to time. We denote by S_d the space of second order symmetric tensors on $\mathbf{R}^d$ $(d = 2, 3)$. To proceed with the variational formulation, we need the following function spaces:

$$H = L^2(\Omega)^d, H_1 = (H^1(\Omega))^d,$$
$$Q = \{\tau = (\tau_{ij}); \tau_{ij} = \tau_{ji} \in L^2(\Omega)\},$$
$$H(\mathrm{div}; \Omega) = \{\sigma \in Q; \operatorname{div} \sigma \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$(u, v)_H = \int_\Omega u_i v_i dx, \quad \langle \sigma, \tau \rangle_Q = \int_\Omega \sigma_{ij} \tau_{ij} dx.$$

The small strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \big(\frac{1}{2}(u_{i,j} + u_{j,i})\big), \quad i, j \in \{1, \dots, d\};$$

$\operatorname{div} \sigma = (\sigma_{ij,j})$ is the divergence of σ. Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $v \in H_1$, we also use the notation v for the trace γv of v on Γ and we denote by v_ν and v_τ the normal and tangential components of v on Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

Let H'_Γ be the dual of H_Γ, for every $\sigma \in H(\operatorname{div} \Omega)$, $\sigma\nu$ can be defined as the element in H'_Γ which satisfies the Green's formula:

$$\langle \sigma, \varepsilon(v) \rangle_Q + (\operatorname{div} \sigma, v)_H = \langle \sigma\nu, v \rangle_{H'_\Gamma \times H_\Gamma} \quad \forall v \in H_1.$$

Denote by σ_ν and σ_τ the normal and tangential traces of σ, respectively. If σ is regular (say C^1), then

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma - \sigma_\nu \nu$$
$$\langle \sigma\nu, v \rangle_{H'_\Gamma \times H_\Gamma} = \int_\Gamma \sigma\nu.v \, da$$

for all $v \in H_1$, where da is the surface measure element. Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1, \ v_\nu = 0 \text{ on } \Gamma_3\}.$$

Since $\operatorname{meas} \Gamma_1 > 0$, the following Korn's inequality holds [7],

$$\|\varepsilon(v)\|_Q \ge c_\Omega \|v\|_{H_1} \quad \forall v \in V \tag{2.7}$$

where the constant c_Ω depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v) \rangle_Q$$

and $\|\cdot\|_V$ is the associated norm. It follows from Korn's inequality (2.7) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V. Then $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover by the Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \le d_\Omega \|v\|_V \quad \forall v \in V \tag{2.8}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we use the notation $C([0,T];X)$ for the space of continuous functions from $[0,T]$ to X; recall that $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

We suppose

$$\phi_1 \in W^{1,\infty}(0,T;H), \quad \phi_2 \in W^{1,\infty}(0,T;L^2(\Gamma_2)^d) \tag{2.9}$$

and we denote by $f(t)$ the element of V defined by

$$(f(t), v)_V = \int_\Omega \phi_1(t).v\,dx + \int_{\Gamma_2} \phi_2(t).v\,da \quad \forall v \in V, \text{ for } t \in [0,T] \tag{2.10}$$

Using (2.9) and (2.10) yields $f \in W^{1,\infty}(0,T;V)$. $\langle .,. \rangle$ shall denote the duality pairing on $H^{1/2}(\Gamma_3) \times H^{-1/2}(\Gamma_3)$, where

$$H^{1/2}(\Gamma_3) = \{w\big|_{\Gamma_3} : w \in H^{1/2}(\Gamma), w = 0 \text{ on } \Gamma_1\}.$$

The normal stress $\sigma_\nu(u(t)) \in H^{-1/2}(\Gamma_3)$ associated with a function $u(t) \in V$ is defined by

$$\begin{gathered} \forall\ w \in H^{1/2}(\Gamma_3) : \\ \langle \sigma_\nu, w\rangle = \langle F(\varepsilon(u(t))), \varepsilon(v)\rangle_Q - (f(t), v)_V \\ \forall\ v \in H_1 : v = 0 \text{ on } \Gamma_1 \text{ and } v_\nu = w, \quad v_\tau = 0 \quad \text{on } \Gamma_3. \end{gathered} \tag{2.11}$$

