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EXISTENCE OF SOLUTIONS FOR SOME NONLINEAR ELLIPTIC EQUATIONS

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Abstract. In this paper, we study the existence of solutions to the following nonlinear elliptic problem in a bounded subset Ω of $\mathbb{R}^N$:

$$\begin{gathered}-\Delta_p u = f(x,u,\nabla u) + \mu \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

where μ is a Radon measure on Ω which is zero on sets of p-capacity zero, $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function that satisfies certain conditions with respect to the one dimensional spectrum.

1. Introduction

We consider the quasilinear elliptic problem

$$\begin{gathered}-\Delta_p u = f(x,u,\nabla u) + \mu \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 2$, $1 < p < +\infty$, μ is a Radon measure on Ω and $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function. We are interested in the existence of solutions to this problem. More precisely, we will prove the existence of a solution $u \in W_0^{1,p}(\Omega)$, if and only if the signed measure μ is zero on sets of capacity zero in Ω. (i.e $\mu(E) = 0$ for every set E such that $\operatorname{cap}_p(E,\Omega) = 0$).

Boccardo, Gallouet and Orsina have proved in [3] the existence of a solution to the problem

$$\begin{gathered}Au + g(x,u,\nabla u) = \mu \quad in \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

where $A(u) = -\operatorname{div}(a(x,\nabla u))$, $a : \Omega\times\mathbb{R}^N \to \mathbb{R}$ and $g : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions such that for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^N$ and for every $s \in \mathbb{R}$,

$$\begin{gathered}a(x,\xi).\xi \geq \alpha|\xi|^p,\\ |a(x,\xi)| \leq l(x) + \beta|\xi|^{p-1},\\ |g(x,s,\xi)| \leq b(|s|)[|\xi|^p + d(x)],\end{gathered}$$

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where α and β are two positive constants, $l \in L^{p'}(\Omega)$, b a real-valued, positive, increasing, continuous function, and d a nonegative function in $L^1(\Omega)$. They assume that for almost every $x \in \Omega$, for every ξ and η in $\mathbb{R}^N$, with $\xi \neq \eta$,

$$[a(x,\xi) - a(x,\eta)].(\xi - \eta) > 0,$$

They require also that for almost every $x \in \Omega$, for every ξ in $\mathbb{R}^N$, for every s in $\mathbb{R}$ such that $|s| \geq \sigma$,

$$g(x,s,\xi)\,\mathrm{sgn}(s) \geq \rho|\xi|^p,$$

where ρ and σ are two positive real numbers and $\mathrm{sgn}(s)$ is the sign of s.

Let $(\beta,\alpha,u) \in \mathbb{R}^N \times \mathbb{R} \times W_0^{1,p}(\Omega)\backslash\{0\}$. If (β,α,u) is a solution of the problem

$$\begin{gathered} -\Delta_p u = \alpha m(x)|u|^{p-2}u + \beta.|\nabla u|^{p-2}\nabla u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $1 < p < \infty$ and $m \in M = \{m \in L^\infty(\Omega) : \mathrm{meas}\{x \in \Omega : m(x) > 0\} \neq 0\}$. In this case, the pair (β,α) is said to be a one dimensional eigenvalue and u the associated eigenfunction. We designate by $\sigma_1(-\Delta_p, m) \subset \mathbb{R}^N \times \mathbb{R}$ the set of one dimensional eigenvalues (β,α) with $\alpha \geq 0$.

Proposition 1.1. *(1) $\sigma_1(-\Delta_p,m)$ contains the union of the sequence of graphs of the functions $\Lambda_n : \mathbb{R}^N \to \mathbb{R}^+$, $n = 1,2,\dots$, where $\Lambda_n(\beta)$ is defined for every $\beta \in \mathbb{R}^N$ by*

$$\frac{1}{\Lambda_n(\beta)} = \sup_{K \in A_n^\beta} \min_{u \in K} \int_\Omega e^{\beta.x} m(x)|u|^p dx.$$

with $A_n^\beta = \{K \subset S_\beta,\ K$ compact symmetrical; $\gamma(K) \geq n\}$,

$$S_\beta = \left\{u \in W_0^{1,p}(\Omega) : \Big(\int_\Omega e^{\beta.x} m(x)|\nabla u|^p dx\Big)^{1/p} = 1\right\}$$

and $\gamma(K)$ indicates the genus of K.

