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WELL-POSEDNESS OF ONE-DIMENSIONAL KORTEWEG MODELS

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Abstract. We investigate the initial-value problem for one-dimensional compressible fluids endowed with internal capillarity. We focus on the isothermal inviscid case with variable capillarity. The resulting equations for the density and the velocity, consisting of the mass conservation law and the momentum conservation with Korteweg stress, are a system of third order nonlinear dispersive partial differential equations. Additionally, this system is Hamiltonian and admits travelling solutions, representing propagating phase boundaries with internal structure. By change of unknown, it roughly reduces to a quasilinear Schrödinger equation. This new formulation enables us to prove local well-posedness for smooth perturbations of travelling profiles and almost-global existence for small enough perturbations. A blow-up criterion is also derived.

Introduction

We are concerned with compressible fluids endowed with internal capillarity. The models we consider are originated from the XIXth century work by van der Waals [25] and Korteweg [21] and were actually derived in their modern form in the 1980s using the second gradient theory, see for instance [26, 16]. They result in dispersive systems of Partial Differential Equations. In fact, special cases of these models also arise in other contexts, *e.g.* quantum mechanics. Our main motivation is about fluids though, especially liquid-vapor mixtures with phase changes. Indeed, Korteweg models allow phase "boundaries" of nonzero thickness that are often called *diffuse interfaces* – by contrast with sharp interfaces in the Laplace-Young's theory. The interest for diffuse interfaces has been renewed in the late 1990s also for numerical purposes, see [1] for a nice review.

The mathematical analysis of Korteweg models is rather recent. One may quote only a few papers [9, 12, 15], in which nonzero viscosity and its regularizing effects play a fundamental role. One should also quote the recent work of Li and Marcati [22], which concerns a similar model in QHD (Quantum HydroDynamics), with weaker dissipation – due to relaxation –included. Here, we concentrate on purely dispersive models, which are still physically meaningful. Although very different from dissipative smoothing, dispersive smoothing is known to exist for various equations, see for instance the seminal work by Kato [19] on the Korteweg-de

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Vries equation, [11, 17] on generalized Schrödinger equations, etc. See also the work by Bedjaoui and Sainsaulieu on a dispersive two-phase flow model [3]. However, up to our knowledge, no result of this kind is known on the most general models we are interested in. This is a direction of research in progress, in connection with the recent work of Kenig, Ponce and Vega [20]. The positive counterpart of neglecting dissipation phenomena is that like most other purely dispersive models (*e.g.* Korteweg-de Vries, Boussinesq, standard Schrödinger equations), the models we consider can be viewed, to some extent, as Hamiltonian systems. The Hamiltonian structure of non-dissipative Korteweg models has been discussed in a companion paper [5].

We address here the local-in-time well-posedness of the non-dissipative Korteweg models. On the one hand, for monotone pressure laws, it has been pointed out by Gavrilyuk and Gouin [14] that these models admit a symmetric form, in the classical sense of hyperbolic systems of conservation laws, at least for their first order part and with non-dissipative – in a sense that we make precise below – higher order terms. Even though this kind of systems enjoy natural L^2 estimates, it is not clear how to show their well-posedness (this is done in [3] by an artificial viscosity method). On the other hand, when concentrating on models with *constant capillarity*, Korteweg models – at least some of them – can be dealt with by Kato's theory of abstract evolution equations [18], disregarding the monotonicity of the pressure. For more general capillarities though, because of the nonlinearity in higher order terms, they are *not* amenable to Kato's theory.

Restricting to one space dimension, where we can use Lagrangian coordinates, we have been able to deal with both a nonmonotone pressure and a nonconstant capillarity. For this, we have introduced an additional unknown that gives rise to a system coupling a transport equation with a variable coefficients Schrödinger equation. Taking advantage of symmetry properties of this system, inspired from a work by Coquel on the numerics of Korteweg models, and introducing suitable gauge functions, inspired from a work by Lim and Ponce [23], we obtain higher order energy estimates *without loss of derivatives*, and eventually prove the local-in-time well-posedness in Sobolev spaces of the one dimensional Korteweg models. As a matter of fact, our main theorem is slightly more general since it also states existence for data pertaining to H^k perturbations of any smooth reference solution whose derivatives have sufficient decay at infinity. Our main motivation for proving this is to investigate the stability of travelling wave solutions for the one-dimensional model. Indeed, typical travelling solutions fail to belong to Sobolev spaces since they have different endstates at $-\infty$ and $+\infty$ (see our companion paper [5] for more details).

We finally derive a blow-up criterion involving the Lipschitz norm of the solution and get a lower bound for the existence time in terms of the norm of the data which entails almost global existence for small perturbations of a global solution (e.g a capillary profile).

1. Derivation of Korteweg models

1.1. **A general model in Eulerian coordinates.** Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but on the gradient of the density. In terms of the free energy for instance, this principle takes the form of a generalized Gibbs relation

$$\mathrm{d}F = -S\,\mathrm{d}T + g\,\mathrm{d}\rho + \phi^* \cdot \mathrm{d}\mathbf{w}\,,$$

where F denotes the free energy per unit volume, S the entropy per unit volume[1], T the temperature, g the chemical potential and, in the additional term, $\mathbf{w}$ stands for $\nabla\rho$. The potential ϕ is most often assumed of the form

$$\phi = K\mathbf{w},$$

where K is called the capillarity coefficient, which may depend on both ρ and T. In this case, F decomposes into a standard part F_0 and an additional term due to gradients of density,

$$F(\rho, T, \nabla\rho) = F_0(\rho, T) + \frac{1}{2}K(\rho, T)\|\nabla\rho\|^2\,,$$

and similar decompositions hold for S and g. We shall use this special form in our subsequent analysis. For the moment we keep the abstract potential ϕ and we define the Korteweg tensor as

$$\mathbf{K} := (\rho\,\mathrm{div}\,\phi)\mathbf{I} - \phi\mathbf{w}^*\,.$$

Neglecting dissipation phenomena, the conservation of mass, momentum and energy read

$$\begin{gathered}
\partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0\,,\\
\partial_t(\rho\mathbf{u}^*) + \mathrm{div}(\rho\mathbf{u}\mathbf{u}^* + p\mathbf{I}) = \mathrm{div}\,\mathbf{K}\,,\\
\partial_t(E + \tfrac{1}{2}\rho|\mathbf{u}|^2) + \mathrm{div}((E + \tfrac{1}{2}\rho|\mathbf{u}|^2 + p)\mathbf{u}) = \mathrm{div}(\mathbf{K}\mathbf{u} + \mathbf{W})\,,
\end{gathered}$$

where $p = \rho g - F$ is the (extended) pressure, $E = F + TS$ is the internal energy per unit volume, and

$$\mathbf{W} := (\partial_t\rho + \mathbf{u}^* \cdot \nabla\rho)\phi = -(\rho\,\mathrm{div}\,\mathbf{u})\phi$$

is the interstitial working that was first introduced by Dunn and Serrin [13]. This supplementary term ensures that the entropy S satisfies the conservation law

$$\partial_t S + \mathrm{div}(S\mathbf{u}) = 0\,.$$

(This is obtained through formal computation, for presumably smooth solutions.) There is also an alternate form of the momentum equation (still for smooth solutions). Using the mass conservation law and the relation

$$\mathrm{d}g = -s\,\mathrm{d}T + v\,\mathrm{d}p + v\phi^* \cdot \mathrm{d}\mathbf{w}\,,$$

with s the specific entropy and $v := 1/\rho$ the specific volume, we arrive at the equation

$$\partial_t\mathbf{u} + (\mathbf{u}^* \cdot \nabla)\mathbf{u} = \nabla(\mathrm{div}\,\phi - g) - s\nabla T\,.$$

The resulting evolution system for $(\rho, \mathbf{u}, S)$ is

$$\begin{gathered}
\partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0\,,\\
\partial_t\mathbf{u} + (\mathbf{u}^* \cdot \nabla)\mathbf{u} = \nabla(\mathrm{div}\,\phi - g) - s\nabla T\,,\\
\partial_t S + \mathrm{div}(S\mathbf{u}) = 0\,.
\end{gathered} \tag{1.1}$$

[1]By convention, extensive quantities per unit volume are denoted by upper case letters and their specific counterparts will be denoted by the same, lower case, letters.

The "Hamiltonian structure" of (1.1) is discussed in [5]. In particular, in the isothermal case this system reduces to

$$\begin{gathered}\partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0\,,\\ \partial_t\mathbf{u} + (\mathbf{u}^*\cdot\nabla)\mathbf{u} = \nabla(\operatorname{div}\phi - g)\,.\end{gathered}\tag{1.2}$$

We shall see that for proving well-posedness, $\mathbf{w}$, or a similar quantity, must be considered at first as an independent unknown. To do so, one also needs an evolution equation for $\mathbf{w}$, which is easily obtained by differentiating the mass conservation law. We get

$$\partial_t\mathbf{w}^* + \operatorname{div}(\mathbf{u}\mathbf{w}^*) + \operatorname{div}(\rho\,\mathrm{D}\mathbf{u}) = 0\,,$$

where $\mathrm{D}\mathbf{u} = (\nabla\mathbf{u})^*$ is by definition the matrix of coefficient $\partial_j u_i$ on the i-th row and j-th column. Therefore, one may also look at the equations of motion as

$$\begin{gathered}\partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0\,,\\ \partial_t(\rho\mathbf{u}^*) + \operatorname{div}(\rho\mathbf{u}\mathbf{u}^* + p\mathbf{I} - \mathbf{K}) = 0\,,\\ \partial_t S + \operatorname{div}(S\mathbf{u}) = 0\,,\\ \partial_t\mathbf{w}^* + \operatorname{div}(\mathbf{u}\mathbf{w}^*) + \operatorname{div}(\rho D\mathbf{u}) = 0\,.\end{gathered}\tag{1.3}$$

It has been pointed out by Gavrilyuk and Gouin [14] that, if the total energy $H := E + \frac{1}{2}\rho\|\mathbf{u}\|^2$ is a convex function of the conservative variable $(\rho,\mathbf{m},S,\mathbf{w})$ – with $\mathbf{m} := \rho\mathbf{u}$ –, then the system (1.3) admits a symmetric form similar to Friedrichs symmetric hyperbolic systems of conservation laws. This symmetrization procedure naturally involves the Legendre transform Π of H, which is a function of the dual variables $(q := g - \frac{1}{2}\|\mathbf{u}\|^2, \mathbf{u}, T, \phi)$. Indeed, one easily finds that

$$\mathrm{d}H = \mathrm{d}E - \frac{1}{2}\|\mathbf{u}\|^2\,\mathrm{d}\rho + \mathbf{u}^*\cdot\mathrm{d}\mathbf{m} = (g - \frac{1}{2}\|\mathbf{u}\|^2)\,\mathrm{d}\rho + \mathbf{u}^*\cdot\mathrm{d}\mathbf{m} + T\,\mathrm{d}S + \phi^*\cdot d\mathbf{w}\,.$$

The Legendre transform Π is by definition such that

$$\mathrm{d}\Pi = \rho\,\mathrm{d}q + \mathbf{m}^*\cdot\mathrm{d}\mathbf{u} + S\,\mathrm{d}T + \mathbf{w}^*\cdot\mathrm{d}\phi\,,$$

and

$$\Pi = \rho q + \mathbf{m}^*\cdot\mathbf{u} + ST + \mathbf{w}^*\cdot\phi - H = p + \mathbf{w}^*\cdot\phi\,.$$

Then it is not difficult to see that (1.3) also reads

$$\begin{gathered}\partial_t\Big(\frac{\partial\Pi}{\partial q}\Big) + \operatorname{div}\Big(\frac{\partial(\Pi\mathbf{u})}{\partial q}\Big) = 0\,,\\ \partial_t\Big(\frac{\partial\Pi}{\partial\mathbf{u}}\Big) + \operatorname{div}\Big(\frac{\partial(\Pi\mathbf{u})}{\partial\mathbf{u}}\Big) - \operatorname{div}\Big(\frac{\partial\Pi}{\partial q}\,D\phi\Big) = 0\,,\\ \partial_t\Big(\frac{\partial\Pi}{\partial T}\Big) + \operatorname{div}\Big(\frac{\partial(\Pi\mathbf{u})}{\partial T}\Big) = 0\,,\\ \partial_t\Big(\frac{\partial\Pi}{\partial\phi}\Big) + \operatorname{div}\Big(\frac{\partial(\Pi\mathbf{u})}{\partial\phi}\Big) + \operatorname{div}\Big(\frac{\partial\Pi}{\partial q}\,D\mathbf{u}\Big) = 0\,.\end{gathered}\tag{1.4}$$

The first-order part in (1.4) is exactly of the Friedrichs symmetric type. An abstract form for this system is

$$\partial_t\mathbf{U} + \sum_{\alpha=1}^d \mathbf{A}_\alpha(\mathbf{U})\partial_\alpha\mathbf{U} + \sum_{\alpha=1}^d \partial_\alpha\Big(\sum_{\beta=1}^d \mathbf{B}_{\alpha,\beta}(\mathbf{U})\,\partial_\beta\mathbf{U}\Big) = 0\,,$$

with the property that for some positive definite symmetric matrix $\Sigma(\mathbf{U})$ all matrices $\Sigma(\mathbf{U})\ \mathbf{A}_\alpha(\mathbf{U})$ are symmetric and

$$\sum_\alpha \sum_\beta\ (\mathbf{X}^\alpha)^*\ \Sigma(\mathbf{U})\ \mathbf{B}_{\alpha,\beta}(\mathbf{U})\ \mathbf{X}^\beta = 0$$

for all vectors $\mathbf{X}^1, \dots, \mathbf{X}^d$ in $\mathbb{R}^{2d+2}$. This is what we mean by non-dissipative second order part of the system, consistently with the usual terminology for parabolic systems of second order conservation laws. One may observe that a similar system (in nonconservative form) is studied in [3]. Here

$$\mathbf{U} = \begin{pmatrix} \rho \\ \mathbf{m} \\ S \\ \mathbf{w} \end{pmatrix}, \quad \Sigma = \mathrm{D}^2 H\,, \quad \mathbf{A}_\alpha = \mathrm{D}^2(\Pi u_\alpha)\ \Sigma\,,$$

where the Hessian $\mathrm{D}^2(\Pi u_\alpha)$ of Πu_α is taken with respect to the dual variables

$$\mathbf{Q} = \begin{pmatrix} q \\ \mathbf{u} \\ T \\ \phi \end{pmatrix}.$$

The actual expression of $\mathbf{B}_{\alpha,\beta}$ also involves the Hessian matrix $\mathrm{D}^2 H$, and more precisely

$$\mathbf{B}_{\alpha,\beta} \frac{\partial}{\partial U_k} = -\rho \frac{\partial^2 H}{\partial w_\alpha \partial U_k}\ \frac{\partial}{\partial m_\beta} + \rho \frac{\partial^2 H}{\partial m_\alpha \partial U_k}\ \frac{\partial}{\partial w_\beta}$$

for all $k \in \{1, 2d+2\}$ (with $U_1 = \rho$, $U_2 = m_1$, $\dots$, $U_{d+1} = m_d$, $U_{d+2} = S$, $U_{d+3} = w_1, \dots, U_{2d+2} = w_d$). The non-dissipativeness of the corresponding second order terms in (1.4) is a calculus exercise, and is equivalent to the fact that these terms formally cancel out in the computation of

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathbf{U}^* \Sigma(\mathbf{U}) \mathbf{U}\, \mathrm{d}\mathbf{x} = 2 \int \mathbf{Q} \partial_t \mathbf{U}\, \mathrm{d}\mathbf{x}$$

along solutions.

Unfortunately, this interesting formulation is limited to monotone pressure laws, since it makes use of the convexity of the total energy.

1.2. **Eulerian capillarity models.** For the rest of this article, we assume that $\phi = K\mathbf{w}$. Then we can write

$$g = g_0 + \tfrac{1}{2} K'_\rho \|\nabla \rho\|^2\,,$$

where g_0 is independent of $\nabla\rho$. In particular, the isothermal model reduces to

$$\begin{gathered} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0\,, \\ \partial_t \mathbf{u} + (\mathbf{u}^* \cdot \nabla)\mathbf{u} = \nabla(K\Delta\rho + \tfrac{1}{2} K'_\rho |\nabla\rho|^2 - g_0)\,, \end{gathered} \tag{1.5}$$

where g_0 and K are given, smooth functions of ρ (with $K > 0$). One may also write this system in conservative form, noting that

$$p = p_0 + \tfrac{1}{2}(\rho K'_\rho - K)\|\nabla\rho\|^2\,, \quad p_0 = \rho g_0 - F_0\,,$$

hence the (complicated) momentum equation

$$\begin{aligned}&\partial_t(\rho\mathbf{u}^*) + \operatorname{div}(\rho\mathbf{u}\mathbf{u}^*) + \nabla p_0\\ &= \nabla(\rho K\Delta\rho + \tfrac12(K+\rho K'_\rho)\|\nabla\rho\|^2) - \operatorname{div}(K\nabla\rho\otimes\nabla\rho).\end{aligned}$$

In one space dimension, our isothermal model reduces to

$$\begin{gathered}\partial_t\rho + \partial_x(\rho u) = 0\,,\\ \partial_t u + u\partial_x u = \partial_x(K\partial^2_{xx}\rho + \tfrac12 K'_\rho\,(\partial_x\rho)^2 - g_0(\rho))\,.\end{gathered}\tag{1.6}$$

We point out that models of this kind actually arise in various other contexts. In the special case

$$K(\rho) = \frac{1}{4\rho}\,,$$

the system (1.5) is equivalent – for irrotational flows – to a nonlinear Schrödinger equation known as the Gross-Pitaevskii equation

$$i\partial_t\psi + \frac12\Delta\psi = g_0(|\psi|^2)\psi$$

for $\psi = \sqrt{\rho}\mathrm{e}^{i\varphi}$, $\nabla\varphi = \mathbf{u}$. See for instance [6], where $g_0(\rho) = \frac14\rho^2$. It is also the case K proportional to $1/\rho$ that is considered in [22], with almost no restriction on g_0. One may also observe that, in one dimension with

$$g_0(\rho) = \tfrac{\rho}{4}\,,\quad K(\rho) = \frac{1}{4\rho}\,,$$

Equations in (1.6) appear as an equivalent form of the filament equation, see [2] p. 353.