$R : H^{-1/2}(\Gamma_3) \to L^2(\Gamma_3)$ is a continuous regularizing operator representing the averaging of the normal stress over a small neighborhood of the contact point. In the case where p is a known function which is independent of σ_ν, i.e., $p(r) = g$, the friction law involved in (2.5) becomes the Tresca friction law, and $H = \mu g$ is the friction bound. By choosing $p(r) = r$ in (2.5), we recover the usual regularized Coulomb friction law used in the literature. The choice $p(r) = r_+(1-\delta r)_+$, where δ is a small positive coefficient related to the wear and hardness of the surface, was employed in [12]. We assume that $R : H^{-1/2}(\Gamma_3) \to L^2(\Gamma_3)$ is a linear compact mapping. The assumptions on the friction function p are:

$$\begin{aligned} &(a) p : \Gamma_3 \times \mathbf{R} \to \mathbf{R}_+ \\ &(b) \text{There exists } L_p > 0 \text{ such that } |p(x,u_1) - p(x,u_2)| \le L_p|u_1 - u_2| \\ &\quad \text{for all } u_1, u_2 \in \mathbf{R}, \text{ a.e. } x \in \Gamma_3. \\ &(c) \text{For each } u \in \mathbf{R}, x \to p(x,u) \text{ is measurable on } \Gamma_3. \\ &(d) \text{The mapping } x \to p(x,0) \in L^2(\Gamma_3). \end{aligned} \tag{2.12}$$

We observe that the above assumptions on p are quite general. Clearly, the functions $p(r) = g$, $p(r) = r$, $p(r) = r_+(1-\delta r)_+$ satisfy these conditions, when g is a known function and δ is a given positive constant. So, the results presented below hold true for the boundary value problems with each one of these tangential functions.

Hypotheses on the nonlinear elasticity operator. As in [13] we assume $F : \Omega \times S_d \to S_d$ satisfies the following conditions:

$$
\begin{aligned}
&(a)\text{There exists } L_1 > 0 \text{ such that } |F(x,\varepsilon_1) - F(x,\varepsilon_2)| \le L_1|\varepsilon_1 - \varepsilon_2|,\\
&\quad \text{for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d, \text{ a.e. } x \text{ in } \Omega.\\
&(b)\text{There exists } L_2 > 0 \text{ such that } (F(x,\varepsilon_1) - F(x,\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \ge L_2|\varepsilon_1 - \varepsilon_2|^2,\\
&\quad \text{for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d, \text{ a.e. } x \text{ in } \Omega.\\
&(c)\text{The mapping } x \to F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega,\\
&\quad \text{for any } \varepsilon \text{ in } S_d.\\
&(d)F(x,0) = 0 \text{ for all } x \text{ in } \Omega.
\end{aligned}
\tag{2.13}
$$

Remark 2.1. $F(x,\tau(x)) \in Q$, for all $\tau \in Q$ and thus it is possible to consider F as an operator defined from Q to Q.

We assume that the friction coefficient satisfies

$$
\mu \in L^\infty(\Gamma_3) \text{ and } \mu \ge 0 \text{ a.e. on } \Gamma_3. \tag{2.14}
$$

Also we assume that the initial data $u_0 \in V$ satisfies

$$
\langle F(\varepsilon(u_0)), \varepsilon(v)\rangle_Q + j(u_0, v) \ge (f(0), v)_V \quad \forall v \in V. \tag{2.15}
$$

Now by assuming the solution to be sufficiently regular, we obtain by using Green's formula and techniques similar to those exposed in [7] that the problem (P_1) has the following variational formulation.

Problem P2. Find a displacement field $u \in W^{1,\infty}(0,T;V)$ such that $u(0) = u_0$ in Ω and for almost all $t \in [0,T]$:

$$
\begin{aligned}
&\langle F(\varepsilon(u(t))), \varepsilon(v) - \varepsilon(\dot u(t))\rangle_Q + j(u(t), v) - j(u(t), \dot u(t))\\
&\ge (f(t), v - \dot u(t))_V \quad \forall v \in V.
\end{aligned}
\tag{2.16}
$$

where

$$
j(u,v) = \int_{\Gamma_3} \mu|R\sigma_\nu(u)||v_\tau|da.
$$

Our main result of this section, which will be established in the next is the following theorem.