(2) $\Lambda_1(.)$ is the first eigensurface of the spectrum of $\sigma_1(-\Delta_p,m)$ in the sense

$$\sigma_1(-\Delta_p,m) \subset \{(\beta,\alpha) \in \mathbb{R}^N \times \mathbb{R};\ \Lambda_1(\beta) \leq \alpha\}$$

The proof of the above proposition can be found in [1]. When $\mu = h \in W^{-1,p'}(\Omega)$, Anane, Chakrone and Gossez have proved in [1] the existence of a solution to (1.1), in the sense

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx = \int_\Omega f(x,u,\nabla u) v\, dx + \langle h, v\rangle$$

for every $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. This is done under the hypotheses of non-resonance with respect to the spectrum of one dimensional $\sigma_1(-\Delta_p,1)$: There exists $(\beta,\alpha) \in \mathbb{R}^N \times \mathbb{R}$ with $\alpha < \Lambda_1(\beta,-\Delta_p,1)$ where $\Lambda_1(.,-\Delta_p,1)$ is the first eigensurface of the spectrum of one dimensional $\sigma_1(-\Delta_p,1)$, such that for all $\delta > 0$ there exists $a_\delta \in L^{p'}(\Omega)$ such that

$$f(x,s,\xi)s \leq \alpha|s|^p + \beta|\xi|^{p-2}\xi s + \delta(|s|^{p-1} + |\xi|^{p-1} + a_\delta(x))|s| \tag{1.2}$$

for almost every $x \in \Omega$ and for all $(\xi,s) \in \mathbb{R}^N \times \mathbb{R}$; and for all $k > 0$ there exist $\phi_k \in L^1(\Omega)$ and $b_k \in \mathbb{R}$ such that

$$\max_{|s| \leq k} |f(x,s,\xi)| \leq b_k|\xi|^p + \phi_k(x) \tag{1.3}$$

for almost every $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$.

Remark 1.2. (1) If $f(x,u,\nabla u) = \alpha m(x)|u|^{p-2}u + \beta.|\nabla u|^{p-2}\nabla u$, then (1.1) has a solution for every $\mu \in W^{-1,p'}(\Omega)$, in the usual sense

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx = \int_\Omega f(x,u,\nabla u)v\, dx + \langle h, v\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}$$

for every $v \in W_0^{1,p}(\Omega)$, if and only if $(\beta,\alpha) \notin \sigma_1(-\Delta_p, m)$.

(2) If $\mu \notin W^{-1,p'}(\Omega)$, problem (1.1) does not have always a solution. Indeed in the case $1 < p \le N$, we have that $L^1(\Omega) \nsubseteq W^{-1,p'}(\Omega) = -\Delta_p(W_0^{1,p}(\Omega))$.

In this work, we assume (1.3) and that μ is a measure. We assume also that for each $\delta > 0$ there exists $a_\delta \in L^{p'}(\Omega)$ such that

$$f(x,s,\xi)s \le -\rho|\xi|^p|s| + \alpha|s|^p + \beta|\xi|^{p-2}\xi s + \delta(|s|^{p-1} + |\xi|^{p-1} + a_\delta(x))|s| \tag{1.4}$$

for almost every $x \in \Omega$ and for all $(\xi,s) \in \mathbb{R}^N \times \mathbb{R}$, where $(\beta,\alpha) \in \mathbb{R}^N \times \mathbb{R}$ satisfies the same conditions as in (1.2) and ρ is a positive real number. In the case $\delta = 1$, there exists $a_1 \in L^{p'}(\Omega)$ such that

$$f(x,s,\xi)\operatorname{sgn}(s) \le -\rho|\xi|^p + \alpha'|s|^{p-1} + \beta'|\xi|^{p-1} + a_1(x) \tag{1.5}$$

for almost every $x \in \Omega$ and for all $(\xi,s) \in \mathbb{R}^N \times \mathbb{R}$, where $\alpha' = \alpha+1$ and $\beta' = |\beta|+1$.

Remark 1.3. (1) The conditions of the sign given in [3] imply (1.4) in the case $\alpha = 0$ and $\beta = 0$.