1.3. **Lagrangian capillarity models.** The one-dimensional isothermal model becomes even simpler in Lagrangian formulation. Introducing y the mass Lagrangian coordinate so that[2] $\mathrm{d}y = \rho\,\mathrm{d}x - \rho u\,\mathrm{d}t$ we obtain with a little piece of calculus the – at least formally – equivalent system

$$\begin{gathered}\partial_t v - \partial_y u = 0\,,\\ \partial_t u + \partial_y p_0 = -\partial_y\left(\kappa\,\partial^2_{yy}v + \tfrac12\kappa'_v(\partial_y v)^2\right)\,,\end{gathered}\tag{1.7}$$

with $v := 1/\rho$ and $\kappa(v) := K(1/v)(1/v)^5$. In the special case $\kappa =$ constant, *i.e.* $K = \mathrm{cst}\rho^{-5}$, the system (1.7) is formally equivalent to the (good) Boussinesq equation and is amenable to the theory of Kato [18], see [7] for more details. Our aim here is to deal with general capillarities, motivated by physical reasons - since there is no reason why K should be proportional to ρ^5 - as well as by the various analogies mentioned above.

Following an idea of Coquel [10], we rewrite the velocity equation as

$$\partial_t u + \partial_y p_0 - \partial_y(\alpha\partial_y w) = 0\,,\quad \alpha = \sqrt{\kappa}\,,\quad w = -\alpha\partial_y v.$$

Applying the differential operator $-\partial_y(\alpha\cdot)$ to the first line of (1.7), we find that w satisfies the equation

$$\partial_t w + \partial_y(\alpha\partial_y u) = 0\,.$$

[2] This change of variable may be justified rigorously for, say, C^1 functions (ρ, u) and ρ bounded away from zero.

Considering w as an additional unknown, we are led to the system

$$\begin{aligned} \partial_t v - \partial_y u = 0\,, \\ \partial_t u + \partial_y p_0 - \partial_y(\alpha \partial_y w) = 0\,, \\ \partial_t w + \partial_y(\alpha \partial_y u) = 0\,. \end{aligned} \tag{1.8}$$

Alternatively, because of the constraint $w = -\alpha \partial_y v$ we may rewrite (1.8) as

$$\begin{aligned} \partial_t v - \partial_y u = 0\,, \\ \partial_t u - \partial_y(\alpha(v) \partial_y w) = q(v) w\,, \\ \partial_t w + \partial_y(\alpha(v) \partial_y u) = 0 \end{aligned} \tag{1.9}$$

with $q(v) := p_0'(v)/\alpha(v)$.

2. Well-posedness results

We are interested in the well-posedness of (1.6) and (1.7) for data with finite specific volumes (*i.e.* away from vacuum) and finite densities. More precisely, as both systems are known to admit global smooth solutions (like of course constant states, but also travelling wave solutions, see for instance [4, 5]), our main purpose is to show the well-posedness of (1.6) and (1.7) in affine spaces about such reference solutions.

Our strategy is to first prove the well-posedness of the system (1.9).

2.1. **Semigroup method.** The special case where α is constant is much easier, and enters the framework of Kato [18]. As a matter of fact, (1.9) can always be put in the abstract form

$$\partial_t \mathbf{v} - \mathcal{A}(v) \cdot \mathbf{v} = \mathcal{F}(v, w)\,,$$

with

$$\mathbf{v} := \begin{pmatrix} v \\ u \\ w \end{pmatrix}, \quad \mathcal{A}(v) := \begin{pmatrix} 0 & \partial_y & 0 \\ \partial_y & 0 & \partial_y \alpha(v) \partial_y \\ 0 & -\partial_y \alpha(v) \partial_y & 0 \end{pmatrix},$$

$$\mathcal{F}(v, w) := \begin{pmatrix} 0 \\ \frac{p_0'(v)+1}{\alpha} w \\ 0 \end{pmatrix}.$$

(In fact, we have used again the expected relation $w = -\alpha \partial_y v$ to obtain a right hand side $\mathcal{F}$ of order 0.) When α is constant, the antisymmetric operator $\mathcal{A}(v)$ of course has constant coefficients, which makes a big difference. Indeed, because $\mathcal{A}$ is the infinitesimal generator of a group of unitary operators on $L^2(\mathbb{R})^3$ (this is due to Stone's theorem, see [24], p. 41), it is possible to apply (a slight adaptation of) Theorem 6 in [18] (p. 36) using the operator $S = (1 - \partial_{yy}^2) I_3$ and the spaces $Y = H^2(\mathbb{R})^3$, $X = L^2(\mathbb{R})^3$, and thus show the following for any p_0 a smooth function of v.

Theorem 2.1. *If $\underline{\mathbf{v}} = (\underline{v}, \underline{u}, \underline{w})^t$ is smooth and exponentially decaying to $(v_\pm, u_\pm, 0)$ when $x \to \pm\infty$, then there is a $T \geq 0$ so that the Cauchy problem associated with*

$$\partial_t \mathbf{v} - \mathcal{A} \mathbf{v} = \mathcal{F}(v, w)$$

and initial data in $\underline{\mathbf{v}}(0) + H^2(\mathbb{R})^3$ admits a unique solution such that $\mathbf{v} - \underline{\mathbf{v}}$ belongs to $\mathcal{C}(0, T; H^2(\mathbb{R})^3) \cap \mathcal{C}^1(0, T; L^2(\mathbb{R})^3)$.

This theorem is essentially the same as Theorem 1 in [7], extended to solutions having nonzero, and possibly different, limits at infinity. Our motivation for this extension is the stability of diffuse interfaces.

For a general function α, the semi-group approach breaks down because the operator $\mathcal{A}(v)$ involves the derivative $\partial_y v$. Thus the Lipschitz estimate requested by Kato's theorem,

$$\|(\mathcal{A}(v_1) - \mathcal{A}(v_2)) \cdot \mathbf{v}\|_X \lesssim \|\mathbf{v}_1 - \mathbf{v}_2\|_X \|\mathbf{v}\|_Y$$

means that the norm of vectors in X should control the first derivative of their first components. This goal does not seem possible to achieve without augmenting also the regularity index of the other components (to keep a semigroup generated by $\mathcal{A}(v)$ in X). For this reason the Lipschitz estimate above is quite unlikely.

2.2. **Alternative method.** We introduce the complex valued function $z = u + iw$ and regard equivalently the system (1.9) as the coupling of the variable coefficient Schrödinger equation

$$\partial_t z + i\partial_y(\alpha(v)\partial_y z) = q(v)\,\mathrm{Im}\, z \tag{2.1}$$

with the compatibility equation

$$\partial_t v - \partial_y \,\mathrm{Re}\, z = 0\,.$$

Let J_v be an open interval of $(0, +\infty)$ and $k \geq 2$ be an integer. Our main assumption is

(H1) Both p_0 and α belong to $W^{k+2,\infty}_{\mathrm{loc}}(J_v)$ and α is positive on J_v.

For a nonconvex pressure law $v \mapsto p_0(v)$, it is easy to show that (1.9) does admit non-constant, bounded, global, smooth solutions, which are in fact travelling wave solutions. This is because of the Hamiltonian structure of (1.9), which implies that the governing equations for the travelling waves are also Hamiltonian, see [4] or our companion paper [5] for more details. In what follows, we assume the existence of a reference global smooth solution, regardless of the convexity properties of p.

Theorem 2.2. *Under assumption* (H1) *with* $k \geq 2$*, let* $(\overline{u}, \overline{v})$ *be a given global classical solution of system* (1.9) *with* $\overline{v}(\mathbb{R}^2) \subset J_v$ *and*

$$\partial_y \overline{u} \in C(\mathbb{R}; H^{k+1}), \quad \partial_y \overline{v} \in C(\mathbb{R}; H^{k+2}).$$

Let $u_0 \in \overline{u}(0) + H^k$ *and* $v_0 \in \overline{v}(0) + H^{k+1}$ *be such that* $v_0(\mathbb{R}) \subset\subset J_v$*. There exists a positive* T *such that the Cauchy problem associated with the system* (1.9) *and initial data* (u_0, v_0) *has a unique solution* (u, v) *with* $v([-T, T] \times \mathbb{R}) \subset\subset J_v$ *and, denoting* $\widetilde{u} := u - \overline{u}$ *and* $\widetilde{v} := v - \overline{v}$*,*

$$(\widetilde{u}, \widetilde{v}) \in \mathcal{C}([-T, T]; H^k \times H^{k+1}) \cap \mathcal{C}^1([-T, T]; H^{k-2} \times H^{k-1}). \tag{2.2}$$

Remark 2.3. Constants are obviously global classical solutions of (1.9). If $(\overline{u}, \overline{v})$ is constant then the theorem can be slightly improved, see Section 4.4.

Remark 2.4. One can find an explicit bound by below for T (see (4.18)). Besides, one can show that for H^k data, the time of existence of a H^k solution is the same as for a H^2 solution (see section 4.3).

In the case where the Sobolev norms of $\partial_y \overline{u}$ and $\partial_y \overline{v}$ are independent of the time, there exists a constant $C = C(\alpha, q, \overline{u}, \overline{v}, v_0(\mathbb{R}))$ such that T may be chosen such that

$$T \geq \frac{1}{C} \log \left(1 + \frac{1}{\|\widetilde{u}_0\|_{H^2} + \|\partial_y \widetilde{v}_0\|_{H^2}} \right) . \tag{2.3}$$

Hence, if $\|\widetilde{u}_0\|_{H^2} + \|\partial_y \widetilde{v}_0\|_{H^2} \sim \varepsilon$ for a small ε, the life span is greater than $C \log \varepsilon^{-1}$.

For simplicity, we shall restrict ourselves to the evolution for *positive* times. As system (1.9) is time reversible, adapting our proof to *negative* is straightforward.

Our approach is quite classical. It consists in deriving first energy estimates *without loss of derivatives* – in sufficiently high order Sobolev spaces – for a linearized version of (2.1), and then solve the nonlinear problem through an iterative scheme. There are some difficulties in both steps that will be pointed out along the detailed proofs.

3. A variable coefficient linear Schrödinger equation

To show the (local) well-posedness of (1.9) by means of an iterative scheme, we shall need the resolution of the linear Schrödinger equation

$$\partial_t z + i\partial_y (a(y,t)\partial_y z) = f(y,t) . \tag{3.1}$$

Whatever the function a (smooth enough and *real*-valued), the operator $i\partial_y a \partial_y$ is obviously antisymmetric on $L^2(\mathbb{R})$. Under suitable assumptions on the asymptotic behavior of a it is not difficult to show that $i\partial_y a \partial_y$ is also skewadjoint [3] in $L^2(\mathbb{R})$. Therefore $i\partial_y a \partial_y$ is the infinitesimal generator of a group of unitary operators, and the standard semigroup theory (see [24], p. 145–147) enables us to prove the following.

Theorem 3.1. *Assuming that a is real valued, belongs to $\mathcal{C}^1([0,T]; W^{1,\infty}(\mathbb{R}))$ and that $a(t)$ has finite limits at $x = \pm\infty$ for all $t \in [0,T]$, the Cauchy problem*

$$\partial_t z + i\partial_y (a(y,t)\partial_y z) = f(y,t) ,$$
$$z(0) = z_0$$

with $z_0 \in H^2(\mathbb{R})$ and $f \in L^1([0,T]; H^2(\mathbb{R})) \cap \mathcal{C}([0,T]; L^2(\mathbb{R}))$ admits a unique (classical) solution $z \in \mathcal{C}([0,T]; H^2(\mathbb{R})) \cap \mathcal{C}^1([0,T]; L^2(\mathbb{R}))$. If $z_0 \in L^2$ and $f \in L^1([0,T]; L^2(\mathbb{R}))$, we get a mild solution $u \in \mathcal{C}([0,T]; L^2(\mathbb{R}))$ given by Duhamel's formula:

$$z(t) = \mathcal{S}(t,0) z_0 + \int_0^t \mathcal{S}(t,s) f(s) \, \mathrm{d}s$$

with $\mathcal{S}(t,s)$ the solution operator of the homogeneous equation.

3.1. **A priori estimates.**

Our aim is to obtain a more precise result, and especially *a priori* estimates in view of our iterative scheme for the nonlinear problem. Of course we immediately have from Duhamel's formula the estimate

$$\|z(t)\|_{L^2} \leq \|z(0)\|_{L^2} + \int_0^t \|f(s)\|_{L^2} \, \mathrm{d}s ,$$

which can also be derived directly from the equation (3.1). Indeed, multiplying (3.1) by $\overline{z}$ and integrating over $\mathbb{R}$ we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{dt}} \|z\|_{L^2}^2 - \Im \int \partial_y (a \partial_y z) \overline{z} \, \mathrm{d}y = \Re \int f \overline{z} \, \mathrm{d}y \tag{3.2}$$

---

[3] It suffices to check that the range of $\partial_y a \partial_y - \lambda$ is dense in $L^2(\mathbb{R})$ for all $\lambda \notin \mathbb{R}$.

and an integration by parts shows that the integral in the left-hand side has no imaginary part since a is *real*. So it follows from Cauchy-Schwarz inequality that (at least formally)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_{L^2} \leq \|f\|_{L^2}$$

hence the result.

However, this L^2 estimate does not provide enough information to solve the non-linear system (1.9), which equivalently reads

$$\begin{aligned} \partial_t v - \partial_y \operatorname{Re} z = 0\,, \\ \partial_t z + i\partial_y(\alpha(v)\partial_y z) = q(v) \operatorname{Im} z\,. \end{aligned} \tag{3.3}$$

The second-order term $\partial_y(\alpha(v)\partial_y z)$ induces us to prove a priori estimates for equation (3.1) in higher order Sobolev spaces, a matter which is not obvious despite the linearity of the equation, neither on the Duhamel formula – because differentiation operators do not commute with the solution operator $\mathcal{S}(t,s)$ – nor in the direct fashion described in L^2. Indeed, let $z^{(k)}$ (resp. $a^{(k)}$ and $f^{(k)}$) denote the k-th order derivative of z (resp. a and f) with respect to y. By Leibniz formula, we have

$$\begin{aligned} \partial_y^k\Big(\partial_y(a\partial_y z)\Big) &= \partial_y\Big(\partial_y^k(a\partial_y z)\Big) \\ &= \partial_y(a\partial_y z^{(k)}) + \sum_{j=0}^{k-1} \tbinom{k}{j}\ \partial_y(z^{(j+1)}a^{(k-j)}) \\ &= \partial_y(a\partial_y z^{(k)}) + k\partial_y a z^{(k+1)} + \sum_{\ell=1}^{k} \tbinom{k+1}{\ell-1}\ z^{(\ell)}a^{(k+2-\ell)} \end{aligned}$$

so we get the following equation for $z^{(k)}$:

$$\partial_t z^{(k)} + i\partial_y(a\partial_y z^{(k)}) = f^{(k)} - ik(\partial_y a)\partial_y z^{(k)} - i\sum_{j=1}^{k} \tbinom{k+1}{j-1} z^{(j)}a^{(k+2-j)}. \tag{3.4}$$

This can be rewritten using Duhamel's formula as

$$\begin{aligned} z^{(k)}(t) &= \mathcal{S}(t,0)z_0^{(k)} \\ &\quad + \int_0^t \mathcal{S}(t,s)\Big(f^{(k)} - ik(\partial_y a)\partial_y z^{(k)} - i\sum_{j=1}^{k} \tbinom{k+1}{j-1} z^{(j)}a^{(k+2-j)}\Big)(s)\,\mathrm{d}s\,. \end{aligned}$$

We see that the time integral involves the derivative $\partial_y z^{(k)}$. This derivative cannot disappear through a direct *a priori* estimate either, because the real part of (3.4) multiplied by $\overline{z}^{(k)}$ contains the term

$$\operatorname{Im}\int (\partial_y a)\partial_y z^{(k)}\overline{z^{(k)}}\,\mathrm{d}y,$$

which cannot be rewritten without a derivative of $z^{(k)}$, unless a is constant or $z^{(k)}$ is real.

This is a well identified problem for *variable* coefficients Schrödinger equations. In [23] for instance, W. Lim and G. Ponce overcome the difficulty by introducing appropriate weighted Sobolev spaces. We are going to show how weights, also called gauges, can help to compensate the loss of derivative in our context.

Let ϕ_k denote the k-th order gauge (to be determined), and multiply the equation (3.4) by ϕ_k. Since

$$\phi_k \partial_y(a\partial_y z^{(k)}) = \partial_y(a\partial_y(\phi_k z^{(k)})) - z^{(k)}\partial_y(a\partial_y\phi_k) - 2a\partial_y\phi_k\partial_y z^{(k)},$$

we find the equation for $\phi_k z^{(k)}$,

$$\begin{aligned}
&\partial_t(\phi_k z^{(k)}) + i\partial_y(a\partial_y(\phi_k z^{(k)}))\\
&= \phi_k f^{(k)} + z^{(k)}\partial_t\phi_k - i\sum_{j=1}^{k-1}\tbinom{k+1}{j-1}\phi_k z^{(j)}a^{(k+2-j)}\\
&+ iz^{(k)}\big(\partial_y(a\partial_y\phi_k) - \frac{k(k+1)}{2}\partial^2_{yy}a\ \phi_k\big) + i\partial_y z^{(k)}\Big(2a\partial_y\phi_k - k\phi_k\partial_y a\Big).
\end{aligned} \tag{3.5}$$

From the above equality, it is now clear that the loss of derivative will be avoided if and only if ϕ_k satisfies

$$2a\partial_y\phi_k - k\phi_k\partial_y a = 0.$$

Choose $\phi_k := a^{k/2}$ so that the last term in (3.5) vanishes. Then multiply (3.5) by $\phi_k\overline{z^{(k)}}$ and integrate in time. If we keep only the real part of the equation, the second order term in the left-hand side vanishes, as well as the last remaining term of the right-hand side, hence the identity

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi_k z^{(k)}\|^2_{L^2} &= \Re\int \phi_k^2 f^{(k)}\overline{z^{(k)}}\,\mathrm{d}y + \int \phi_k\partial_t\phi_k|z^{(k)}|^2\,\mathrm{d}y\\
&+ \sum_{j=1}^{k-1}\tbinom{k+1}{j-1}\Im\int a^{(k+2-j)}\phi_k^2 z^{(j)}\overline{z^{(k)}}\,\mathrm{d}y.
\end{aligned} \tag{3.6}$$

We can now state estimates in H^k for the variable coefficient Schrödinger equation.