Theorem 2.2. *Let $T > 0$ and assume that* (2.9), (2.12), (2.13), (2.14), *and* (2.15) *hold. Then problem (P_2) has at least a solution for a small enough friction coefficient μ. Moreover, there exists a constant $C > 0$ such that*

$$
\|u\|_{W^{1,\infty}(0,T;V)} \le C\|f\|_{W^{1,\infty}(0,T;V)}.
$$

3. Existence of solutions

This evolution problem can be integrated in time by an implicit scheme as in [13]. Let $0 = t_0 < t_1 < \dots < t_n = T$ be a uniform partition of the time interval $[0,T]$, i.e., $t_i = i\Delta t$, $i = 0,1,\dots,n$, $\Delta t = \frac{T}{n}$ the step-size. We denote by u^i the approximation of u at the time t_i and by the symbol Δu^i the backward difference $u^{i+1} - u^i$. We obtain a sequence of incremental problems, for $u^0 \in V$, defined as

Problem (P_n^i)**.** Find $u^{i+1}\in V$ such that

$$\begin{aligned}
&\langle F(\varepsilon(u^{i+1})),\varepsilon(w)-\varepsilon(u^{i+1})\rangle_Q + j(u^{i+1},w-u^i) - j(u^{i+1},u^{i+1}-u^i)\\
&\geq (f^{i+1},w-u^{i+1})_V \quad \forall w\in V
\end{aligned}\tag{3.1}$$

where $u^0=u_0$, $f^{i+1}=f(t_{i+1})$.

Lemma 3.1. *There exists a positive constant $\mu_1>0$ such that for $\|\mu\|_{L^\infty(\Gamma_3)}<\mu_1$, problem (P_n^i) admits a unique solution.*

For the proof of the above lemma, see [13] in the case of unilateral contact.

Lemma 3.2. *We have the following estimates: For a positive constant $\mu_2>0$, when $\|\mu\|_{L^\infty(\Gamma_3)}<\mu_2$, there exists $d_i>0$, $i=1,2$, such that*

$$\|u^{i+1}\|_V \leq d_1\|f^{i+1}\|_V \tag{3.2}$$

$$\|\Delta u^i\|_V \leq d_2\|\Delta f^i\|_V \tag{3.3}$$

Proof. By setting $w=0$ in the inequality (3.1) and by using hypothesis (2.13)(b), hypothesis (2.12)(b), (2.11) and the properties of j, there exists $c_1>0$ such that for $\|\mu\|_{L^\infty(\Gamma_3)}<c_1$, we deduce that there exists $d_1>0$ such that (3.2) is satisfied.

To show the inequality (3.3) we consider the translated inequality of (3.1) at the time t_i that is

$$\begin{aligned}
&\langle F(\varepsilon(u^i)),\varepsilon(w)-\varepsilon(u^i)\rangle_Q + j(u^i,v-u^{i-1}) - j(u^i,u^i-u^{i-1})\\
&\geq (f^i,w-u^i)_V, \quad \forall w\in V
\end{aligned}\tag{3.4}$$

By setting $w=u^i$ in (3.1) and $w=u^{i+1}$ in (3.4) and add them up, we obtain the inequality

$$\begin{aligned}
&-\langle F(\varepsilon(u^{i+1}))-F(\varepsilon(u^i)),\varepsilon(\Delta u^i)\rangle_Q - j(u^{i+1},\Delta u^i)\\
&+j(u^i,u^{i+1}-u^{i-1}) - j(u^i,u^i-u^{i-1})\\
&\geq (-\Delta f^i,\Delta u^i)_V\,.
\end{aligned}$$

Furthermore, using the inequality

$$||u_\tau^{i+1}-u_\tau^{i-1}|-|u_\tau^i-u_\tau^{i-1}|| \leq |u_\tau^{i+1}-u_\tau^i|$$

we have

$$j(u^i,u^{i+1}-u^{i-1}) - j(u^i,u^i-u^{i-1}) \leq j(u^i,\Delta u^i).$$

Therefore,

$$-\langle F(\varepsilon(u^{i+1}))-F(\varepsilon(u^i)),\varepsilon(\Delta u^i)\rangle_Q + j(u^i,\Delta u^i) - j(u^{i+1},\Delta u^i) \geq (-\Delta f^i,\Delta u^i)_V$$

Using hypothesis (2.12)(b), (2.11), hypothesis (2.13) (b) and the properties of j, we deduce that there exists two positive constants d_3 and d_4 such that

$$L_2\|\Delta u^i\|_V^2 \leq d_3\|\mu\|_{L^\infty(\Gamma_3)}\|\Delta u^i\|_V^2 + d_4\|\Delta f^i\|_V\|\Delta u^i\|_V.$$