(2) The hypothesis (1.3) and (1.4) are satisfied for example if

$$f(x,s,\xi) = -\rho|\xi|^p\operatorname{sgn}(s) + \alpha|s|^{p-2}s + \beta|\xi|^{p-2}\xi + g(x,s,\xi) + l(x,s,\xi)$$

where g and l satisfy

$$\begin{gathered} g(x,s,\xi)s \le 0, \\ |g(x,s,\xi)| \le b(|s|)(|x|^p + c(x)), \\ sl(x,s,\xi) \le C(|s|^{q-1} + |x|^{q-1} + d(x))|s| \end{gathered}$$

with b continuous, $c(x) \in L^1(\Omega)$, $q < p$, $d(x) \in L^{p'}(\Omega)$ and C a constant.

For every compact subset K of Ω, the p-capacity of K with respect to Ω is defined as

$$\operatorname*{cap}_p(K,\Omega) = \inf\{\int_\Omega |\nabla u|^p dx,\ u \in C_0^\infty(\Omega) \text{ and } u \ge \chi_K\}$$

where χ_K is the characteristic function of K; we will use the convention that $\inf(\emptyset) = +\infty$. The p-capacity of any open subset U of Ω is defined by $\operatorname{cap}_p(U,\Omega) = \sup\{\operatorname{cap}_p(K,\Omega),\ K \text{ compact and } K \subseteq U\}$. Also the p-capacity of any subset $B \subseteq \Omega$ by $\operatorname{cap}_p(B,\Omega) = \inf\{\operatorname{cap}_p(U,\Omega),\ U \text{ open and } B \subseteq U\}$. We will denote by $\mathcal{M}_b(\Omega)$ the space of all signed measures on Ω and by $\mathcal{M}_0^p(\Omega)$ the space of all measures μ in $\mathcal{M}_b(\Omega)$ such that $\mu(E) = 0$ for every set E such that $\operatorname{cap}_p(E,\Omega) = 0$.

Our main result is stated as follows.

Theorem 1.4. *Assume* (1.3), (1.4) *and that μ is a measure in $\mathcal{M}_b(\Omega)$. Then, there exists a solution u of*

$$\begin{gathered} -\Delta_p u = f(x,u,\nabla u) + \mu \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.6}$$

in the sense that $u \in W_0^{1,p}(\Omega)$, $f(x,u,\nabla u) \in L^1(\Omega)$, *and*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx = \int_\Omega f(x,u,\nabla u)v\,dx + \int_\Omega v\,d\mu,$$

for every $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, *if and only if* $\mu \in \mathcal{M}_0^p(\Omega)$.

2. Proof of Main result

The notation $\langle .,. \rangle$ stands hereafter for the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. We define, for s and k in $\mathbb{R}$, with $k > 0$,

$$T_k(s) = \begin{cases} k\,\mathrm{sgn}(s) & \text{if } |s| > k, \\ s & \text{if } |s| \le k, \end{cases}$$

and $G_k(s) = s - T_k(s)$.

Lemma 2.1. *Let* $g \in L^\infty(\Omega)$ *and* $F \in (L^{p'}(\Omega))^N$. *Under the hypotheses* (1.3) *and* (1.4), *the problem*

$$\begin{gathered} -\Delta_p u = f(x,u,\nabla u) + g - \operatorname{div} F \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

admits a solution $u \in W_0^{1,p}(\Omega)$ *in the sense that* $f(x,u,\nabla u)$ *and* $f(x,u,\nabla u)u$ *are in* $L^1(\Omega)$, *and that*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx = \int_\Omega f(x,u,\nabla u)v\,dx + \int_\Omega gv + \int_\Omega F\nabla v$$

for every $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *and for* $v = u$.

Proof. Letting $l = g - \operatorname{div} F$, we have $l \in W^{-1,p'}(\Omega)$. Then (1.4) implies (1.2), and Lemma 2.1 is a particular case of a result in [1]. $\square$

Lemma 2.2. $\mathcal{M}_0^p(\Omega) = L^1(\Omega) + W^{-1,p'}(\Omega)$ *for every* $1 < p < +\infty$.

For the proof of the above lemma see [4].

Lemma 2.3. *Let* a, b *be two nonnegative numbers, and let* $\varphi(s) = se^{\theta s^2}$ *with* $\theta = b^2/(4a^2)$. *Then for all* $s \in \mathbb{R}$, $a\varphi'(s) - b|\varphi(s)| \ge a/2$.