Proposition 3.2. *Let z be a solution of (3.1) on $\mathbb{R}\times[0,T]$. Assume in addition that a is bounded away from zero by $\underline{a}$ and is bounded by $\widetilde{a}$. Then for all $t\in[0,T]$, we have*

$$\|z(t)\|_{L^2} \le \|z_0\|_{L^2} + \int_0^t \|f(\tau)\|_{L^2}\,d\tau. \tag{3.7}$$

Besides, denoting for $k\in\mathbb{N}^*$,

$$Z_k(t) := \Big(\sum_{j=1}^k \|(a^{\frac{j}{2}}z^{(j)})(t)\|^2_{L^2}\Big)^{1/2} \quad \text{and} \quad F_k(t) := \Big(\sum_{j=1}^k \|(a^{\frac{j}{2}}f^{(j)})(t)\|^2_{L^2}\Big)^{1/2},$$

we have

$$Z_k(t) \le \mathrm{e}^{A_k(t)}\Big(Z_k(0) + \int_0^t \mathrm{e}^{-A_k(\tau)}F_k(\tau)\,d\tau\Big), \tag{3.8}$$

with $A_k(t) := \int_0^t\Big(\frac{k}{2}\|\partial_t\log a(\tau)\|_{L^\infty} + C_k\big(\|\partial_y^3 a(\tau)\|_{H^{k-2}} + \|\partial_y^2 a(\tau)\|_{L^\infty}\big)\Big)\,d\tau$ for some positive constant C_k depending only on k, $\underline{a}$ and $\widetilde{a}$ if $k\ge 2$, and $C_1 = 0$.

Proof. The L^2 estimate was already pointed out. In order to prove the k-th order estimate, we sum equalities (3.6) for $j=1,\dots,k$. This implies by Cauchy-Schwarz inequality,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}Z_k^2 \le Z_k\Big(F_k + \frac{k}{2}\|\partial_t\log a\|_{L^\infty}\ Z_k + \sum_{j=2}^k\sum_{\ell=1}^{j-1}\tbinom{j+1}{\ell-1}\widetilde{a}^{\frac{j}{2}}\|a^{(j+2-\ell)}z^{(\ell)}\|_{L^2}\Big). \tag{3.9}$$

Note that in the case $k=1$, the last term vanishes so that straightforward calculations yield (3.8). Let us assume from now on that $k\geq 2$. Since $a^{(j+2-\ell)}z^{(\ell)}=\big(\partial_y^{(j-1)-(\ell-1)}a''\big)\big(\partial_y^{\ell-1}z'\big)$, inequality (5.1) in the appendix ensures that

$$\|a^{(j+2-\ell)}z^{(\ell)}\|_{L^2}\leq C_{j,\ell}\Big(\|a''\|_{L^\infty}\|z'\|_{H^{j-1}}+\|z'\|_{L^\infty}\|a^{(j+1)}\|_{L^2}\Big).$$

Now, because

$$\|z'\|^2_{L^\infty}\leq \|z'\|_{L^2}\|z''\|_{L^2}\leq \underline{a}^{-\frac{3}{2}}\|a^{1/2}z'\|_{L^2}\|az''\|_{L^2},$$

we conclude that[4] whenever $2\leq j\leq k$ and $1\leq \ell\leq j-1$,

$$\|a^{(j+2-\ell)}z^{(\ell)}\|_{L^2}\leq C_{j,\ell}\left(\|a''\|_{L^\infty}+\|a^{(3)}\|_{H^{k-2}}\right)Z_k,$$

for some constant $C_{j,\ell}$ depending only on j, ℓ and $\underline{a}$. Plugging this latter inequality in (3.9), we end up with

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}Z_k^2\leq F_kZ_k+\Big(\frac{k}{2}\|\partial_t\log a\|_{L^\infty}+C_k\big(\|a''\|_{L^\infty}+\|a^{(3)}\|_{H^{k-2}}\big)\Big)Z_k^2.$$

Then Gronwall lemma entails inequality (3.8). □

3.2. **Existence of regular solutions.** This section is devoted to the regularity of solutions of (3.1), namely we want to prove the following theorem.

Theorem 3.3. *Let k be an integer such that $k\geq 2$, Let $a=a(y,t)$ be bounded by $\widetilde{a}$ and bounded away from zero by $\underline{a}>0$ on $\mathbb{R}\times[0,T]$, and satisfy $\partial_y^2a\in L^1(0,T;H^{k-1}(\mathbb{R}))$ and $\partial_ta\in L^1(0,T;L^\infty(\mathbb{R}))$. Let z_0 be in $H^k(\mathbb{R})$ and f in $L^1(0,T;H^k(\mathbb{R}))$. Then the Cauchy problem*

$$\begin{gathered}\partial_tz+i\partial_y(a(y,t)\partial_yz)=f(y,t)\,,\\ z(0)=z_0\end{gathered}\tag{3.10}$$

admits a unique solution $z\in\mathcal{C}([0,T];H^k(\mathbb{R}))$. If, besides, $f\in\mathcal{C}([0,T];H^{k-2}(\mathbb{R}))$ then z also belongs to $\mathcal{C}^1([0,T];H^{k-2}(\mathbb{R}))$.

Proof. The proof is based on the following fourth-order regularization of equation (3.10):

$$\begin{gathered}\partial_tz_\varepsilon+i\partial_y(a(y,t)\partial_yz_\varepsilon)+\varepsilon\partial_y^4z_\varepsilon=f\,,\\ z_\varepsilon(0)=z_0,\end{gathered}\tag{3.11}$$

where $\varepsilon>0$ stands for a positive parameter bound to go to zero.

We shall see in the proof however that this regularization does not enable us to pass to the limit in the very space $\mathcal{C}([0,T];H^k)$. For doing so, we shall adapt the method by J. Bona and R. Smith in [8] which amounts to smoothing out conveniently the data and the variable coefficients.

Let us briefly describe the main steps of the proof:

(1) Getting bounds for z_ε similar to (3.8) and independent of ε.
(2) Stating well-posedness in $H^k(\mathbb{R})$ for the problem (3.11) with $\varepsilon>0$.
(3) Showing that the family of solutions (z_ε) to (3.11) with regularized data and coefficient converges to some z in the space $\mathcal{C}([0,T];H^k(\mathbb{R}))$ when ε approaches 0.

[4] remark that we always have $j+1\geq 3$.

Step 1: uniform a priori estimates. Let $p \in \mathbb{N}$. We assume that we are given a solution $z \in \mathcal{C}([0,T]; H^p(\mathbb{R}))$ to (3.11)[5] corresponding to data $z_0 \in H^p(\mathbb{R})$ and $f \in L^1([0,T]; H^p(\mathbb{R}))$. We claim that z satisfies an estimate similar to (3.8) with a constant C *independent* of ε.

Let ℓ be an integer such that $0 \le \ell \le p$ and $\phi_\ell = a^{\frac{\ell}{2}}$ be the ℓ-th order gauge introduced previously. Applying Leibniz formula to $\partial_y^4(\alpha\beta)$ with $\alpha := 1/\phi_\ell$ and $\beta = \phi_\ell z^{(\ell)}$, one obtains the following expansion:

$$\begin{aligned}\phi_\ell \partial_y^4 z^{(\ell)} &= \partial_y^4(\phi_\ell z^{(\ell)}) + 4\phi_\ell \partial_y(\phi_\ell^{-1})\partial_y^3(\phi_\ell z^{(\ell)}) + 6\phi_\ell\partial_y^2(\phi_\ell^{-1})\partial_y^2(\phi_\ell z^{(\ell)}) \\ &\quad + 4\phi_\ell\partial_y^3(\phi_\ell^{-1})\partial_y(\phi_\ell z^{(\ell)}) + \phi_\ell \partial_y^4(\phi_\ell^{-1})(\phi_\ell z^{(\ell)}).\end{aligned}$$

Hence, applying the operator $\phi_\ell \partial_y^\ell$ to the evolution equation in (3.11) and denoting $\widetilde{z}_\ell := a^{\ell/2} z^{(\ell)}$, we get

$$\begin{aligned}&\partial_t \widetilde{z}_\ell + i\partial_y(a\partial_y \widetilde{z}_\ell) + \varepsilon \partial_y^4 \widetilde{z}_\ell \\ &= a^{\frac{\ell}{2}} f^{(\ell)} + z^{(\ell)} \partial_t a^{\frac{\ell}{2}} + i\widetilde{z}_\ell \Big(a^{-\frac{\ell}{2}} \partial_y(a\partial_y a^{\frac{\ell}{2}}) - \frac{\ell(\ell+1)}{2}\partial_y^2 a\Big) \\ &\quad - i\sum_{j=1}^{\ell-1} \tbinom{\ell+1}{j-1} a^{\frac{\ell}{2}} z^{(j)} a^{(\ell+2-j)} - \varepsilon a^{\ell/2}\Big(4\partial_y(a^{-\ell/2})\partial_y^3 \widetilde{z}_\ell \\ &\quad + 6\partial_y^2(a^{-\ell/2})\partial_y^2\widetilde{z}_\ell + 4\partial_y^3(a^{-\ell/2})\partial_y\widetilde{z}_\ell + \partial_y^4(a^{-\ell/2})\widetilde{z}_\ell\Big).\end{aligned} \tag{3.12}$$

Now, multiplying (3.12) by $\overline{\widetilde{z}_\ell}$ and performing a time integration, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{z}_\ell\|_{L^2}^2 + \varepsilon\|\partial_y^2\widetilde{z}_\ell\|_{L^2}^2 &= \Re\int a^\ell f^{(\ell)}\overline{z^{(\ell)}}\,\mathrm{d}y + \frac{\ell}{2}\int \partial_t \log a\, |\widetilde{z}_\ell|^2\,\mathrm{d}y \\ &\quad + \sum_{j=1}^{\ell-1}\tbinom{\ell+1}{j-1}\Im\int a^{(\ell+2-j)} a^\ell z^{(j)}\overline{z^{(\ell)}}\,\mathrm{d}y \\ &\quad - \varepsilon\Re\big(4R_1 + 6R_2 + 4R_3 + R_4\big)\end{aligned} \tag{3.13}$$

with

$$R_1 := \int a^{\ell/2}\partial_y(a^{-\ell/2})\,(\partial_y^3\widetilde{z}_\ell)\,\overline{\widetilde{z}_\ell}\,\mathrm{d}y, \quad R_2 := \int a^{\ell/2}\partial_y^2(a^{-\ell/2})(\partial_y^2\widetilde{z}_\ell)\,\overline{\widetilde{z}_\ell}\,\mathrm{d}y,$$
$$R_3 := \int a^{\ell/2}\partial_y^3(a^{-\ell/2})\,(\partial_y\widetilde{z}_\ell)\,\overline{\widetilde{z}_\ell}\,\mathrm{d}y, \quad R_4 := \int a^{\ell/2}\partial_y^4(a^{-\ell/2})|\widetilde{z}_\ell|^2\,\mathrm{d}y.$$

To bound R_1, we perform an integration by parts:

$$\begin{aligned}R_1 &= -\int \big(\partial_y(a^{\ell/2})\partial_y(a^{-\ell/2}) + a^{\ell/2}\partial_y^2(a^{-\ell/2})\big)(\partial_y^2\widetilde{z}_\ell)\overline{\widetilde{z}_\ell}\,\mathrm{d}y \\ &\quad - \int a^{\ell/2}\partial_y(a^{-\ell/2})(\partial_y^2\widetilde{z}_\ell)\partial_y\overline{\widetilde{z}_\ell}\,\mathrm{d}y\,.\end{aligned}$$

Using Hölder inequalities and applying the interpolation inequality

$$\|F'\|_{L^\infty}^2 \le 2\|F\|_{L^\infty}\|F''\|_{L^\infty},$$

to $F = a^{\pm\ell/2}$, we get by Young's inequality

$$|4R_1| \le \frac{1}{8}\|\partial_y^2\widetilde{z}_\ell\|_{L^2}^2 + C\|\partial_y^2 a\|_{L^\infty}^2\|\widetilde{z}_\ell\|_{L^2}^2$$

[5]For notational convenience, we drop the indices ε in this step

for some constant C depending only on $\widetilde{a} \geq \|a\|_{L^\infty}$ and $\underline{a}^{-1} \geq \|a^{-1}\|_{L^\infty}$. We get a similar bound for R_2 using only Hölder and Young inequalities.

We claim that the same bound holds true for R_3 and R_4. This is only a matter of using suitable integration by parts and standard inequalities like above. Indeed, we may rewrite

$$\begin{aligned} R_3 = -\int \big(&\partial_y(a^{\ell/2})\partial_y^2(a^{-\ell/2})(\partial_y\widetilde{z}_\ell)\overline{\widetilde{z}_\ell} + a^{\ell/2}\partial_y^2(a^{-\ell/2})|\partial_y\widetilde{z}_\ell|^2 \\ &+ a^{\ell/2}\partial_y^2(a^{-\ell/2})(\partial_y^2\widetilde{z}_\ell)\overline{\widetilde{z}_\ell}\big)\,\mathrm{d}y \end{aligned}$$

and

$$\begin{aligned} R_4 = -\int \big(&\partial_y^2(a^{\ell/2})\partial_y^2(a^{-\ell/2})|\widetilde{z}_\ell|^2 + 2\partial_y(a^{\ell/2})\partial_y^2(a^{-\ell/2})\partial_y\big(|\widetilde{z}_\ell|^2\big) \\ &+ a^{\ell/2}\partial_y^2(a^{-\ell/2})\partial_y^2\big(|\widetilde{z}_\ell|^2\big)\big)\,\mathrm{d}y. \end{aligned}$$

Therefore, we eventually find a constant C depending only on $\widetilde{a}$ and $\underline{a}$ so that

$$\Big|4R_1 + 6R_2 + 4R_3 + R_4\Big| \leq \frac{1}{2}\|\partial_y^2\widetilde{z}_\ell\|_{L^2}^2 + C\|\partial_y^2 a\|_{L^\infty}^2\|\widetilde{z}_\ell\|_{L^2}^2.$$

The other terms appearing in the right-hand side of (3.13) may be bounded like in the case $\varepsilon = 0$. Hence

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{z}_\ell\|_{L^2}^2 + \frac{\varepsilon}{2}\|\partial_y^2\widetilde{z}_\ell\|_{L^2}^2 \\ &\leq \|\widetilde{z}_\ell\|_{L^2}\|a^{\ell/2}f^{(\ell)}\|_{L^2} + \big(\frac{\ell}{2}\|\partial_t\log a\|_{L^\infty} + C\varepsilon\|\partial_y^2 a\|_{L^\infty}^2\big)\|\widetilde{z}_\ell\|_{L^2}^2 \\ &\quad + C\big(\|\partial_y^2 a\|_{L^\infty} + \|\partial_y^3 a\|_{H^{\ell-2}}\big)\|z\|_{H^\ell}^2. \end{aligned}$$

Summing the above inequalities for $\ell = 0, \dots, p$ and applying the usual Gronwall type argument, we deduce that for t in $[0,T]$ and $\varepsilon \geq 0$, we have

$$\Big(Z_p^2(t) + \varepsilon\sum_{\ell=0}^{p}\|\partial_y^2\big(a^{\ell/2}z^{(\ell)}\big)\|_{L^2}^2\Big)^{1/2} \leq \mathrm{e}^{A_{p,\varepsilon}(t)}\Big(Z_p(0) + \int_0^t F_p(\tau)\mathrm{e}^{-A_{p,\varepsilon}(\tau)}\,d\tau\Big) \tag{3.14}$$

with

$$Z_p^2(t) = \sum_{\ell=0}^{p}\|a^{\ell/2}\partial_y^{(\ell)}z(t)\|_{L^2}^2, \quad F_p^2(t) = \sum_{\ell=0}^{p}\|a^{\ell/2}\partial_y^{(\ell)}f(t)\|_{L^2}^2,$$

$$A_{p,\varepsilon}(t) = \int_0^t \big(\frac{p}{2}\|\partial_t\log a\|_{L^\infty} + C_p\big(\|\partial_y^2 a\|_{L^\infty} + \|\partial_y^3 a\|_{H^{p-2}} + \varepsilon\|\partial_y^2 a\|_{L^\infty}^2\big)\big)d\tau$$

where $C_0 = C_1 = 0$ and C_p depends only on p, $\underline{a}$ and $\widetilde{a}$ for $p \geq 2$.

Step 2: solving the regularized equation. This step is devoted to the proof of the following result.

Proposition 3.4. *Let k, a, z_0 and f satisfy the hypotheses of theorem* 3.3*. System* (3.11) *has a unique solution z_ε in the space*

$$\mathcal{C}([0,T];H^k) \cap L^2(0,T;H^{k+2}).$$

Proof. Denote by $S(t)$ the analytic semi-group generated by ∂_y^4 and, for $t > 0$,

$$E_t^\varepsilon := \mathcal{C}([0,t];H^k) \cap L^2(0,t;H^{k+2})$$

endowed with the norm

$$\|z\|_{E_t^\varepsilon} := \|z\|_{L_t^\infty(H^k)} + \varepsilon^{1/2}\|\partial_y^2 z\|_{L_t^2(H^k)}.$$

We claim that for suitably small $t > 0$, the operator Φ defined for $\tau \in [0,t]$ by

$$\Phi(z)(\tau) = S(\varepsilon\tau)z_0 + \int_0^\tau S(\varepsilon(\tau - s))\Big(f(s) - i\partial_y(a(y,s)\partial_y z(y,s))\Big)ds \qquad (3.15)$$

has a fixed point in E_t^ε.