Then we deduce that there exists a constant $c_2>0$ such that if $\|\mu\|_{L^\infty(\Gamma_3)}<c_2$, there exists $d_2>0$ such that

$$\|\Delta u^i\|_V \leq d_2\|\Delta f^i\|_V$$

It suffices to take $\mu_2=\min(c_1,c_2)$ and the lemma is proved. $\square$

The proof of Theorem 2.2 is done as in [4], but in L^∞. For the next lemma, we define the continuous function u^n on $[0,T] \to V$ by

$$u^n(t) = u^i + \frac{(t-t_i)}{\Delta t}\Delta u^i \quad \text{on } [t_i, t_{i+1}], \quad i = 0, \dots, n-1.$$

Lemma 3.3. *There exists a function $u \in W^{1,\infty}(0,T;V)$ such that passing to a subsequence still denoted (u^n) we have*

$$u^n \to u \ \textit{weak} * \textit{in} W^{1,\infty}(0,T;V).$$

Proof. From (3.2), the sequence (u^n) is bounded in $C([0,T];V)$ and there exists a constant $c_3 > 0$ such that

$$\max_{0 \le t \le T} \|u^n(t)\|_V \le c_3 \|f\|_{C([0,T];V)}\,.$$

From (3.3), the sequence $(\dot{u}^n)$ is bounded in $L^\infty(0,T;V)$ and there exists $c_4 > 0$ such that

$$\|\dot{u}^n\|_{L^\infty(0,T;V)} = \max_{0 \le i \le n-1} \|\frac{\Delta u^{t_i}}{\Delta t}\|_V \le c_4 \|\dot{f}\|_{L^\infty(0,T;V)}\,.$$

Consequently, the sequence (u^n) is bounded in $W^{1,\infty}(0,T;V)$, and thus we can extract a subsequence still denoted (u^n) such that $u^n \to u$ in $W^{1,\infty}(0,T;V)$ weak $*$ as $n \to \infty$, and satisfying

$$\|u\|_{W^{1,\infty}(0,T;V)} \le C\|f\|_{W^{1,\infty}(0,T.V)}.$$

$\square$

As in [11] let's introduce the functions $\widetilde{u}^n : [0,T] \to V$, $\widetilde{f}^n : [0,T] \to V$ defined by

$$\widetilde{u}^n(t) = u^{i+1} = u(t_{i+1}), \quad \widetilde{f}^n(t) = f(t_{i+1}) \quad \forall t \in (t_i, t_{i+1}], i = 0, \dots, n-1$$

As in [13] we have the following result.

Lemma 3.4. *There exists a subsequence still denoted by $(\tilde{u}^n)$ such that*

(i) $\widetilde{u}^n \to u$ *weak $*$ in* $L^\infty(0,T;V)$
(ii) $\widetilde{u}^n(t) \to u(t)$ *weakly in V a.e.* $t \in [0,T]$

Remark 3.5. Since $f \in W^{1,\infty}(0,T;V)$, $u \in W^{1,\infty}(0,T;V)$, we have

$$\widetilde{f}^n \to f \quad \text{strongly in } L^2(0,T;V) \tag{3.5}$$

$$\widetilde{u}^n \to u \quad \text{strongly in } L^2(0,T;V) \tag{3.6}$$

To prove that u is a solution of the problem, in the inequality of problem (P_n^i), for $v \in V$, we set $w = u^i + v\Delta t$ and divide by Δt. We obtain

$$\begin{aligned}&\langle F(\varepsilon(u^{i+1})), \varepsilon(v) - \varepsilon(\frac{\Delta u^i}{\Delta t})\rangle_Q + j(u^{i+1}, v) - j(u^{i+1}, \frac{\Delta u^i}{\Delta t})\\ &\ge \Big(f(t_{i+1}), v - \frac{\Delta u^i}{\Delta t}\Big)_V \quad \forall v \in V\end{aligned}$$

Whence for any $v \in L^2(0,T;V)$, we have

$$\begin{aligned}&\langle F(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t))\rangle_Q + j(\widetilde{u}^n(t), v(t)) - j(\widetilde{u}^n(t), \dot{u}^n(t))\\ &\ge \Big(\widetilde{f}^n(t), v(t) - \dot{u}^n(t)\Big)_V\,.\end{aligned}$$