Proof. For $s \in \mathbb{R}$ let $\psi(s) = a\varphi'(s) - b|\varphi(s)|$. Then

$$\psi(s) = e^{\theta s^2}[a(1+2\theta s^2) - b|s|] = ae^{\theta s^2}[(1+2\theta s^2) - 2\sqrt{\theta}|s|],$$

Then ψ is even, and assuming that $s \ge 0$, we obtain that for every $s \ge 0$,

$$\psi(s) = 2ae^{\theta s^2}\big[(\sqrt{\theta}s - \frac{1}{2})^2 + \frac{1}{4}\big] \ge \frac{a}{2}.$$

$\square$

Remark 2.4. Let $\mu \in \mathcal{M}_0^p(\Omega)$. If $p > N$, then $L^1(\Omega) \subset W^{-1,p'}(\Omega)$; therefore, $\mathcal{M}_0^p(\Omega) = W^{-1,p'}(\Omega)$. Then the existence of a solution of (1.6) is a consequence of [1, Theorem 7.1]. That is why, we assume that $1 < p \le N$.

Proof of the Theorem 1.4. Note that if $u \in W_0^{1,p}(\Omega)$ is a solution of (1.6), then
$$\mu = -\Delta_p u - f(x, u, \nabla u)$$
with $\Delta_p u \in W^{-1,p'}(\Omega)$ and $f(x, u, \nabla u) \in L^1(\Omega)$; So by Lemma 2.2, $\mu \in \mathcal{M}_0^p(\Omega)$.

Conversely, suppose that $\mu \in \mathcal{M}_0^p(\Omega)$, so by Lemma 2.2 there exists $g \in L^1(\Omega)$ and $F \in (L^{p'}(\Omega))^N$ such that $\mu = g-\operatorname{div} F$. There exists a sequence $(g_n)_n$ of $L^\infty(\Omega)$ that converges strongly to g in $L^1(\Omega)$ and $\widetilde{g} \in L^1(\Omega)$ such that $|g_n(x)| \le |\widetilde{g}(x)|$ for every $n \in \mathbb{N}$ and for almost every $x \in \Omega$.

By Lemma 2.1, the problem
$$\begin{gathered} -\Delta_p u_n = f(x, u_n, \nabla u_n) + g_n - \operatorname{div} F \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$
admits a solution $u_n \in W_0^{1,p}(\Omega)$ in the sense that
$$f(x, u_n, \nabla u_n),\ f(x, u_n, \nabla u_n)u_n \in L^1(\Omega), \tag{2.3}$$
and
$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v\, dx = \int_\Omega f(x, u_n, \nabla u_n)v\, dx + \int_\Omega g_n v + \int_\Omega F\nabla v, \tag{2.4}$$
for every $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and for $v = u_n$.

Lemma 2.5. *The sequence $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Let us choose $v = \varphi(T_1(u_n))$ as a test function in (2.4), where $\varphi(s) = se^{\theta s^2}$ with $\theta = \frac{b^2}{4a^2}$, $a = 1$ and $b = b_1$ ($b_1 \ge 0$ is given for $k = 1$ by (1.3)). Setting
$$\begin{gathered} a(\xi) = |\xi|^{p-2}\xi \quad \forall \xi \in \mathbb{R}^N, \\ \varphi_1 = \varphi(T_1(u_n)), \quad \varphi_1' = \varphi'(T_1(u_n)), \end{gathered}$$
we have
$$\begin{aligned} \int_\Omega a(\nabla u_n)\nabla[\varphi(T_1(u_n))]dx &= \int_\Omega f(x, u_n, \nabla u_n)\varphi(T_1(u_n))dx \\ &\quad + \int_\Omega g_n\varphi(T_1(u_n))dx + \int_\Omega F\nabla[\varphi(T_1(u_n))]dx. \end{aligned} \tag{2.5}$$
On the other hand,
$$\begin{aligned} \int_\Omega a(\nabla u_n)\nabla[\varphi(T_1(u_n))]dx &= \int_\Omega a(\nabla u_n)\varphi_1'\nabla(T_1(u_n))dx \\ &= \int_\Omega \varphi_1'|\nabla(T_1(u_n))|^p dx. \end{aligned}$$
Since φ' is an even function in $\mathbb{R}$, φ' is increasing in $\mathbb{R}^+$ and $|T_1(u_n)| \le 1$, we have
$$\begin{aligned} \int_\Omega F\nabla[\varphi(T_1(u_n))]dx &\le \|F\|_{L^{p'}}\|\varphi(T_1(u_n))\|_{1,p} \\ &\le \|F\|_{L^{p'}}(\int_\Omega |\varphi_1'\nabla(T_1(u_n))|^p dx)^{1/p} \\ &\le \|F\|_{L^{p'}}\varphi'(1)\|T_1(u_n)\|_{1,p}. \end{aligned}$$
Since φ is increasing in $\mathbb{R}$, we get
$$\int_\Omega g_n\varphi(T_1(u_n))dx \le \varphi(1)\int_\Omega |g_n|dx \le \varphi(1)\|\widetilde{g}\|_{L^1}.$$