Obviously Φ maps E_t^ε in E_t^ε. Indeed, on the one hand, using standard properties of $S(\tau)$, the terms pertaining to z_0 and f belong to E_t^ε. On the other hand, by virtue of (5.2), we have

$$\|a\partial_y z\|_{H^{k+1}} \lesssim \|a\|_{L^\infty}\|\partial_y z\|_{H^{k+1}} + \|\partial_y z\|_{L^\infty}\|\partial_y^2 a\|_{H^{k-1}}.$$

Hence, by standard computations relying on Hölder inequality,

$$\|a\partial_y z\|_{L_t^1(H^{k+1})} \le C\Big(\varepsilon^{-\frac12}\|a\|_{L_t^2(L^\infty)} + \|a\|_{L_t^1(L^\infty)} + \|\partial_y^2 a\|_{L_t^1(H^{k-1})}\Big)\|z\|_{E_t^\varepsilon}.$$

This entails that $\partial_y(a\partial_y z) \in L^1(0,t;H^k)$, hence the Duhamel term pertaining to $\partial_y(a\partial_y z)$ also belongs to E_t^ε.

Therefore, if z_2 and z_1 both belong to E_t^ε, we have for some constant C depending only on k,

$$\begin{aligned}
&\|\Phi(z_2) - \Phi(z_1)\|_{E_t^\varepsilon}\\
&\le C\|a\partial_y(z_2 - z_1)\|_{L_t^1(H^{k+1})},\\
&\le C\Big(\varepsilon^{-\frac12}\|a\|_{L_t^2(L^\infty)} + \|a\|_{L_t^1(L^\infty)} + \|\partial_y^2 a\|_{L_t^1(H^{k-1})}\Big)\|z_2 - z_1\|_{E_t^\varepsilon}.
\end{aligned}$$

Choosing t so small as to satisfy

$$2C\big(\varepsilon^{-\frac12}\|a\|_{L_t^2(L^\infty)} + \|a\|_{L_t^1(L^\infty)} + \|\partial_y^2 a\|_{L_t^1(H^{k-1})}\big) \le 1,$$

we conclude that Φ is a contractive map so that it has a unique fixed point in E_t^ε. Whence (3.11) has a unique solution in E_t on the time interval $[0,t]$.

Obviously, the above proof may be repeated starting from time t. We end up with a solution on the whole interval $[0,T]$. Uniqueness stems from estimate (3.14). □

Step 3. Passing to the limit. In this part, we are given a nonnegative smooth function θ with support in $[-1,1]$ and such that $\int_{\mathbb{R}} \theta(y)\,dy = 1$, and a smooth radial function χ whose Fourier transform is supported in $[-1,1]$. For $\eta > 0$, we denote $\theta_\eta := \eta^{-1}\theta(\eta^{-1}\cdot)$ and $\chi_\eta := \eta^{-1}\chi(\eta^{-1}\cdot)$.

We shall make use repeatedly of the following two lemmas the proof of which is left to the reader.

Lemma 3.5. *Let X be a Banach space and $u \in L^1_{\rm loc}(\mathbb{R};X)$. Then $\theta_\eta * u$ tends to u in $L^1_{\rm loc}(\mathbb{R};X)$ when η approaches 0. Besides, for all $q \ge 1$, $a < b$ and $\eta > 0$, the function $\theta_\eta * u$ belongs to $L^q(a,b;X)$ and we have*

$$\|\theta_\eta * u\|_{L^q(a,b;X)} \le \eta^{\frac1q - 1}\|u\|_{L^1(a-\eta,b+\eta;X)}.$$

Lemma 3.6. *Let u be in $H^k(\mathbb{R};\mathbb{C})$. Then $\chi_\eta * u$ belongs to $H^\infty(\mathbb{R};\mathbb{C})$ and tends to u in $H^k(\mathbb{R};\mathbb{C})$ when η approaches zero. Besides, there exists a constant C depending only on p and k and such that*

$$\|\chi_\eta * u\|_{H^p} \le C\eta^{k-p}\|u\|_{H^k} \quad \text{for } p \ge k.$$

If $u \in \mathcal{S}'(\mathbb{R};\mathbb{C})$ satisfies $\partial_y^2 u \in H^{k-2}(\mathbb{R};\mathbb{C})$ then $\chi_\eta * u - u \in H^k(\mathbb{R};\mathbb{C})$ and

$$\|\chi_\eta * u - u\|_{H^p} \leq C\eta^{k-p}\|\partial_y^2 u\|_{H^{k-2}} \quad \textit{for } p \leq k.$$

For $\varepsilon > 0$, mollify the data and the coefficient a by setting

$$z_{0,\varepsilon} := \chi_{\eta(\varepsilon)} *_y z_0, \quad f_\varepsilon := \chi_{\eta(\varepsilon)} *_y f, \quad a_\varepsilon := \theta_\varepsilon *_t \big(\chi_{\eta(\varepsilon)} *_y a\big),$$

where η is a positive function tending to 0 in 0 to be chosen hereafter, and with the convention that $a(y,t) = 0$ if $t > T$.

According to lemma 3.6 and Sobolev embeddings, we have $\underline{a}/2 \leq a_\varepsilon(y,t) \leq 2\widetilde{a}$ for small enough ε. This fact will be used repeatedly in the sequel.

Theorem 3.4 provides a solution $z_\varepsilon \in \mathcal{C}([0,T];H^k) \times L^2(0,T;H^{k+2})$ for the system

$$\begin{gathered} \partial_t z_\varepsilon + i\partial_y(a_\varepsilon(y,t)\partial_y z_\varepsilon) + \varepsilon\partial_y^4 z_\varepsilon = f_\varepsilon\,, \\ z_\varepsilon(0) = z_{0,\varepsilon}. \end{gathered} \tag{3.16}$$

We claim that $(z_\varepsilon)_{\varepsilon>0}$ is a *Cauchy sequence* in $\mathcal{C}([0,T];H^k)$. We first notice that (z_ε) is uniformly bounded in $\mathcal{C}([0,T];H^k)$. Indeed, combining lemmas 3.5 and 3.6 with estimate (3.14) leads to

$$\begin{aligned} \|z_\varepsilon(t)\|_{H^k} \leq C\Big(\|z_0\|_{H^k} + \|f(\tau)\|_{L^1_t(H^k)}\Big)\exp\Big(C\big(\|\partial_y^2 a\|^2_{L^1_{t+\varepsilon}(L^\infty)} \\ + \|\partial_t \log a\|_{L^1_{t+\varepsilon}(L^\infty)} + \|\partial_y^2 a\|_{L^1_{t+\varepsilon}(H^{k-1})}\big)\Big) \end{aligned} \tag{3.17}$$

for some C depending only on k, $\underline{a}$ and $\widetilde{a}$. Let $0 < \delta \leq \varepsilon$ and $\zeta_\varepsilon^\delta := z_\varepsilon - z_\delta$. Since ζ_ε^δ satisfies

$$\begin{gathered} \partial_t \zeta_\varepsilon^\delta + i\partial_y(a_\delta \partial_y \zeta_\varepsilon^\delta) + \delta\partial_y^4\zeta_\varepsilon^\delta = \big(f_\varepsilon - f_\delta\big) + (\delta - \varepsilon)\partial_y^4 z_\varepsilon + i\partial_y\big((a_\delta - a_\varepsilon)\partial_y z_\varepsilon\big), \\ \zeta^\delta_{\varepsilon\,|t=0} = z_{0,\varepsilon} - z_{0,\delta}, \end{gathered}$$

inequality (3.14) and lemmas 3.5, 3.6 ensure that

$$\begin{aligned} \|\zeta_\varepsilon^\delta(t)\|_{H^k} \leq C\exp\Big(C\big(\|\partial_y^2 a\|^2_{L^1_{t+\varepsilon}(L^\infty)} + \|\partial_t \log a\|_{L^1_{t+\varepsilon}(L^\infty)} + \|\partial_y^2 a\|_{L^1_{t+\varepsilon}(H^{k-1})}\big)\Big) \\ \times\Big(\|z_{0,\varepsilon} - z_{0,\delta}\|_{H^k} + \|f_\varepsilon - f_\delta\|_{L^1_t(H^k)} + \varepsilon\|\partial_y^4 z_\varepsilon\|_{L^1_t(H^k)} \\ + \|(a_\delta - a_\varepsilon)\partial_y z_\varepsilon\|_{L^1_t(H^{k+1})}\Big) \end{aligned} \tag{3.18}$$

for some C depending only on k, $\underline{a}$ and $\widetilde{a}$. By lemmas 3.5, 3.6, the first two terms of the right-hand side tend to zero as ε and δ approach zero.

Let us admit for a while the following lemma.

Lemma 3.7. *There exists a constant C depending only on k, $\underline{a}$ and $\widetilde{a}$ and such that*

$$\begin{aligned} &\|z_\varepsilon(t)\|_{H^{k+2}} + \varepsilon^{1/2}\|z_\varepsilon\|_{L^2_t(H^{k+4})} \\ &\leq C\frac{1}{\eta(\varepsilon)^2}\Big(\|z_0\|_{H^k} + \|f\|_{L^1_t(H^k)} + \|\partial_y z_\varepsilon\|_{L^\infty_t(L^\infty)}\|\partial_y^3 a\|_{L^1_{t+\varepsilon}(H^{k-2})}\Big) \\ &\times\exp\Big(C\big(\|\partial_y^2 a\|^2_{L^1_{t+\varepsilon}(L^\infty)} + \|\partial_t \log a\|_{L^1_{t+\varepsilon}(L^\infty)} + \|\partial_y^2 a\|_{L^1_{t+\varepsilon}(L^\infty)}\big)\Big). \end{aligned}$$

On the one hand, since we assumed that $k \geq 2$, Sobolev embedding combined with inequality (3.17) implies that $\partial_y z_\varepsilon$ is uniformly bounded in $L^\infty(0,T;L^\infty)$. Lemma 3.7 then supplies a constant $C_T \geq 0$ such that for all $\varepsilon > 0$, we have

$$\varepsilon \int_0^t \|\partial_y^4 z_\varepsilon\|_{H^k}\,d\tau = C_T[\eta(\varepsilon)]^{-2}\sqrt{\varepsilon}. \tag{3.19}$$

On the other hand, according to inequality (5.2),

$$\begin{aligned}&\|(a_\delta - a_\varepsilon)\partial_y z_\varepsilon\|_{L^1_t(H^{k+1})}\\ &\lesssim \|a_\delta - a_\varepsilon\|_{L^1_t(L^\infty)}\|\partial_y z_\varepsilon\|_{L^\infty_t(H^{k+1})} + \|\partial_y z_\varepsilon\|_{L^\infty_t(L^\infty)}\|\partial_y^2(a_\delta - a_\varepsilon)\|_{L^1_t(H^{k-1})}.\end{aligned}$$

Since $\partial_y z_\varepsilon$ is uniformly bounded in $L^\infty(0,T;L^\infty)$ and $\partial_y^2 a \in L^1(0,T;H^{k-1})$, the Lebesgue dominated convergence theorem entails that the last term approaches 0 when $\varepsilon, \delta \to 0$.

The estimation of $\|a_\delta - a_\varepsilon\|_{L^1_T(L^\infty)}$ relies on the identity

$$a_\delta - a_\varepsilon = \chi_{\eta(\varepsilon)} *_y \big[(\theta_\delta - \theta_\varepsilon) *_t a\big] + \theta_\delta *_t \big[(\chi_{\eta(\delta)} - \chi_{\eta(\varepsilon)}) *_y a\big].$$

From lemma 3.5, we gather

$$\|a_\delta - a_\varepsilon\|_{L^1_T(L^\infty)} \leq \|(\theta_\delta - \theta_\varepsilon) *_t a\|_{L^1_T(L^\infty)} + \|(\chi_{\eta(\delta)} - \chi_{\eta(\varepsilon)}) *_y a\|_{L^1_T(L^\infty)}.$$

The first term may be bounded by

$$\alpha(\varepsilon) := \sup_{0<\delta\leq\delta'\leq\varepsilon} \|(\theta_\delta - \theta_{\delta'}) *_t a\|_{L^1_T(L^\infty)}$$

which, in view of lemma 3.5 is a nondecreasing positive function tending to 0 in 0.

To bound the other term, one can argue by interpolation and write

$$\begin{aligned}&\|(\chi_{\eta(\delta)} - \chi_{\eta(\varepsilon)}) *_y a\|_{L^\infty}\\ &\leq \|(\chi_{\eta(\delta)} - \chi_{\eta(\varepsilon)}) *_y a\|_{L^2}^{1/2}\|(\chi_{\eta(\delta)} - \chi_{\eta(\varepsilon)}) *_y \partial_y a\|_{L^2}^{1/2},\\ &\leq C\|(\chi_{\eta(\delta)} - \chi_{\eta(\varepsilon)}) *_y a\|_{L^2}^{\frac{2k+1}{2k+2}}\|(\chi_{\eta(\delta)} - \chi_{\eta(\varepsilon)}) *_y \partial_y^2 a\|_{H^{k-1}}^{\frac{1}{2k+2}}.\end{aligned}$$

Hence, using lemma 3.6,

$$\|(\chi_{\eta(\delta)} - \chi_{\eta(\varepsilon)}) *_y a\|_{L^\infty} \leq C[\eta(\varepsilon)]^{k+\frac12}\|\partial_y^2 a\|_{H^{k-1}}.$$

Combining this with lemma 3.7, we conclude that for some $C_T \geq 0$,

$$\|a_\delta - a_\varepsilon\|_{L^1_t(L^\infty)}\|\partial_y z_\varepsilon\|_{L^\infty_t(H^{k+1})} \leq C_T\Big([\eta(\varepsilon)]^{k-\frac32} + \alpha(\varepsilon)[\eta(\varepsilon)]^{-2}\Big).$$

Of course, with no loss of generality, one can assume that $\alpha(\varepsilon) \geq \varepsilon$. Now, choosing $\eta(\varepsilon) = \alpha(\varepsilon)^{\frac16}$ and coming back to (3.19), we conclude that (z_ε) has some limit z in $\mathcal{C}([0,T];H^k)$ when ε approaches 0. □

For the proof of Lemma 3.7, the starting point is (3.13) with $0 \leq \ell \leq k+2$. All the terms of the right-hand side are going to be bounded like in the proof of (3.14) except for

$$\sum_{j=1}^{\ell-1} \binom{\ell+1}{j-1} \Im \int a_\varepsilon^{(\ell+2-j)} a_\varepsilon^\ell z_\varepsilon^{(j)} \overline{z_\varepsilon^{(\ell)}}\,\mathrm{d}y$$

that we are now going to estimate for $2 \leq \ell \leq k+2$. By Hölder inequality and (5.1), we have for some C depending only on k, $\underline{a}$ and $\widetilde{a}$,

$$\begin{aligned}&\Big|\int a_\varepsilon^{(\ell+2-j)} a_\varepsilon^\ell z^{(j)} \overline{z_\varepsilon^{(\ell)}}\,\mathrm{d}y\Big|\\ &\leq C\|a_\varepsilon^{\ell/2} z_\varepsilon^{(\ell)}\|_{L^2}\Big(\|\partial_y^2 a_\varepsilon\|_{L^\infty}\|z_\varepsilon\|_{H^\ell} + \|\partial_y z_\varepsilon\|_{L^\infty}\|\partial_y^3 a_\varepsilon\|_{H^{\ell-2}}\Big).\end{aligned}$$

Hence, summing inequalities (3.13) for $\ell = 0,\dots,k+2$ and using the above inequality,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{\ell=0}^{k+2}\|a_\varepsilon^{\ell/2} z_\varepsilon^{(\ell)}\|_{L^2}^2 + \frac{\varepsilon}{2}\|\partial_y^2\big(a_\varepsilon^{\ell/2} z_\varepsilon^{(\ell)}\big)\|_{L^2}^2\\ &\lesssim \Re\sum_{\ell=0}^{k+2}\int a_\varepsilon^\ell z_\varepsilon^{(\ell)} f_\varepsilon^{(\ell)}\,\mathrm{d}y + \|\partial_y z_\varepsilon\|_{L^\infty}\|\partial_y^3 a_\varepsilon\|_{H^k}\|z_\varepsilon\|_{H^{k+2}}\\ &\quad + \big(\|\partial_y^2 a_\varepsilon\|_{L^\infty} + \varepsilon\|\partial_y^2 a_\varepsilon\|_{L^\infty}^2 + \|\partial_t \log a_\varepsilon\|_{L^\infty}\big)\|z_\varepsilon\|_{H^{k+2}}^2.\end{aligned}$$

Since $\partial_y^2\big(a_\varepsilon^{\ell/2} z_\varepsilon^{(\ell)}\big) = \partial_y^2\big(a_\varepsilon^{\ell/2}\big) z_\varepsilon^{(\ell)} + 2\partial_y\big(a_\varepsilon^{\ell/2}\big)\partial_y z_\varepsilon^{(\ell)} + a_\varepsilon^{\ell/2}\partial_y^2 z_\varepsilon^{(\ell)}$, for some C depending only on k, $\underline{a}$ and $\widetilde{a}$, we clearly have

$$\|z_\varepsilon\|_{H^{k+4}}^2 \leq C\Big(\|\partial_y^2 a_\varepsilon\|_{L^\infty}^2\sum_{\ell=0}^{k+2}\|a_\varepsilon^{\ell/2} z_\varepsilon^{(\ell)}\|_{L^2}^2 + \sum_{\ell=0}^{k+2}\|\partial_y^2\big(a_\varepsilon^{\ell/2} z_\varepsilon^{(\ell)}\big)\|_{L^2}^2\Big).$$

Hence, denoting $Z_\varepsilon^2 := \sum_{\ell=0}^{k+2}\|a_\varepsilon^{\ell/2} z_\varepsilon^{(\ell)}\|_{L^2}^2$ and $\kappa := C^{-1}$, we end up with

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}Z_\varepsilon^2 + \frac{\kappa\varepsilon}{2}\|z_\varepsilon\|_{H^{k+4}}^2 &\lesssim Z_\varepsilon\Big(\|f_\varepsilon\|_{H^{k+2}} + \|\partial_y z_\varepsilon\|_{L^\infty}\|\partial_y^3 a_\varepsilon\|_{H^k}\Big)\\ &\quad + \big(\varepsilon\|\partial_y^2 a_\varepsilon\|_{L^\infty}^2 + \|\partial_y^2 a_\varepsilon\|_{L^\infty} + \|\partial_t \log a_\varepsilon\|_{L^\infty}\big)Z_\varepsilon^2.\end{aligned}$$

According to lemmas 3.5 and 3.6, we have

$$\begin{gathered}\int_0^t\Big(\varepsilon\|\partial_y^2 a_\varepsilon\|_{L^\infty}^2 + \|\partial_y^2 a_\varepsilon\|_{L^\infty} + \|\partial_t \log a_\varepsilon\|_{L^\infty}\Big)\,d\tau\\ \leq C\left(\int_0^{t+\varepsilon}\|\partial_y^2 a\|_{L^\infty}^2\,d\tau\right)^2 + C\int_0^{t+\varepsilon}\Big(\|\partial_y^2 a\|_{L^\infty} + \|\partial_t \log a\|_{L^\infty}\Big)\,d\tau,\\ \|\partial_y^3 a_\varepsilon\|_{L_t^1(H^k)} \leq C[\eta(\varepsilon)]^{-2}\|\partial_y^3 a\|_{L_{t+\varepsilon}^1(H^{k-2})},\\ Z_\varepsilon(0) \leq C[\eta(\varepsilon)]^{-2}\|z_0\|_{H^k},\\ \|f_\varepsilon\|_{L_t^1(H^{k+2})} \leq C[\eta(\varepsilon)]^{-2}\|f\|_{L_t^1(H^k)},\end{gathered}$$

so that Gronwall lemma yields the desired inequality.