Integrating both sides of the previous inequality on $(0,T)$, we obtain

$$\begin{aligned} &\int_0^T \langle F(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t))\rangle_Q dt \\ &+ \int_0^T j(\widetilde{u}^n(t), v(t))dt - \int_0^T j(\widetilde{u}^n(t), \dot{u}^n(t))dt \\ &\geq \int_0^T (\widetilde{f}^n(t), v - \dot{u}^n(t))_V dt \end{aligned} \tag{3.7}$$

Lemma 3.6. *For every $v \in L^2(0,T;V)$ we have following properties*

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T \langle F(\varepsilon(\widetilde{u}^n(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}^n(t))\rangle_Q dt \\ &= \int_0^T \langle F(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}(t))\rangle_Q dt, \end{aligned} \tag{3.8}$$

$$\lim_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), v(t))dt = \int_0^T j(u(t), v(t))dt, \tag{3.9}$$

$$\liminf_{n\to\infty} \int_0^T j(\widetilde{u}^n(t), \dot{u}^n(t))dt \geq \int_0^T j(u(t), \dot{u}(t))dt, \tag{3.10}$$

$$\lim_{n\to\infty} \int_0^T (\widetilde{f}^n(t), v(t) - \dot{u}^n(t))_V dt = \int_0^T (f(t), v(t) - \dot{u}(t))_V dt\,. \tag{3.11}$$

Proof. For (3.8), see [13]. For (3.9) it suffices to use (3.6). To prove (3.10), we write

$$j(\widetilde{u}^n(t), \dot{u}^n(t)) = (j(\widetilde{u}^n(t), \dot{u}^n(t)) - j(u(t), \dot{u}^n(t))) + j(u(t), \dot{u}^n(t))$$

then by (2.12)(b), we have

$$\begin{aligned} &\Big|\int_0^T (j(\widetilde{u}^n(t), \dot{u}^n(t) - j(u(t), \dot{u}^n(t))))dt\Big| \\ &\leq L_p \|\mu\|_{L^\infty(\Gamma_3)} \|R(\sigma_\nu(\widetilde{u}^n) - \sigma_\nu(u))\|_{L^2(0,T;L^2(\Gamma_3))} \|\dot{u}_\tau^n\|_{L^2(0,T;L^2(\Gamma_3)^d)}\,. \end{aligned}$$

Since the mapping R is compact, we have

$$\lim_{n\to\infty} \|R(\sigma_\nu(\widetilde{u}^n) - \sigma_\nu(u))\|_{L^2(0,T:L^2(\Gamma_3))} = 0\,,$$

$$\liminf_{n\to\infty} \int_0^T j(u(t), \dot{u}^n(t))dt \geq \int_0^T j(u(t), \dot{u}(t))dt$$

by Mazur's lemma. For proving (3.11) it suffices to use (3.5). Passing to the limit in inequality (3.7), we obtain

$$\begin{aligned} &\int_0^T \langle F(\varepsilon(u(t))), \varepsilon(v(t)) - \varepsilon(\dot{u}(t))\rangle_Q dt + \int_0^T j(u(t), v(t))dt - \int_0^T (j(u(t), \dot{u}(t)))dt \\ &\geq \int_0^T (f(t), v(t) - \dot{u}(t))_V dt\,. \end{aligned}$$

In this inequality we set

$$v(s) = \begin{cases} z & \text{for } s \in (t, t+\lambda) \\ \dot{u}(s) & \text{elsewhere.} \end{cases}$$

Then we obtain

$$\begin{aligned}&\frac{1}{\lambda}\int_t^{t+\lambda}(F\langle(\varepsilon(u(s))),\varepsilon(z)-\varepsilon(\dot{u}(s))\rangle_Q+j(u(s),z)-j(u(s),\dot{u}(s)))ds\\&\geq\frac{1}{\lambda}\int_t^{t+\lambda}(f(s),z-\dot{u}(s))_V ds\,.\end{aligned}$$

Passing to the limit, one obtains that u satisfies (2.16). □

Conclusion. In this article we have obtained the existence of a weak solution of the quasistatic bilateral contact problem for nonlinear elastic materials under a smallness assumption of the friction coefficient. The uniqueness of solution represents, as far as we know, an open question.

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