Writing

$$\begin{aligned}&\int_\Omega f(x,u_n,\nabla u_n)\varphi(T_1(u_n))dx\\&=\int_{\{|u_n|\le 1\}}\varphi_1 f(x,u_n,\nabla u_n)dx+\int_{\{|u_n|>1\}}\varphi_1 f(x,u_n,\nabla u_n)dx.\end{aligned}$$

By (1.3), we have

$$\begin{aligned}|\int_{\{|u_n|\le 1\}}\varphi_1 f(x,u_n,\nabla u_n)dx| &\le \int_{\{|u_n|\le 1\}}|\varphi_1||f(x,u_n,\nabla u_n)|dx\\&\le \int_{\{|u_n|\le 1\}}|\varphi_1|[b_1|\nabla u_n|^p+\phi_1(x)]dx\\&\le b_1\int_{\{|u_n|\le 1\}}|\varphi_1||\nabla u_n|^p dx+\varphi(1)\|\phi_1\|_{L^1}\\&\le b_1\int_\Omega|\varphi_1||\nabla(T_1(u_n))|^p dx+\varphi(1)\|\phi_1\|_{L^1}.\end{aligned}$$

On the other hand, on $\{|u_n|>1\}$, $T_1(u_n)=\operatorname{sgn}(u_n)$, so $\varphi(T_1(u_n))=\operatorname{sgn}(u_n)\ e^\theta$ and by (1.5), we get

$$\begin{aligned}&\int_{\{|u_n|>1\}}\varphi_1 f(x,u_n,\nabla u_n)dx\\&=\int_{\{|u_n|>1\}}e^\theta f(x,u_n,\nabla u_n)\operatorname{sgn}(u_n)dx\\&\le e^\theta\int_{\{|u_n|>1\}}[-\rho|\nabla u_n|^p+\alpha'|u_n|^{p-1}+\beta'|\nabla u_n|^{p-1}+a_1(x)]dx.\end{aligned}$$

Adding the above inequalities, by (2.5), we obtain

$$\begin{aligned}&\int_\Omega[\varphi_1'-b_1|\varphi_1|]|\nabla(T_1(u_n))|^p dx+\rho e^\theta\int_{\{|u_n|>1\}}|\nabla u_n|^p dx\\&\le\|F\|_{L^{p'}}\varphi'(1)\|T_1(u_n)\|_{1,p}+\varphi(1)\|\widetilde{g}\|_{L^1}+\varphi(1)\|\phi_1\|_{L^1}\\&\quad+e^\theta\int_{\{|u_n|>1\}}[\alpha'|u_n|^{p-1}+\beta'|\nabla u_n|^{p-1}+a_1(x)]dx.\end{aligned}\tag{2.6}$$

Using Hölder's inequality, we have

$$\begin{gathered}\int_{\{|u_n|>1\}}|\nabla u_n|^{p-1}dx\le\|u_n\|_{1,p}^{p-1}(\operatorname{meas}(\Omega))^{1/p},\\\int_{\{|u_n|>1\}}|u_n|^{p-1}dx\le\|u_n\|_p^{p-1}(\operatorname{meas}(\Omega))^{1/p}.\end{gathered}$$

By Poincaré's inequality, there exists $c>0$ such that

$$\|u_n\|_p\le c\|\nabla u_n\|_p.$$

So

$$\int_{\{|u_n|>1\}}|u_n|^{p-1}dx\le c^{p-1}\|u_n\|_{1,p}^{p-1}(\operatorname{meas}(\Omega))^{1/p}.$$