4. The Korteweg model

This section is devoted to the proof of local well-posedness for (1.9).

4.1. **Uniqueness and continuity with respect to the data.** Let us start with the proof of uniqueness in theorem 2.2. This is a straightforward corollary of the proposition below.

Proposition 4.1. *Under the assumption* (H1) *with* $k = 2$, *let* (u_1, v_1) *and* (u_2, v_2) *be two solutions of* (1.9) *on* $[0,T]\times\mathbb{R}$ *with* v_1 *and* v_2 *both taking values in a compact subset* K *of* J_v. *Let* $\delta u := u_2 - u_1$ *and* $\delta v := v_2 - v_1$. *Assume that* δu *belongs to* $\mathcal{C}([0,T];L^2)$, δv *belongs to* $\mathcal{C}([0,T];H^1)$ *and that in addition,* $\partial_y u_1 \in L^1(0,T;H^1)$, $\partial^2_{yy} v_1 \in L^1(0,T;H^1)$ *and* $\partial_y v_1 \in L^\infty(\mathbb{R}\times[0,T])$. *Denote* $z_j = u_j + iw_j$ *with* $w_j := -\alpha(v_j)\partial_y v_j$ *for* $j = 1, 2$, *and* $\delta z := z_2 - z_1$. *There exists a constant* C *depending only on* K *and on the functions* α *and* q, *such that*

$$\|\delta v(t)\|^2_{L^2} + \|\delta z(t)\|^2_{L^2} \le \Big(\|\delta v(0)\|^2_{L^2} + \|\delta z(0)\|^2_{L^2}\Big) e^{C\int_0^t (1+\|\partial_y v_1\|_{L^\infty})(1+\|\partial_y z_1\|_{H^1})\,d\tau}.$$

Proof. Taking the difference of the systems satisfied by (v_1, z_1) and (v_2, z_2) we easily compute that

$$\partial_t \delta v = \partial_y \delta u, \tag{4.1}$$

$$\partial_t \delta z + i\partial_y(\alpha(v_2)\partial_y \delta z) = f \tag{4.2}$$

with

$$\begin{aligned} f &= (q(v_2) - q(v_1))\Im z_1 + q(v_2)\Im\delta z \\ &\quad + i\partial_y(\alpha(v_1) - \alpha(v_2))\partial_y z_1 + i(\alpha(v_1) - \alpha(v_2))\partial^2_{yy} z_1. \end{aligned}$$

Multiplying (4.1) by δv, and integrating by parts the right-hand side term, we readily get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta v(t)\|^2_{L^2} &= -\int \delta u \partial_y \delta v \,\mathrm{d}y \\ &= \int \frac{\delta w}{\alpha(v_2)}\delta u \,\mathrm{d}y + \int w_1\Big(\frac{1}{\alpha(v_2)} - \frac{1}{\alpha(v_1)}\Big)\delta u \,\mathrm{d}y. \end{aligned} \tag{4.3}$$

On the other hand, Equation (4.2) is of the form (3.1) with $z = \delta z$, $a = \alpha(v_2)$ so we can apply the identity (3.2). This yields

$$\begin{aligned} \frac{1}{2}\frac{d}{\mathrm{d}t}\|\delta z(t)\|^2_{L^2} &= \int \Big(q(v_2) - q(v_1)\Big) w_1 \delta u \,\mathrm{d}y + \int q(v_2)\delta w \delta u \,\mathrm{d}y \\ &\quad + \Im\int \partial_y(\alpha(v_2) - \alpha(v_1))\partial_y z_1 \overline{\delta} z \,\mathrm{d}y \\ &\quad + \Im\int (\alpha(v_2) - \alpha(v_1))\partial^2_{yy} z_1 \overline{\delta} z \,\mathrm{d}y. \end{aligned}$$

Adding (4.3) to this equality, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\delta v(t)\|^2_{L^2} + \|\delta z(t)\|^2_{L^2}\Big) \\ &\le \|\frac{1}{\alpha(v_2)}\|_{L^\infty}\|\delta w\|_{L^2}\|\delta u\|_{L^2} + \|w_1\|_{L^\infty}\|\frac{1}{\alpha(v_2)} - \frac{1}{\alpha(v_1)}\|_{L^2}\|\delta u\|_{L^2} \\ &\quad + \|q(v_2) - q(v_1)\|_{L^2}\|w_1\|_{L^\infty}\|\delta z\|_{L^2} + \|q(v_2)\|_{L^\infty}\|\delta z\|^2_{L^2} \\ &\quad + \|\partial_y(\alpha(v_2) - \alpha(v_1))\|_{L^2}\|\partial_y z_1\|_{L^\infty}\|\delta z\|_{L^2} \\ &\quad + \|\alpha(v_2) - \alpha(v_1)\|_{L^\infty}\|\partial^2_{yy} z_1\|_{L^2}\|\delta z\|_{L^2}. \end{aligned}$$

All terms of the type $\|F(v_2) - F(v_1)\|_{L^2}$ can be bounded by the mean value theorem. Furthermore, we have

$$\partial_y(\alpha(v_2) - \alpha(v_1)) = -\frac{\alpha'(v_2)}{\alpha(v_2)}\delta w - \Big(\frac{\alpha'(v_2)}{\alpha(v_2)} - \frac{\alpha'(v_1)}{\alpha(v_1)}\Big) w_1$$

so that

$$\|\partial_y(\alpha(v_2) - \alpha(v_1))\|_{L^2} \le C(\|\delta w\|_{L^2} + \|w_1\|_{L^\infty} \, \|\delta v\|_{L^2}) \, .$$

Using the Sobolev embedding $H^1 \hookrightarrow L^\infty$, we eventually get the inequality

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\delta v(t)\|_{L^2}^2 + \|\delta z(t)\|_{L^2}^2\Big) \\ &\le C\Big(\|\delta v(t)\|_{L^2}^2 + \|\delta z(t)\|_{L^2}^2\Big)\Big(1 + \|w_1\|_{L^\infty}\Big)\Big(1 + \|\partial_y z_1\|_{H^1}\Big) \end{aligned}$$

for some constant C depending only on the functions α and q, and on K. Applying Gronwall lemma completes the proof. □

Remark 4.2. Let $\overline{z}$ be a smooth solution to (2.1). Combined with the existence theorem, Proposition 4.1 shows the Lipschitz continuity of the mapping

$$\begin{aligned} \overline{z}(0) + H^2 \quad &\longrightarrow \quad \overline{z} + \mathcal{C}([0,T]; L^2) \\ z_0 \qquad &\longmapsto \quad z \quad \text{solution of (2.1) with data } z_0 \, . \end{aligned}$$

Actually, by adapting the method of [8], the above map may be shown to be continuous from $\overline{z}(0) + H^k$ to $\overline{z} + \mathcal{C}([0,T]; H^k)$.

4.2. **Existence of a local solution.** Our aim here is to prove existence in Theorem 2.2. Uniqueness under condition (2.2) is given by Proposition 4.1. Remark that it actually holds in a class which is much larger than the one defined in (2.2). The existence proof proceeds in a classical way through the four main steps:

(a) Construction of approximate solutions,
(b) Uniform a priori estimates in large norm,
(c) Convergence in small norm,
(d) Continuity results.

To simplify the presentation however, we shall introduce the auxiliary function $\lambda := \Lambda(v)$ where Λ stands for a primitive of α.

We obviously have $w = -\partial_y \lambda$. Besides, observing that Λ is by assumption on α monotonically increasing, we may also use its reciprocal $\Lambda^{-1} : J_\lambda \longrightarrow J_v$ and we have $v = \Lambda^{-1}(\lambda)$. Therefore, as far as v is valued in J_v, solving system (1.7) amounts to solving

$$\begin{aligned} \partial_t \lambda - \alpha_\sharp(\lambda)\partial_y u = 0 \, , \\ \partial_t z + i\partial_y \left(\alpha_\sharp(\lambda)\partial_y z\right) = q_\sharp(\lambda) w, \end{aligned} \tag{4.4}$$

with $\alpha_\sharp = \alpha \circ \Lambda^{-1}$, $q_\sharp := q \circ \Lambda^{-1}$, $z = u + iw$, and under the constraint $w = -\partial_y \lambda$.

In what follows, we show that system (4.4) has a local solution (u, λ) with λ valued in J_λ and such that $(\widetilde{u} := u - \overline{u}, \widetilde{\lambda} := \lambda - \overline{\lambda})$ belongs to the space

$$E_T^k := \mathcal{C}\big([0,T]; H^k \times H^{k+1}\big) \cap \mathcal{C}^1\big([0,T]; H^{k-2} \times H^{k-1}\big).$$

According to corollaries 5.5 and 5.6, this gives theorem 2.2.

(a) Construction of approximate solutions. Our approximate scheme will of course take advantage of the linear estimates in Proposition 3.2. The most natural way of computing the iterate z^{n+1} in terms of $(\lambda^n, z^n = u^n + iw^n)$ is to consider the equation

$$\partial_t z^{n+1} + i\partial_y(\alpha_\sharp(\lambda^n)\partial_y z^{n+1}) = q_\sharp(\lambda^n) w^n \, . \tag{4.5}$$

We remark however that since we did not assume that the data belong to a Sobolev space, theorem 3.1 does not supply a solution for (4.5). Hence we are going to work with $\widetilde{z}^n := z^n - \overline{z}$ (with the obvious notation $\overline{z} = \overline{u} + i\overline{w}$ and $\overline{w} = -\alpha(\overline{v})\partial_y \overline{v}$) rather than with z^n.

Regarding the computation of λ^{n+1}, we must keep in mind that we expect in the limit that $w = -\partial_y \lambda$. Therefore it will be suitable to have also $w^n = -\partial_y \lambda^n$ for all $n \in \mathbb{N}$. This induces us to set

$$\lambda^{n+1}(y,t) := \lambda_0(y) + \int_0^t \Big(\alpha_\sharp(\lambda^n)\partial_y u^{n+1}\Big)(y,\tau)d\tau \quad \text{where} \quad \lambda_0(y) := \Lambda(v_0(y)).$$

Indeed, the term $\alpha_\sharp(\lambda^n)\partial_y u^{n+1}$ will be continuous in y and t (by Sobolev embeddings) so that differentiating the above inequality with respect to t yields

$$\partial_t \lambda^{n+1} = \alpha_\sharp(\lambda^n)\partial_y u^{n+1} \tag{4.6}$$

in the classical sense. Then, differentiating with respect to y, we get

$$\partial_t \partial_y \lambda^{n+1} = \partial_y \partial_t \lambda^{n+1} = \partial_y \Big(\alpha_\sharp(\lambda^n)\partial_y u^{n+1}\Big) = -\partial_t w^{n+1}$$

in the weak sense – in fact, this equality will be true in H^{k-2} because $\partial_y \lambda^{n+1} + w^{n+1}$ will belong to $\mathcal{C}^1([0,T]; H^{k-2})$. So, if the initial data for w^{n+1} is chosen so that $w^{n+1}(y,0) = w_0 = -\alpha(v_0)\partial_y v_0 = -\partial_y \lambda_0$, we shall have the identity

$$\partial_y \lambda^{n+1} + w^{n+1} \equiv 0. \tag{4.7}$$

Finally, our scheme is as follows. For the first term $(\lambda^0, z^0 = u^0 + iw^0)$ of the sequence, we just set for all $(y,t) \in \mathbb{R}^2$,

$$\lambda^0(y,t) := \lambda_0(y)\,,\ u^0(y,t) := u_0(y) \quad \text{and} \quad w^0(y,t) := w_0 := -\partial_y \lambda_0(y,t).$$

Obviously λ^0 is valued in J_λ and $(\widetilde{u}^0 := u^0 - \overline{u}^0, \widetilde{\lambda}^0 := \lambda^0 - \overline{\lambda}^0)$ belongs to $\cap_{T>0} E^k_T$.

Then we define $(\lambda^n, z^n = u^n + iw^n)$ inductively in the following way. Suppose (λ^n, z^n) has been defined in such a way that λ^n is valued in J_λ, $w^n = -\partial_y \lambda^n$ and, for some $T > 0$, $(\widetilde{u}^n := u^n - \overline{u}, \widetilde{\lambda}^n := \lambda^n - \overline{\lambda})$ belongs to E^k_T. Then inequality (5.2) and corollaries 5.4 and 5.6 insure that the right-hand side of the first line of the following system

$$\begin{gathered} \partial_t \widetilde{z}^{n+1} + i\partial_y(\alpha_\sharp(\lambda^n)\partial_y \widetilde{z}^{n+1}) = F_1^n + F_2^n + F_3^n + F_4^n, \\ \widetilde{z}^n_{|t=0} = \widetilde{u}_0 + i\widetilde{w}_0, \end{gathered} \tag{4.8}$$

with

$$\begin{gathered} F_1^n := i\partial^2_{yy}\overline{z}\big(\alpha_\sharp(\overline{\lambda}) - \alpha_\sharp(\lambda^n)\big), \quad F_2^n := i\partial_y\overline{\lambda}\partial_y\overline{z}\big(\alpha'_\sharp(\overline{\lambda}) - \alpha'_\sharp(\lambda^n)\big), \\ F_3^n := \big(q_\sharp(\lambda^n) - q_\sharp(\overline{\lambda})\big)\overline{w}, \quad F_4^n := \big(q_\sharp(\lambda^n) + i\partial_y\overline{z}\alpha'_\sharp(\lambda^n)\big)\widetilde{w}^n, \end{gathered}$$

belongs to $\mathcal{C}([0,T]; H^k) \cap \mathcal{C}^1([0,T]; H^{k-2})$.

Hence Theorem 3.3 ensures that system (4.8) has a unique solution $\widetilde{z}^{n+1}$ in $\mathcal{C}([0,T]; H^k) \cap \mathcal{C}^1([0,T]; H^{k-2})$. Then we set $\widetilde{u}^{n+1} := \Re\widetilde{z}^{n+1}$, $u^{n+1} := \widetilde{u}^{n+1} + \overline{u}$, $\widetilde{w}^{n+1} := \Im\widetilde{z}^{n+1}$, $w^{n+1} := \widetilde{w}^{n+1} + \overline{w}$ and $z^{n+1} := u^{n+1} + iw^{n+1}$ so that z^{n+1} satisfies (4.5) as required. Finally, we set

$$\lambda^{n+1}(t) := \lambda_0 + \int_0^t \alpha_\sharp(\lambda^n)\partial_y u^{n+1}\, d\tau.$$

Of course we have to check that λ^{n+1} is valued in J_λ. This will be the case for small enough time.

(b) Uniform a priori estimates in large norm. Let K_λ be a compact subset of J_λ containing $\lambda_0(\mathbb{R})$ and such that $\delta := d(\lambda_0(\mathbb{R}), {}^cK_\lambda) > 0$. Throughout this section, we shall denote by C a "constant" – which may change from line to line – depending only on k, q, α, δ and K_λ. Let $\alpha^n := \alpha_\sharp(\lambda^n)$, $\alpha_0 = \alpha_\sharp(\lambda_0)$,

$$\widetilde{X}^0(t) := \widetilde{X}_0 := \Big(\sum_{j=0}^{k} \|\alpha_0^{\frac{j}{2}} \partial_y^j \widetilde{z}_0\|_{L^2}^2\Big)^{1/2},$$

$$\widetilde{X}^n(t) := \Big(\sum_{j=0}^{k} \|\big((\alpha^{n-1})^{\frac{j}{2}} \partial_y^j \widetilde{z}^n\big)(t)\|_{L^2}^2\Big)^{1/2}$$

for $n \geq 1$. Further define $\widetilde{Y}^n(t) := \|\widetilde{\lambda}^n(t)\|_{L^2} + \widetilde{X}^n(t)$ and $\widetilde{Y}_0 := \|\widetilde{\lambda}_0\|_{L^2} + \widetilde{X}_0$.

We introduce the induction hypothesis

(H_n^T) $\widetilde{z}^n \in \mathcal{C}([0,T]; H^k) \cap \mathcal{C}^1([0,T]; H^{k-2})$, $\widetilde{\lambda}^n \in \mathcal{C}([0,T]; H^{k+1})$ and $\lambda^n(\mathbb{R} \times [0,T]) \subset K_\lambda$ with the inequality

$$\widetilde{Y}^n(t) \leq \frac{\widetilde{Y}_0 \mathrm{e}^{2C_0 \int_0^t \overline{Z}(\tau)\, d\tau}}{1 - \widetilde{Y}_0 \int_0^t \mathrm{e}^{2C_0 \int_0^\tau \overline{Z}(\tau')\, d\tau'}\, d\tau}$$

for some constant $C_0 \geq 0$ depending only on k, α, q and K_λ to be determined afterwards and

$$\overline{Z} := \big(1 + \|\partial_y \overline{u}\|_{H^{k+1}} + \|\overline{w}\|_{H^{k+2}}\big)^3.$$

We are going to show there is a positive time $T > 0$ such that the scheme described in §a) yields a sequence (λ^n, z^n) satisfying (H_n^T) for all $n \in \mathbb{N}$.

Obviously (H_0^T) is satisfied for all T and any $C_0 \geq 0$. Now we fix $n \in \mathbb{N}$ and assume that (H_p^T) is true for all $p \leq n$. According to Proposition 3.2, we have that for all $t \in \mathbb{R}^+$,

$$\widetilde{X}^{n+1}(t) \leq \mathrm{e}^{A_k^n(t)}\Big(\widetilde{X}_0 + \int_0^t \mathrm{e}^{-A_k^n(\tau)} h^n(\tau)\, d\tau\Big), \tag{4.9}$$

with

$$A_k^n(t) := C \int_0^t \Big(\|\partial_t \log \alpha^n(\tau)\|_{L^\infty} + \|\partial_y \alpha^n(\tau)\|_{H^k}\Big)\, d\tau,$$

$$h^n(t) = C \sum_{j=1}^{4} \|F_j^n\|_{H^k}.$$

We need a bound for $A_k^n(t)$. For $n = 0$, the first term $\partial_t \log \alpha^0(\tau)$ is zero. Otherwise, for $n \geq 1$, we have by definition of v^n and α^n and according to (4.6),

$$\partial_t \log \alpha^n = [\log \alpha_\sharp]'(\lambda^n) \alpha_\sharp(\lambda^{n-1}) \partial_y u^n = \frac{\alpha_\sharp'(\lambda^n)}{\alpha_\sharp(\lambda^n)}\, \alpha_\sharp(\lambda^{n-1}) \partial_y u^n\,.$$

Since $\|\partial_y \widetilde{u}^n\|_{L^\infty} \lesssim \|\widetilde{u}^n\|_{H^k}$, using ($\mathrm{H}_{n-1}^T$) and ($\mathrm{H}_n^T$), we have that for all $t \in [0,T]$,

$$\|\partial_t \log \alpha^n\|_{L^\infty} \leq C\big(\widetilde{X}^n(t) + \|\partial_y \overline{u}(t)\|_{L^\infty}\big)$$

for some $C = C(k, \alpha, \underline{\lambda}, \widetilde{\lambda})$. The second term in $A_k^n(t)$, $\partial_y \alpha^n = \alpha_\sharp'(\lambda^n)\partial_y \lambda^n$ can be bounded using inequality (5.5) in the appendix, which yields

$$\|\partial_y \alpha^n\|_{H^k} \le C\|w^n\|_{H^k} \sum_{j=0}^{k} \|\lambda^n\|_{L^\infty}^j \|\alpha_\sharp^{(j+1)}(\lambda^n)\|_{L^\infty} \le C\|w^n\|_{H^k}.$$

Hence, we eventually obtain

$$\forall t \in [0,T],\ A_k^n(t) \le C \int_0^t \Big(\widetilde{X}^n(\tau) + \|\partial_y \overline{u}(\tau)\|_{L^\infty} + \|\overline{w}\|_{H^k} \Big)\, d\tau. \tag{4.10}$$

Let us now bound the source term $h^n(t)$. Using the fact that H^k is an algebra and corollary 5.6, we get

$$\begin{aligned}\|F_1^n\|_{H^k} &\lesssim \|\partial_{yy}^2 \overline{z}\|_{H^k} \|\alpha_\sharp(\overline{\lambda}) - \alpha_\sharp(\lambda^n)\|_{H^k}, \\ &\lesssim \|\partial_{yy}^2 \overline{z}\|_{H^k} \big(\|\widetilde{\lambda}^n\|_{H^k} + \|\widetilde{\lambda}^n\|_{L^\infty} (\|\overline{w}\|_{H^{k-1}} + \|\widetilde{w}^n\|_{H^{k-1}}) \big).\end{aligned}$$

Hence, as $H^k \hookrightarrow L^\infty$,

$$\|F_1^n\|_{H^k} \lesssim \|\partial_{yy}^2 \overline{z}\|_{H^k} \big(1 + \|\overline{w}\|_{H^{k-1}}\big) \|\widetilde{\lambda}^n\|_{H^k}. \tag{4.11}$$

From similar computations, we get

$$\|F_2^n\|_{H^k} \lesssim \|\partial_y \overline{z}\|_{H^k} \|\overline{w}\|_{H^k} \big(1 + \|\overline{w}\|_{H^{k-1}}\big) \|\widetilde{\lambda}^n\|_{H^k}, \tag{4.12}$$

$$\|F_3^n\|_{H^k} \lesssim \|\overline{w}\|_{H^k} \big(1 + \|\overline{w}\|_{H^{k-1}}\big) \|\widetilde{\lambda}^n\|_{H^k}. \tag{4.13}$$

We remark that here $\alpha_\sharp$ (resp. $q_\sharp$) has $k+2$ (resp. $k+1$) locally bounded derivatives. To bound F_4^n, we make use of inequality (5.6) which yields

$$\begin{aligned}\|q_\sharp(\lambda^n)\widetilde{w}^n\|_{H^k} &\lesssim \|\widetilde{w}^n\|_{H^k} + \|\widetilde{\lambda}^n\|_{L^\infty} \|w^n\|_{H^k}, \\ &\lesssim \|\widetilde{w}^n\|_{H^k} + \|\overline{w}\|_{H^k} \|\widetilde{\lambda}^n\|_{H^{k+1}}.\end{aligned}$$

Similar computations enable us to handle the term $\alpha_\sharp'(\lambda^n)\widetilde{w}^n$ so that we end up with

$$\|F_4^n\|_{H^k} \lesssim \big(1 + \|\partial_y \overline{z}\|_{H^k}\big)\big(\|\widetilde{w}^n\|_{H^k} + \|\overline{w}\|_{H^k} \|\widetilde{\lambda}^n\|_{H^{k+1}}\big). \tag{4.14}$$

Plugging inequalities (4.10), (4.11), (4.12), (4.13) and (4.14) in (4.9), we conclude that

$$\begin{aligned}\widetilde{X}^{n+1}(t) \le \exp\Big(C \int_0^t \big(\widetilde{X}^n + \|\partial_y \overline{u}\|_{L^\infty} + \|\overline{w}\|_{H^k}\big)\, d\tau \Big) \\ \times \Big(\widetilde{X}_0 + C \int_0^t \mathrm{e}^{-C \int_0^\tau \big(\widetilde{X}^n + \|\partial_y \overline{u}\|_{L^\infty} + \|\overline{w}\|_{H^k}\big)\, d\tau'} \\ \times \Big(\widetilde{X}^n + \big(1 + \|\overline{w}\|_{H^{k-1}}\big)\big(\|\overline{w}\|_{H^k}\big(1 + \|\overline{w}\|_{H^{k-1}} \\ + \|\partial_y \overline{z}\|_{H^k}\big) + \|\partial_{yy}^2 \overline{z}\|_{H^k}\big)\widetilde{Y}^n \Big)\, d\tau \Big).\end{aligned} \tag{4.15}$$

To complete the estimates, we still have to find a bound for $\|\widetilde{\lambda}^n\|_{L^2}$. This is quite straightforward. Indeed, we have

$$\widetilde{\lambda}^{n+1}(t) = \widetilde{\lambda}_0 + \int_0^t \Big(\big(\alpha_\sharp(\lambda^n) - \alpha_\sharp(\overline{\lambda})\big)\partial_y \overline{u} + \alpha_\sharp(\lambda^n)\partial_y \widetilde{u}^{n+1} \Big)\, d\tau,$$

whence

$$\|\widetilde{\lambda}^{n+1}(t)\|_{L^2} \le \|\widetilde{\lambda}_0\|_{L^2} + C \int_0^t \Big(\|\partial_y \overline{u}\|_{L^\infty} \|\widetilde{\lambda}^n\|_{L^2} + \|\partial_y \widetilde{u}^{n+1}\|_{L^2} \Big)\, d\tau.$$

Adding this latter inequality to (4.15), we get

$$\begin{aligned}&\widetilde{Y}^{n+1}(t)\\&\leq \mathrm{e}^{C\int_0^t\left(\widetilde{Y}^n+\overline{Z}\right)\,d\tau}\left(\widetilde{Y}_0+C\int_0^t \mathrm{e}^{-C\int_0^\tau\left(\widetilde{Y}^n+\overline{Z}\right)\,d\tau'}\overline{Z}\widetilde{Y}^n\,d\tau\right)+C\int_0^t \widetilde{Y}^{n+1}(\tau)\,d\tau,\end{aligned}$$

with $\overline{Z}:=\left(1+\|\partial_y\overline{u}\|_{H^{k+1}}+\|\overline{w}\|_{H^{k+2}}\right)^3$.

Now, applying Gronwall lemma, we get, up to a change of C,

$$\widetilde{Y}^{n+1}(t)\leq \mathrm{e}^{C\int_0^t\left(\widetilde{Y}^n+\overline{Z}\right)\,d\tau}\left(\widetilde{Y}_0+C\int_0^t \mathrm{e}^{-C\int_0^\tau\left(\widetilde{Y}^n+\overline{Z}\right)\,d\tau'}\overline{Z}\widetilde{Y}^n\,d\tau\right). \tag{4.16}$$

We choose for C_0 the constant C appearing in the above inequality (note that this choice is *independent* of n) and we assume that T satisfies

$$C_0\widetilde{Y}_0\int_0^T \mathrm{e}^{2C_0\int_0^t\overline{Z}(\tau)\,d\tau}\,dt<1. \tag{4.17}$$

Taking advantage of (H_n^T), straightforward calculations yield for $0\leq s\leq t\leq T$,

$$\mathrm{e}^{C_0\int_s^t\widetilde{Y}^n(\tau)\,d\tau}\leq \frac{1-C_0\widetilde{Y}_0\int_0^s \mathrm{e}^{2C_0\int_0^\tau\overline{Z}\,d\tau'}\,d\tau}{1-C_0\widetilde{Y}_0\int_0^t \mathrm{e}^{2C_0\int_0^\tau\overline{Z}\,d\tau'}\,d\tau}.$$

Inserting the above inequality in (4.16), we get

$$\widetilde{Y}^{n+1}(t)\leq \frac{\widetilde{Y}_0\mathrm{e}^{C_0\int_0^\tau\overline{Z}\,d\tau'}\,d\tau}{1-C_0\widetilde{Y}_0\int_0^t \mathrm{e}^{2C_0\int_0^\tau\overline{Z}\,d\tau'}\,d\tau}+C_0\widetilde{Y}_0\int_0^t\frac{\overline{Z}\mathrm{e}^{C_0\int_s^t\overline{Z}\,d\tau'}\mathrm{e}^{2C_0\int_0^s\overline{Z}\,d\tau'}}{1-C_0\widetilde{Y}_0\int_0^t \mathrm{e}^{2C_0\int_0^\tau\overline{Z}\,d\tau'}\,d\tau}\,ds,$$

whence

$$\widetilde{Y}^{n+1}(t)\leq \frac{\widetilde{Y}_0\mathrm{e}^{2C_0\int_0^\tau\overline{Z}\,d\tau'}\,d\tau}{1-C_0\widetilde{Y}_0\int_0^t \mathrm{e}^{2C_0\int_0^\tau\overline{Z}\,d\tau'}\,d\tau}$$

as required. With our definition of λ^{n+1}, we have (up to a change of C_0)

$$\begin{aligned}\|\widetilde{\lambda}^{n+1}(t)-\widetilde{\lambda}_0\|_{L^\infty}&\leq C_0\int_0^t\left(\|\widetilde{\lambda}^n\|_{L^\infty}\|\partial_y\overline{u}\|_{L^\infty}+\|\partial_y\widetilde{u}^{n+1}\|_{L^\infty}\right)ds,\\&\leq C_0\int_0^t\left(1+\|\partial_y\overline{u}\|_{L^\infty}\right)\left(\widetilde{Y}^n+\widetilde{Y}^{n+1}\right)ds,\\&\leq\int_0^t\frac{C_0\overline{Z}\widetilde{Y}_0\mathrm{e}^{2C_0\int_0^s\overline{Z}\,d\tau'}}{1-C_0\widetilde{Y}_0\int_0^s \mathrm{e}^{2C_0\int_0^\tau\overline{Z}\,d\tau'}\,d\tau}\,ds,\\&\leq -\sup_{\tau\in[0,t]}\overline{Z}(\tau)\,\log\left(1-C_0\widetilde{Y}_0\int_0^t \mathrm{e}^{2C_0\int_0^\tau\overline{Z}\,d\tau'}\,d\tau\right).\end{aligned}$$

Therefore, the condition

$$\int_0^T \mathrm{e}^{2C_0\int_0^t\overline{Z}\,d\tau}\,dt\leq\frac{1-exp(-\delta/\sup_{t\in[0,T]}\overline{Z}(t))}{C_0\widetilde{Y}_0} \tag{4.18}$$

which is a stronger condition than (4.17) ensures that $\lambda^{n+1}\subset K_\lambda$ on $[0,T]$.

(c) Convergence in small norm. We aim at proving that $(z^n)_{n\in\mathbb{N}}$ is convergent in the affine space $\overline{z}+\mathcal{C}([0,T];L^2)$ and that $(\lambda^n)_{n\in\mathbb{N}}$ is convergent in the affine space $\overline{\lambda}+\mathcal{C}([0,T];L^2)$.

Let $\delta z^n := z^{n+1}-z^n$, $\delta u^n := u^{n+1}-u^n$, $\delta w^n := w^{n+1}-w^n$ and $\delta\lambda^n := \lambda^{n+1}-\lambda^n$. On the one hand, we have

$$\begin{aligned}
&\partial_t\delta z^n + i\partial_y(\alpha_\sharp(\lambda^n)\partial_y\delta z^n)\\
&= -i\partial_y z^n\partial_y(\alpha_\sharp(\lambda^n)-\alpha_\sharp(\lambda^{n-1})) - i\partial^2_{yy}z^n(\alpha_\sharp(\lambda^n)-\alpha_\sharp(\lambda^{n-1}))\\
&\quad + q_\sharp(\lambda^n)\delta w^{n-1} + (q_\sharp(\lambda^n)-q_\sharp(\lambda^{n-1}))w^{n-1}.
\end{aligned}$$

Hence, using the basic L^2 estimate

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta z^n\|^2_{L^2} \le \|\delta z^n\|_{L^2}\Big(&\|\partial_y z^n\|_{L^\infty}\|\partial_y(\alpha_\sharp(\lambda^n)-\alpha_\sharp(\lambda^{n-1}))\|_{L^2}\\
&+ \|\partial^2_{yy}z^n\|_{L^2}\|\alpha_\sharp(\lambda^n)-\alpha_\sharp(\lambda^{n-1})\|_{L^\infty} + \|q_\sharp(\lambda^n)\|_{L^\infty}\|\delta w^{n-1}\|_{L^2}\\
&+ \|q_\sharp(\lambda^n)-q_\sharp(\lambda^{n-1})\|_{L^2}\|w^{n-1}\|_{L^\infty}\Big).
\end{aligned}$$

By the Sobolev embedding $H^1\hookrightarrow L^\infty$, this implies

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta z^n\|^2_{L^2}\\
&\le \|\delta z^n\|_{L^2}\Big(\|\partial_y z^n\|_{H^1}\|\alpha_\sharp(\lambda^n)-\alpha_\sharp(\lambda^{n-1})\|_{H^1}\\
&\quad + \|q_\sharp(\lambda^n)\|_{L^\infty}\|\delta w^{n-1}\|_{L^2} + \|q_\sharp(\lambda^n)-q_\sharp(\lambda^{n-1})\|_{L^2}\|w^{n-1}\|_{L^\infty}\Big).
\end{aligned} \tag{4.19}$$

On the other hand, by (4.6), we have

$$\partial_t\delta\lambda^n = (\alpha_\sharp(\lambda^n)-\alpha_\sharp(\lambda^{n-1}))\partial_y u^{n+1} + \alpha_\sharp(\lambda^{n-1})\partial_y\delta u^n$$

so that taking the L^2-scalar product with $\delta\lambda^n$ and integrating by parts in the last term,

$$\begin{aligned}
\frac{1}{2}\frac{d}{\mathrm{d}t}\|\delta\lambda^n\|^2_{L^2} = \int\Big(&(\alpha_\sharp(\lambda^n)-\alpha_\sharp(\lambda^{n-1}))\partial_y u^{n+1}\delta\lambda^n\\
&- \delta u^n\alpha'_\sharp(\lambda^{n-1})\partial_y\lambda^{n-1}\delta\lambda^n - \delta u^n\alpha_\sharp(\lambda^{n-1})\partial_y\delta\lambda^n\Big)\,\mathrm{d}y.
\end{aligned}$$

Now, as $\partial_y\lambda^{n-1} = -w^{n-1}$, straightforward computations yield

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\lambda^n\|^2_{L^2} \le &\|\alpha_\sharp(\lambda^n)-\alpha_\sharp(\lambda^{n-1})\|_{L^2}\|\partial_y u^{n+1}\|_{L^\infty}\|\delta\lambda^n\|_{L^2}\\
&+ \|\delta u^n\|_{L^2}\|\alpha'_\sharp(\lambda^{n-1})\|_{L^\infty}\|w^{n-1}\|_{L^\infty}\|\delta\lambda^n\|_{L^2}\\
&+ \|\delta u^n\|_{L^2}\|\alpha_\sharp(\lambda^{n-1})\|_{L^\infty}\|\delta w^n\|_{L^2}.
\end{aligned} \tag{4.20}$$

Since the sequence $(\lambda^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}\times 0,T)$, corollary 5.6 ensures that there is a constant C such that

$$\max\big(\|\alpha_\sharp(\lambda^n)-\alpha_\sharp(\lambda^{n-1})\|_{L^2}, \|q_\sharp(\lambda^n)-q_\sharp(\lambda^{n-1})\|_{L^2}\big) \le C\|\delta\lambda^{n-1}\|_{L^2},$$

$$\|\alpha_\sharp(\lambda^n)-\alpha_\sharp(\lambda^{n-1})\|_{H^1} \le C\Big(\|w^{n-1}\|_{L^2}+\|w^n\|_{L^2}\Big)\Big(\|\delta\lambda^{n-1}\|_{L^2}+\|\delta w^{n-1}\|_{L^2}\Big).$$

Hence, adding (4.19) and (4.20) yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(Y^n)^2 \leq CY^n\Big((1+\|\partial_y z^n\|_{H^1}+\|\partial_y u^{n+1}\|_{L^\infty}+\|w^{n-1}\|_{L^\infty}\\ +\|w^{n-1}\|_{L^2}+\|w^n\|_{L^2})Y^{n-1}+(1+\|w^{n-1}\|_{L^\infty})Y^n\Big)\end{aligned}$$

with $Y^n(t) := \big(\|\delta\lambda^n\|_{L^2}^2+\|\delta z^n\|_{L^2}^2)\big)^{1/2}$.