Replacing this in (2.6) and using that $\varphi_1' - b_1|\varphi_1| \geq \frac{1}{2}$, we obtain

$$\frac{1}{2}\int_\Omega |\nabla(T_1(u_n))|^p dx + \rho e^\theta \int_{\{|u_n|>1\}} |\nabla u_n|^p dx \leq c_1\|u_n\|_{1,p} + c_2\|u_n\|_{1,p}^{p-1} + c_3,$$

where $c_1 = \|F\|_{L^{p'}}\varphi'(1)$, $c_2 = e^\theta[\alpha' c^{p-1} + \beta'](\text{meas}(\Omega))^{\frac{1}{p}}$ and $c_3 = \varphi(1)\|\widetilde{g}\|_{L^1} + \varphi(1)\|\phi_1\|_{L^1} + e^\theta\|a_1(x)\|_{L^1}$. Set $c_4 = \min(\frac{1}{2}, \rho e^\theta)$, we have

$$c_4\|u_n\|_{1,p}^p \leq c_1\|u_n\|_{1,p} + c_2\|u_n\|_{1,p}^{p-1} + c_3,$$

since $p > 1$, $(u_n)_n$ is a bounded sequence in $W_0^{1,p}(\Omega)$. ☐

For a subsequence, still denoted by $(u_n)_n$, we have

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \\ u_n \to u \quad \text{strongly in } \ L^p(\Omega), \\ u_n(x) \to u(x) \quad \text{for almost every } x \in \Omega. \end{gathered} \tag{2.7}$$

Lemma 2.6. *For every $k > 0$, the sequence $(T_k(u_n))_n$ converges strongly to $T_k(u)$ in $W_0^{1,p}(\Omega)$.*

Proof. Let $k > 0$. Consider $\varphi(s) = se^{\theta s^2}$ with $\theta = \frac{b^2}{4a^2}$, $a = 1$ and $b = a_k$ ($a_k \geq 0$ is given by (1.3). Setting

$$a(\xi) = |\xi|^{p-2}\xi, \quad \forall \xi \in \mathbb{R}^N, \varphi_n = \varphi(T_k(u_n) - T_k(u)), \quad \varphi_n' = \varphi'(T_k(u_n) - T_k(u)).$$

By (2.7), the continuity of φ and φ', and the dominated convergence theorem, we have

$$\begin{gathered} \varphi_n \rightharpoonup 0 \quad \text{and} \quad \varphi_n' \rightharpoonup 1 \quad \text{weak-}* \text{ in } L^\infty(\Omega) \text{ and a. e. } x \in \Omega, \\ \varphi_n \to 0 \quad \text{and} \quad \varphi_n' \to 1 \quad \text{in } L^q(\Omega) \text{ for every } q \geq 1. \end{gathered} \tag{2.8}$$

We will denote by ε_n any quantity which converges to zero as n tends to infinity. Let $v = \varphi_n$, be a test function in (2.4). Then

$$\begin{aligned} &\int_\Omega a(\nabla u_n)\nabla(T_k(u_n) - T_k(u))\varphi_n' dx \\ &= \int_\Omega f(x, u_n, \nabla u_n)\varphi_n dx + \int_\Omega g_n\varphi_n dx + \int_\Omega F\nabla(T_k(u_n) - T_k(u))\varphi_n' \\ &:= A + B + C + D \end{aligned} \tag{2.9}$$

For the third term on the right-hand side: Since $\varphi_n \rightharpoonup 0$ weak-$*$ in $L^\infty(\Omega)$ and $g_n \to g$ in $L^1(\Omega)$, we have $\int_\Omega g_n\varphi_n dx \to 0$ so that

$$C = \varepsilon_n. \tag{2.10}$$

For the forth term on the right-hand side: It is clear that $F\varphi_n' \to F$ in $(L^{p'}(\Omega))^N$ and $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega)$, so that

$$D = \varepsilon_n. \tag{2.11}$$

For the second term on the right-hand side:

$$\begin{aligned} &\int_\Omega f(x, u_n, \nabla u_n)\varphi_n dx \\ &= \int_{\{|u_n|>k\}} f(x, u_n, \nabla u_n)\varphi_n dx + \int_{\{|u_n|\leq k\}} f(x, u_n, \nabla u_n)\varphi_n dx := B_1 + B_2. \end{aligned}$$