Because $(\partial_y u^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H^1)$ and $(w^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H^2)$, we eventually get

$$Y^n(t) \leq C_1\int_0^t Y^{n-1}(\tau)\,d\tau + C_2\int_0^t Y^n(\tau)\,d\tau,$$

whence

$$\mathrm{e}^{-C_2 t}Y^n(t) \leq C_1\int_0^t \mathrm{e}^{-C_2\tau}Y^{n-1}(\tau)\,d\tau.$$

A standard induction argument enables us to conclude that

$$Y^n(t) \leq \frac{C_1^n}{n!}\mathrm{e}^{C_2 t}Y^0(t)\,.$$

The series $\sum_n C_1^n/n!$ being convergent, this shows that $(\widetilde{z}^n)_{n\in\mathbb{N}}$ and $(\widetilde{\lambda}^n)_{n\in\mathbb{N}}$ are Cauchy sequences in the Banach space $\mathcal{C}([0,T];L^2)$. We conclude that $(z^n)_{n\in\mathbb{N}}$ tends to some function z in $\overline{z}+\mathcal{C}([0,T];L^2)$, and that $(\lambda^n)_{n\in\mathbb{N}}$ tends to some function λ in $\overline{\lambda}+\mathcal{C}([0,T];L^2)$. Passing to the limit in the linear equation (4.7) we readily get

$$\partial_y\lambda = -w\,.$$

Furthermore, we have in the limit $\lambda\in J_\lambda$. So we can define $v := \Lambda^{-1}(\lambda)$, and we have $(v-\overline{v})\in\mathcal{C}([0,T];L^2)$. Now, by using the uniform bounds of §b), we have in addition

$$\widetilde{z} := z-\overline{z}\in L^\infty(0,T;H^k)\cap \mathrm{Lip}(0,T;H^{k-2}). \tag{4.21}$$

An interpolation argument shows that for $\varepsilon>0$, $\widetilde{z}^n\to\widetilde{z}$ in $\mathcal{C}([0,T];H^{k-\varepsilon})$ and $v^n\to v$ in $\overline{v}+\mathcal{C}([0,T];H^{k+1-\varepsilon})$. This suffices to show that $w=-\partial_y\Lambda(v) = -\alpha(v)\partial_y v$ and to pass to the limit in (4.6) and (4.8), thus obtaining

$$\begin{gathered}\partial_t\Lambda(v) = \alpha(v)\partial_y u\\ \partial_t z + i\partial_y(\alpha(v)\partial_y z) = q(v)w\,,\\ z_{|t=0} = v_0+iw_0,\end{gathered}$$

Simplifying by $\alpha(v)$ in the first equation, we get $\partial_t v = \partial_y u$.

(d) Continuity results. Using (4.21) and the fact that both z and $\widetilde{z}$ are solutions of (2.1), it can be easily shown that $\partial_t\widetilde{z}+i\partial_y(\alpha(v)\partial_y\widetilde{z})$ belongs to $L^\infty(0,T;H^k)$ (see equation 4.23 below). Hence Theorem 3.3 ensures that $\widetilde{z}$ belongs to $\mathcal{C}([0,T];H^k)$. This new result implies that $\partial_t\widetilde{z}\in\mathcal{C}([0,T];H^{k-2})$. Therefore $\widetilde{z}$ also belongs to $\mathcal{C}^1([0,T];H^{k-2})$. Now, since

$$\partial_t\widetilde{\lambda} = \big(\alpha_\sharp(\lambda)-\alpha_\sharp(\overline{\lambda})\big)\partial_y\overline{u}+\alpha_\sharp(\lambda)\partial_y\widetilde{u},$$

Corollaries 5.4 and 5.6 guarantee that the right-hand side above belongs to the space $\mathcal{C}([0,T];H^{k-1})$ so that $\widetilde{\lambda}\in\mathcal{C}^1([0,T];H^{k-1})$. Since moreover $\partial_y\lambda=-w$ and

$w \in \mathcal{C}([0,T];H^k)$, we thus have $\lambda \in \mathcal{C}([0,T];\overline{\lambda}+H^{k+1})$. Applying corollary 5.6, we conclude that

$$v \in \overline{v} + \mathcal{C}([0,T];H^{k+1}) \cap \mathcal{C}^1([0,T];H^{k-1}).$$

Remark 4.3. The lower bound for the existence time given by (4.18) depends on k, $\underline{b}$, $\widetilde{b}$, α, q, $\|u_0\|_{H^k}$ and $\|\partial_y v_0\|_{H^k}$. In the next subsection, we shall see that for H^k data ($k \geq 2$), the time of existence in H^k is the same as in H^2. Hence inequality (4.18) with $k=2$ provides a lower bound. This proves remark 2.4.

4.3. **Continuation results and life span.** This section is devoted to the proof of a continuation criterion for H^k solutions to (1.7). Let us first explain what we mean by an H^k solution.

Definition 4.4. Under assumption (H1) with $k \geq 2$, assume that (u,v) is a couple of functions of $(y,t) \in \mathbb{R} \times [0,T)$ such that v is valued in J_v. We shall say that (u,v) is a H^k solution of (1.7) on the time interval $[0,T)$ if (u,v) satisfies (1.7) on $\mathbb{R} \times [0,T)$ (in the weak sense) and

$$(u-\overline{u}, v-\overline{v}) \in \mathcal{C}([0,T);H^k \times H^{k+1}) \cap \mathcal{C}^1([0,T);H^{k-2} \times H^{k-1}) \tag{4.22}$$

where $(\overline{u},\overline{v})$ stands for a classical solution of (1.7) on $\mathbb{R}\times\mathbb{R}$ such that for all $(y,t) \in \mathbb{R}^2$,

$$\overline{v}(y,t) \in J_v, \quad \partial_y \overline{u} \in \mathcal{C}(\mathbb{R};H^{k+1}) \quad \text{and} \quad \partial_y \overline{v} \in \mathcal{C}(\mathbb{R};H^{k+2}).$$

For given data (u_0,v_0) such that $u_0 - \overline{u}(0) \in H^k$ and $v_0 - \overline{v}(0) \in H^{k+1}$, we define the lifespan of a H^k solution as the supremum of all T such that (1.7) has a H^k solution on $[0,T)$.

Our main continuation result is based on the following lemma.

Lemma 4.5. *Under assumption* (H1), *let* (u,v) *be an* H^k *solution of* (3.1) *on* $\mathbb{R}\times[0,T)$ *with* v *valued in* $K_v \subset\subset J_v$. *Denote by* $(\overline{u},\overline{v})$ *a classical solution of* (3.1) *such that* (4.22) *is fulfilled. Let* Λ *be a primitive of* α, $\lambda := \Lambda(v)$, $w = -\partial_y\lambda$ *and* $z = u + iw$. *Further define* $\overline{\lambda} := \Lambda(\overline{v})$, $\overline{w} = -\partial_y\overline{\lambda}$, $\overline{z} = \overline{u} + i\overline{w}$, $\widetilde{\lambda} := \lambda - \overline{\lambda}$, $\widetilde{w} := w - \overline{w}$, $\widetilde{u} := u - \overline{u}$ *and* $\widetilde{z} := z - \overline{z}$. *Let*

$$\widetilde{Y}_k(t) := \Big(\|\widetilde{\lambda}\|_{L^2}^2 + \sum_{j=0}^{k} \|(\alpha(v)^{\frac{j}{2}}\widetilde{z}^{(j)})(t)\|_{L^2}^2\Big)^{1/2}.$$

Then there exists a constant C *depending only on* k, q, α *and* K_v, *and such that*

$$\widetilde{Y}_k(t) \leq \widetilde{Y}_k(0) e^{Ct} e^{C\int_0^t \|\partial_y\widetilde{z}(\tau)\|_{L^\infty}\,d\tau} e^{C\int_0^t \{\|\partial_{yy}^2\overline{z}\|_{H^k} + \|\overline{w}\|_{H^k}(1+\|\overline{w}\|_{H^{k-1}})(1+\|\partial_y\overline{z}\|_{H^k})\}\,d\tau}.$$

Proof. Denoting $\alpha_\sharp = \alpha\circ\Lambda^{-1}$ and $q_\sharp = q\circ\Lambda^{-1}$, one easily find that $\widetilde{z}$ solves

$$\begin{aligned}\partial_t\widetilde{z} + i\partial_y(\alpha_\sharp(\lambda)\partial_y\widetilde{z}) &= \underbrace{i\partial_{yy}^2\overline{z}\big(\alpha_\sharp(\overline{\lambda}) - \alpha_\sharp(\lambda)\big)}_{F_1} + \underbrace{i\partial_y\overline{\lambda}\partial_y\overline{z}\big(\alpha'_\sharp(\overline{\lambda}) - \alpha'_\sharp(\lambda)\big)}_{F_2} \\ &\quad + \underbrace{\big(q_\sharp(\lambda) - q_\sharp(\overline{\lambda})\big)\overline{w}}_{F_3} + \underbrace{\big(q_\sharp(\lambda) + i\partial_y\overline{z}\alpha'_\sharp(\lambda)\big)\widetilde{w}}_{F_4}.\end{aligned} \tag{4.23}$$

Hence, summing equalities (3.6) for $j=0,\dots,k$, we get by Cauchy-Schwarz inequality

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{X}_k^2 \le \max(1,\widetilde{\alpha})^{\frac{k}{2}}\widetilde{X}_k\sum_{j=1}^{4}\|F_j\|_{H^k} + \frac{k}{2}\|\partial_t\log\alpha_\sharp(\lambda)\|_{L^\infty}\widetilde{X}_k^2 \\ + \sum_{j=2}^{k}\sum_{\ell=1}^{j-1}\tbinom{j+1}{\ell-1}\widetilde{\alpha}^{\frac{j}{2}}\|\alpha^{\frac{j}{2}}\partial_y^j\widetilde{z}\|_{L^2}\|\partial_y^{j+2-\ell}\big(\alpha_\sharp(\lambda)\big)\partial_y^\ell\widetilde{z}\|_{L^2}\end{aligned} \tag{4.24}$$

with

$$\widetilde{X}_k(t) := \Big(\sum_{j=0}^{k}\|(\alpha(v)^{\frac{j}{2}}\widetilde{z}^{(j)})(t)\|_{L^2}^2\Big)^{1/2}.$$

As $w=-\partial_y\lambda$, $\overline{w}=-\partial_y\overline{\lambda}$ and $\widetilde{w}=-\partial_y\widetilde{\lambda}$, corollary 5.6 combined with the embedding $H^k\hookrightarrow L^\infty$ yields for some constant C depending only on k, $\underline{b}$, $\widetilde{b}$ and α,

$$\begin{aligned}\|F_1\|_{H^k} &\le C\|\partial_{yy}^2\overline{z}\|_{H^k}\|\alpha_\sharp(\lambda)-\alpha_\sharp(\overline{\lambda})\|_{H^k}, \\ &\le C\|\partial_{yy}^2\overline{z}\|_{H^k}\big(1+\|\overline{w}\|_{H^{k-1}}\big)\|\widetilde{\lambda}\|_{H^k}.\end{aligned} \tag{4.25}$$

Similar computations yield

$$\|F_2\|_{H^k} \le C\|\partial_y\overline{z}\|_{H^k}\|\overline{w}\|_{H^k}\big(1+\|\overline{w}\|_{H^{k-1}}\big)\|\widetilde{\lambda}\|_{H^k}, \tag{4.26}$$

$$\|F_3\|_{H^k} \le C\|\overline{w}\|_{H^k}\big(1+\|\overline{w}\|_{H^{k-1}}\big)\|\widetilde{\lambda}\|_{H^k}. \tag{4.27}$$

Regarding F_4, we apply inequality (5.6) to $q_\sharp(\lambda)\partial_y\widetilde{\lambda}$ and $\alpha'_\sharp(\lambda)\partial_y\widetilde{\lambda}$ so that we get

$$\|F_4\|_{H^k} \le C\big(1+\|\partial_y\overline{z}\|_{H^k}\big)\Big(\|\widetilde{w}\|_{H^k}+\|\overline{w}\|_{H^k}\|\widetilde{\lambda}\|_{H^{k+1}}\Big). \tag{4.28}$$

Because $\partial_t\log\alpha_\sharp(\lambda)=\alpha'_\sharp(\lambda)\partial_y u$, we obviously have

$$\|\partial_t\log\alpha_\sharp(\lambda)\|_{L^\infty} \le C\|\partial_y u\|_{L^\infty}. \tag{4.29}$$

For bounding the last term in (4.24), we first use inequality (5.1) in the appendix which implies

$$\|\partial_y^{j+2-\ell}\big(\alpha_\sharp(\lambda)\big)\partial_y^\ell\widetilde{z}\|_{L^2} \lesssim \|\partial_y\widetilde{z}\|_{L^\infty}\|\partial_{yy}^2\big(\alpha_\sharp(\lambda)\big)\|_{H^{j-1}} + \|\partial_{yy}^2\big(\alpha_\sharp(\lambda)\big)\|_{L^\infty}\|\widetilde{z}\|_{H^j}. \tag{4.30}$$

On the one hand,

$$\|\partial_{yy}^2\big(\alpha_\sharp(\lambda)\big)\|_{H^{j-1}}^2 = \sum_{m=2}^{j+1}\|\partial_y^m\alpha_\sharp(\lambda)\|_{L^2}^2,$$

hence according to lemma 5.3,

$$\|\partial_{yy}^2\big(\alpha_\sharp(\lambda)\big)\|_{H^{j-1}} \lesssim \|w\|_{H^j},$$

on the other hand, $\partial_{yy}^2\big(\alpha_\sharp(\lambda)\big)=\alpha''_\sharp(\lambda)(\partial_y\lambda)^2+\alpha'_\sharp(\lambda)\partial_{yy}^2\lambda$ so that, since $\|\partial_y\lambda\|_{L^\infty}^2 \lesssim \|\lambda\|_{L^\infty}\|\partial_{yy}^2\lambda\|_{L^\infty}$, we get

$$\|\partial_{yy}^2\big(\alpha_\sharp(\lambda)\big)\|_{L^\infty} \lesssim \|\partial_y w\|_{L^\infty}.$$

Coming back to (4.30), we end up with

$$\|\partial_y^{j+2-\ell}\big(\alpha_\sharp(\lambda)\big)\partial_y^\ell z\|_{L^2} \lesssim \big(\|\partial_y\widetilde{z}\|_{L^\infty}+\|\overline{w}\|_{H^k}\big)\|\widetilde{z}\|_{H^k}. \tag{4.31}$$

Plugging inequalities (4.25), (4.26), (4.27), (4.28), (4.29) and (4.31) in (4.24), we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{X}_k^2 &\lesssim \big(1+\|\partial_y\widetilde{z}\|_{L^\infty}+\|\partial_y\overline{u}\|_{L^\infty}\big)\widetilde{X}_k^2\\ &\quad+\big(1+\|\overline{w}\|_{H^{k-1}}\big)\big(\|\partial_{yy}^2\overline{z}\|_{H^k}+\|\overline{w}\|_{H^k}(1+\|\partial_y\overline{z}\|_{H^k})\big)\widetilde{X}_k\widetilde{Y}_k.\end{aligned}\tag{4.32}$$

To conclude, we still have to bound $\|\widetilde{\lambda}\|_{L^2}$. For doing so, we use the fact that

$$\partial_t\widetilde{\lambda}=\big(\alpha_\sharp(\lambda)-\alpha_\sharp(\overline{\lambda})\big)\partial_y\overline{u}+\alpha_\sharp(\lambda)\partial_y\widetilde{u},$$

whence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\lambda}\|_{L^2}^2\lesssim\|\partial_y\overline{u}\|_{L^\infty}\|\widetilde{\lambda}\|_{L^2}^2+\|\widetilde{\lambda}\|_{L^2}\|\partial_y\widetilde{u}\|_{L^2}.$$

Adding this last inequality to (4.32), we eventually get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{Y}_k^2\lesssim\Big(1+\|\partial_y\widetilde{z}\|_{L^\infty}+\big(1+\|\overline{w}\|_{H^{k-1}}\big)\big(\|\partial_{yy}^2\overline{z}\|_{H^k}+\|\overline{w}\|_{H^k}(1+\|\partial_y\overline{z}\|_{H^k})\big)\Big)\widetilde{Y}_k^2.$$

Then Gronwall's lemma completes the proof. □

One can now state a continuation result which is very similar to the standard one for quasi-linear hyperbolic symmetric systems.

Proposition 4.6. *Under assumption* (H1)*, let* (u,v) *be a* H^k *solution of* (3.1) *on* $\mathbb{R}\times[0,T)$*. Assume in addition that*

$$\int_0^T\Big(\|\partial_y u(\tau)\|_{L^\infty}+\|\partial_{yy}^2 v(\tau)\|_{L^\infty}\Big)\,d\tau<\infty\quad\text{and}\quad v(\mathbb{R}\times[0,T))\subset\subset J_v.\tag{4.33}$$

Then (u,v) *may be continued beyond* T *into a smooth solution of* (1.7)*.*

Proof. Let (u,v) satisfy the hypotheses of the proposition and denote by $(\overline{u},\overline{v})$ a classical solution of (3.1) such that (4.22) is fulfilled. Introducing $w=-\partial_y\big(\Lambda(v)\big)$, a straightforward interpolation shows that

$$\begin{aligned}&\int_0^T\|\partial_y w(t)\|_{L^\infty}\,dt\\ &\le C\int_0^T\Big(\|\alpha(v)\|_{L^\infty}\|\partial_{yy}^2 v\|_{L^\infty}+\|\alpha'(v)\|_{L^\infty}\|v\|_{L^\infty}\|\partial_{yy}^2 v\|_{L^\infty}\Big)\,dt<\infty.\end{aligned}$$

Since $\partial_y\overline{v}\in\mathcal{C}(\mathbb{R};H^{k+2})$ and $\partial_y\overline{u}\in\mathcal{C}(\mathbb{R};H^{k+1})$, $\int_0^T\big(\|\partial_y\overline{u}(t)\|_{L^\infty}+\|\partial_y\overline{w}(t)\|_{L^\infty}\big)\,dt<\infty$. Therefore, lemma 4.5 may be applied. From it, we get (with an obvious notation)

$$\widetilde{u}\in L^\infty(0,T;H^k),\quad\widetilde{w}\in L^\infty(0,T;H^k)\quad\text{and}\quad\partial_y\widetilde{v}\in L^\infty(0,T;H^k).$$

Let η be a positive time which satisfies (4.18) with $\|\widetilde{z}\|_{L^\infty(0,T;H^k)}$ instead of $\widetilde{Y}_0$. Theorem 2.2 supplies a solution on the time interval $[0,\eta]$ for (1.7) with data $(u(T-\frac{\eta}{2}),v(T-\frac{\eta}{2}))$. By virtue of uniqueness, this solution is a continuation of (u,v) beyond T. □

Because $H^2\hookrightarrow\mathrm{Lip}$, we conclude that the H^k regularity is controlled by the H^2 regularity so that **the time of existence in H^2 is the same as in H^k.**

4.4. **Further comments on the case of a constant profile.** In this section, we briefly review how theorem 2.2 and blow-up criteria may be improved if we restrict ourselves to the case of a *constant profile* $(\overline{u}, \overline{v})$. The main improvement is that we do not have to suppose that v has a limit at $-\infty$ and $+\infty$. Only assumptions on $\partial_y v$ are needed.

Going along the lines of the proof of theorem 2.2, one can observe that the scheme reduces to solving

$$\partial_t \widetilde{z}^{n+1} + i\partial_y(\alpha_\sharp(\lambda^n)\partial_y \widetilde{z}^{n+1}) = q_\sharp(\lambda^n)\widetilde{w}^n,$$
$$\widetilde{z}^n_{|t=0} = \widetilde{v}_0 + i\widetilde{w}_0,$$

with $\widetilde{w}_0 := -\partial_y \lambda_0$, $\lambda_0 = \Lambda(v_0)$ and $\widetilde{u}_0 = u_0 - \overline{u}$. Then we set $u^{n+1} := \overline{u} + \Re z^{n+1}$, $w^{n+1} := \Im z^{n+1}$ and

$$\lambda^{n+1} = \Lambda(v_0) + \int_0^t \alpha(v^n)\partial_y u^{n+1}.$$

Therefore, most of the terms in F_1^n, F_2^n, F_3^n and F_4^n vanish and we end up with the inequality

$$\widetilde{X}^{n+1}(t) \le \mathrm{e}^{C\int_0^t \widetilde{X}^n(\tau)\,d\tau}\Big(\widetilde{X}_0 + C\int_0^t \mathrm{e}^{-C\int_0^\tau \widetilde{X}^n(\tau')\,d\tau'}\widetilde{X}^n(\tau)\,d\tau\Big)$$

for some constant C depending only on k, $\underline{b}$, $\widetilde{b}$, α and q.

Also one has to assume only that q and α have $k+1$ bounded derivatives (instead of $k+2$ in the general case) and no control on $\|\widetilde{\lambda}^n\|_{L^2}$ is needed to close the estimates. Therefore, we eventually get the following existence theorem.

Theorem 4.7. *Let $\overline{u} \in \mathbb{R}$ and $k \ge 2$. Under assumption* (H1) *with $k-1$, let $K_v \subset\subset J_v$, $u_0 \in \overline{u} + H^k$ and $v_0 \in L^\infty$ with $\partial_y v_0 \in H^k$ and $v_0(\mathbb{R}) \subset K_v$. There exists a positive T such that the Cauchy problem associated with the system* (1.9) *and initial data $(u(0), v(0)) = (u_0, v_0)$ has a unique solution (u, v) which satisfies*

$$u - \overline{u}\,,\ \partial_y v \ \in\ \mathcal{C}([0,T]; H^k) \cap \mathcal{C}^1([0,T]; H^{k-2}) \quad \textit{and} \quad v(\mathbb{R}\times[0,T]) \subset J_v \tag{4.34}$$

with moreover,

$$(v - v_0) \in \mathcal{C}([0,T]; H^{k+1}) \cap \mathcal{C}^1([0,T]; H^{k-1}). \tag{4.35}$$

Besides, there exists a constant $C = C(\alpha, q, K_v)$ such that T may be chosen such that

$$T \ge \frac{1}{C}\log\big(1 + \frac{1}{\|u_0 - \overline{u}\|_{H^2} + \|\partial_y v_0\|_{H^2}}\big).$$

Proof. Under the assumptions of theorem 4.7, the estimate of lemma 4.5 reduces to

$$\widetilde{X}_k(t) \le \widetilde{X}_k(0)\mathrm{e}^{Ct}\mathrm{e}^{C\int_0^t \|\partial_y z(\tau)\|_{L^\infty}\,d\tau} \tag{4.36}$$

with C depending only on k, q, α and J_v, $z := (u - \overline{u}) + iw$ and

$$X_k(t) := \Big(\sum_{j=0}^k \|(\alpha(v)^{\frac{j}{2}} z^{(j)})(t)\|_{L^2}^2\Big)^{1/2}.$$

From the above estimate, we gather that the blow-up criterion stated in (4.33) remains true under the assumptions of theorem 4.7 and that the time of existence in H^k is the same as in H^2. Indeed, from (4.36) and Sobolev embeddings we get

$$\widetilde{X}_k(t) \le \widetilde{X}_k(0)\mathrm{e}^{Ct}\mathrm{e}^{C\int_0^t X_2(\tau)\,d\tau}.$$

Hence, $\widetilde{X}_k$ remains bounded as long as $\widetilde{X}_2$ does. Thus the lifespan in H^k is the same as in H^2. Now, the above inequality with $k = 2$ yields

$$\widetilde{X}_2(t) \leq \frac{\widetilde{X}_2(0)e^{Ct}}{1 - \widetilde{X}_2(0)(e^{Ct} - 1)} \quad \text{while} \quad \widetilde{X}_2(0)(e^{Ct} - 1) < 1.$$

This gives the desired lower bound for the life span in H^2. □

5. Appendix

In this section, we state some technical estimates for products or composition of functions which have been used repeatedly throughout the paper.

Most of them are based on the following Gagliardo-Nirenberg inequality.

Lemma 5.1. *Let $k \in \mathbb{N}$ and $j \in \{0, \dots, k\}$. There exists a constant $C_{j,k}$ depending only on j and k such that*

$$\|\partial_y^j v\|_{L^{\frac{2k}{j}}} \leq C_{j,k} \|v\|_{L^\infty}^{1-\frac{j}{k}} \|\partial_y^k v\|_{L^2}^{\frac{j}{k}}.$$

We can now state some tame estimates for the product of two functions.

Lemma 5.2. *Let $k \in \mathbb{N}$ and $j \in \{0, \dots, k\}$. There exist a constant $C_{j,k}$ depending only on j and on k and a constant C_k depending only on k, and such that*

$$\|\partial_y^j u \partial_y^{k-j} v\|_{L^2} \leq C_{j,k} \Big(\|u\|_{L^\infty} \|\partial_y^k v\|_{L^2} + \|v\|_{L^\infty} \|\partial_y^k u\|_{L^2} \Big), \tag{5.1}$$

$$\|uv\|_{H^k} \leq C_k \Big(\|u\|_{L^\infty} \|v\|_{H^k} + \|v\|_{L^\infty} \|\partial_y^k u\|_{L^2} \Big). \tag{5.2}$$

Proof. Because

$$\|uv\|_{H^k} \leq C_k \Big(\|uv\|_{L^2} + \|\partial_y^k (uv)\|_{L^2} \Big),$$

Leibniz formula entails that

$$\|uv\|_{H^k} \leq C_k \Big(\|u\|_{L^\infty} \|v\|_{L^2} + \sum_{j=0}^{k} \|\partial_y^j u \partial_y^{k-j} v\|_{L^2} \Big)$$

so that inequality (5.1) (used repeatedly) yields (5.2). Let us focus on the proof of (5.1). According to Hölder inequality, we have

$$\|\partial_y^j u \partial_y^{k-j} v\|_{L^2} \leq \|\partial_y^j u\|_{L^{\frac{2k}{j}}} \|\partial_y^{k-j} v\|_{L^{\frac{2k}{k-j}}}.$$

This obviously yields (5.1) if $j = 0$ or k. Else, using lemma 5.1, we get

$$\|\partial_y^j u \partial_y^{k-j} v\|_{L^2} \leq C_{j,k} \Big(\|u\|_{L^\infty} \|\partial_y^k v\|_{L^2} \Big)^{1-\frac{j}{k}} \Big(\|v\|_{L^\infty} \|\partial_y^k u\|_{L^2} \Big)^{j/k},$$

and Young inequality leads to (5.1). □

Let us now state estimates in Sobolev spaces for the composition of functions.

Lemma 5.3. *Let $k \geq 1$ and F be in $W^{k,\infty}_{\rm loc}(\mathbb{R})$. There exists a constant C_k such that for all $v \in L^\infty$ such that $\partial_y^k v \in L^2$, there holds*

$$\|\partial_y^k \big(F(v)\big)\|_{L^2} \leq C_k \|\partial_y^k v\|_{L^2} \sum_{j=0}^{k-1} \|v\|_{L^\infty}^j \|F^{(j+1)}(v)\|_{L^\infty}.$$

Proof. This inequality stems from Faá-di-Bruno's formula:

$$\partial_y^k\big(F(v)\big)=\sum\frac{k!}{i_1!\cdots i_k!}\left(\frac{\partial_y v}{1!}\right)^{i_1}\cdots\left(\frac{\partial_y^k v}{k!}\right)^{i_k}F^{(i_1+\cdots+i_k)}(v)$$

where the sum is over all the $(i_1,\dots,i_k)\in\mathbb{N}^k$ such that $i_1+2i_2+\cdots+ki_k=k$. On the one hand, Hölder inequality gives

$$\|\big(\partial_y v\big)^{i_1}\cdots\big(\partial_y^k v\big)^{i_k}F^{(i_1+\cdots+i_k)}(v)\|_{L^2}\le\|F^{(i_1+\cdots+i_k)}(v)\|_{L^\infty}\prod_{\ell=1}^k\|\partial_y^\ell v\|_{L^{\frac{2k}{\ell}}}^{i_\ell}. \tag{5.3}$$

On the other hand, lemma 5.1 yields for $1\le\ell\le k$,

$$\|\partial_y^\ell v\|_{L^{\frac{2k}{\ell}}}\le C_{\ell,k}\|v\|_{L^\infty}^{1-\frac{\ell}{k}}\|\partial_y^k v\|_{L^2}^{\frac{\ell}{k}}.$$

Inserting this inequality in (5.3) completes the proof of lemma 5.3. □

Corollary 5.4. *Let $F\in W^{k,\infty}_{\rm loc}$, $k\in\mathbb{N}$. There exists a constant C_k so that*

$$\|F(v)w\|_{H^k}\le C_k\Big(\|F(v)\|_{L^\infty}\|w\|_{H^k}+\|w\|_{L^\infty}\|\partial_y^k v\|_{L^2}\sum_{j=0}^{k-1}\|F^{(j+1)}(v)\|_{L^\infty}\|v\|_{L^\infty}^j\Big). \tag{5.4}$$

$$\|F(v)\partial_y v\|_{H^k}\le C_k\|\partial_y v\|_{H^k}\sum_{j=0}^{k}\|F^{(j)}(v)\|_{L^\infty}\|v\|_{L^\infty}^j. \tag{5.5}$$

$$\begin{aligned}&\|F(v)\partial_y w\|_{H^k}\\&\le C_k\Big(\|F(v)\|_{L^\infty}\|\partial_y w\|_{H^k}+\|w\|_{L^\infty}\|\partial_y v\|_{H^k}\sum_{j=0}^{k}\|F^{(j+1)}(v)\|_{L^\infty}\|v\|_{L^\infty}^j\Big).\end{aligned} \tag{5.6}$$

Proof. The three results are obvious if $k=0$ so let us assume that $k\ge1$. Then applying inequality (5.2) yields

$$\|F(v)w\|_{H^k}\le C_k\Big(\|F(v)\|_{L^\infty}\|w\|_{H^k}+\|w\|_{L^\infty}\|\partial_y^k\big(F(v)\big)\|_{L^2}\Big).$$

Lemma 5.3 enables us to bound the last term in the right-hand side. This yields (5.4).

For proving (5.5), we introduce $\mathcal{F}$ a primitive of F. We have

$$\|F(v)\partial_y v\|_{H^k}\le C_k\left(\|F(v)\partial_y v\|_{L^2}+\|\partial_y^{k+1}\big(\mathcal{F}(v)\big)\|_{L^2}\right).$$

Now, applying lemma 5.3 yields the desired result.

For proving (5.6), we first notice that by virtue of Leibniz formula, we have

$$\|F(v)\partial_y w\|_{H^k}\le C_k\sum_{0\le\ell\le j\le k}\|\partial_y^\ell\big(F(v)\big)\partial_y^{j+1-\ell}w\|_{L^2}.$$

Hence, according to (5.1),

$$\begin{aligned}\|F(v)\partial_y w\|_{H^k}&\le C_k\sum_{0\le j\le k}\Big(\|F(v)\|_{L^\infty}\|\partial_y^{j+1}w\|_{L^2}+\|w\|_{L^\infty}\|\partial_y^{j+1}\big(F(v)\big)\|_{L^2}\Big),\\&\le C_k\Big(\|F(v)\|_{L^\infty}\|\partial_y w\|_{H^k}+\|w\|_{L^\infty}\sum_{m=1}^{k+1}\|\partial_y^m\big(F(v)\big)\|_{L^2}\Big),\end{aligned}$$

and applying lemma 5.3 achieves the proof of (5.6). □

Corollary 5.5. *Let I be a bounded interval of $\mathbb{R}$ and F be in $W^{k,\infty}(\mathbb{R};I)$ for some $k\in\mathbb{N}$ and satisfy $F(0)=0$. Let $v\in H^k$ be a I-valued function. There exists a constant $C=C_k$ such that*

$$\|F(v)\|_{H^k}\le C_k\Big(\|F'\|_{L^\infty(I)}\|v\|_{L^2}+\|\partial_y^k v\|_{L^2}\sum_{j=0}^{k-1}\|v\|_{L^\infty}^j\|F^{(j+1)}\|_{L^\infty(I)}\Big).$$

Proof. We first use the fact that

$$\|F(v)\|_{H^k}\le C_k\Big(\|F(v)\|_{L^2}+\|\partial_y^k(F(v))\|_{L^2}\Big).$$

The last term in the right-hand side may be bounded according to lemma 5.3. For bounding the first one, we take advantage of first order Taylor's formula

$$F(v)=v\int_0^1 F'(\tau v)\,d\tau.$$

which obviously implies $\|F(v)\|_{L^2}\le\|F'\|_{L^\infty(I)}\|v\|_{L^2}$. □

Corollary 5.6. *Let I be a bounded interval of $\mathbb{R}$ and F be in $W^{k+1,\infty}(\mathbb{R};I)$ for some $k\in\mathbb{N}$. Let v and w be two I-valued functions such that $\partial_y v$ and $\partial_y w\in H^{k-1}$ and $w-v\in H^k$. Then $F(w)-F(v)$ belongs to H^k and there exists a constant $C=C_k$ such that*

$$\begin{aligned}\|F(w)-F(v)\|_{H^k}\le C_k\Big(&\|F'\|_{L^\infty(I)}\|w-v\|_{H^k}+\|w-v\|_{L^\infty}\Big(\|\partial_y^k v\|_{L^2}+\|\partial_y^k w\|_{L^2}\Big)\\&\times\sum_{j=0}^{k-1}\Big(\|v\|_{L^\infty}+\|w\|_{L^\infty}\Big)^j\|F^{(j+2)}\|_{L^\infty(I)}\Big).\end{aligned}$$

Proof. Arguing by density, it suffices to prove the inequality for smooth I-valued functions. According to first order Taylor's formula, we have

$$F(w)-F(v)=\int_0^1(w-v)F'(v+\tau(w-v))\,d\tau.$$

Therefore,

$$\|F(w)-F(v)\|_{H^k}\le\int_0^1\|(w-v)F'(v+\tau(w-v))\|_{H^k}\,d\tau.$$

Fix a $\tau\in[0,1]$. From corollary 5.4, we get

$$\begin{aligned}&\|(w-v)F'(v+\tau(w-v))\|_{H^k}\\&\le C_k\Big(\|F'(v+\tau(w-v))\|_{L^\infty}\|w-v\|_{H^k}+\|w-v\|_{L^\infty}\|\partial_y^k v+\tau(\partial_y^k w-\partial_y^k v)\|_{L^2}\\&\quad\times\sum_{j=0}^{k-1}\|F^{(j+2)}\|_{L^\infty(I)}\|v+\tau(w-v)\|_{L^\infty}^j\Big)\end{aligned}$$

whence the desired inequality follows. □

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