On the set $\{|u_n| > k\}$, φ_n has the same sign as u_n, so by (1.5),

$$\begin{aligned}
&f(x, u_n, \nabla u_n)\varphi_n \\
&\leq -\rho|\nabla u_n|^p|\varphi_n| + \alpha'|u_n|^{p-1}|\varphi_n| + \beta'|\nabla u_n|^{p-1}|\varphi_n| + a_1(x)|\varphi_n| \\
&\leq [\alpha'|u_n|^{p-1} + \beta'|\nabla u_n|^{p-1} + a_1(x)]|\varphi_n|.
\end{aligned}$$

By Lemma 2.5 and (2.8), we have $B_1 \leq \varepsilon_n$, so that

$$\int_\Omega f(x, u_n, \nabla u_n)\varphi_n dx \leq \int_{\{|u_n|\leq k\}} f(x, u_n, \nabla u_n)\varphi_n dx + \varepsilon_n.$$

By (1.3), we have

$$\begin{aligned}
|\int_{\{|u_n|\leq k\}} f(x, u_n, \nabla u_n)\varphi_n dx| &\leq \int_{\{|u_n|\leq k\}} |f(x, u_n, \nabla u_n)||\varphi_n| dx \\
&\leq \int_{\{|u_n|\leq k\}} [b_k|\nabla u_n|^p + \phi_k(x)]|\varphi_n| dx \\
&\leq b_k \int_\Omega |\nabla T_k(u_n)|^p|\varphi_n| dx + \int_\Omega \phi_k(x)|\varphi_n| dx,
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |\nabla T_k(u_n)|^p|\varphi_n| dx &= \int_\Omega a(\nabla T_k(u_n))\nabla T_k(u_n)|\varphi_n| dx \\
&= \int_\Omega (a(\nabla T_k(u_n)) - a(\nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))|\varphi_n| dx \\
&\quad + \int_\Omega a(\nabla T_k(u_n))\nabla T_k(u)|\varphi_n| dx \\
&\quad + \int_\Omega a(\nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u))|\varphi_n| dx.
\end{aligned}$$

By (2.8), since $(T_k(u_n))_n$ is bounded in $W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega f(x, u_n, \nabla u_n)\varphi_n dx \\
&\leq \varepsilon_n + b_k \int_\Omega \big(a(\nabla T_k(u_n)) - a(\nabla T_k(u))\big)\big(\nabla T_k(u_n) - \nabla T_k(u)\big)|\varphi_n| dx.
\end{aligned} \tag{2.12}$$

For the firs term on the right-hand side (A): We verify easily that $a(\nabla T_k(u_n)) + a(\nabla G_k(u_n)) = a(\nabla u_n)$, so that

$$\begin{aligned}
&\int_\Omega a(\nabla u_n)\nabla(T_k(u_n) - T_k(u))\varphi_n' dx \\
&= \int_\Omega a(\nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u))\varphi_n' dx + \int_\Omega a(\nabla G_k(u_n))\nabla(T_k(u_n) \\
&\quad - T_k(u))\varphi_n' dx := A_1 + A_2.
\end{aligned}$$

We have $\nabla(T_k(u_n)) = 0$ if $\nabla(G_k(u_n)) \neq 0$, so

$$\begin{aligned}
A_2 &= -\int_\Omega a(\nabla G_k(u_n))\nabla(T_k(u))\varphi_n' dx \\
&= -\int_\Omega a(\nabla G_k(u_n))\nabla(T_k(u))\chi_{\{|u_n|\geq k\}}\varphi_n' dx.
\end{aligned}$$

Since $\nabla T_k(u) = 0$ on the set $\{|u| \geq k\}$, $\nabla T_k(u)\chi_{\{|u_n|\geq k\}} \to 0$ for almost every $x \in \Omega$, so, by Lebesgue theorem $A_2 = \varepsilon_n$. For (A_1), we have

$$\begin{aligned}
&\int_\Omega a(\nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u))\varphi_n' dx \\
&= \int_\Omega [a(\nabla T_k(u_n)) - a(\nabla T_k(u))]\nabla(T_k(u_n) - T_k(u))\varphi_n' dx \\
&\quad + \int_\Omega a(\nabla T_k(u))\nabla(T_k(u_n) - T_k(u))\varphi_n' dx := A_{1.1} + A_{1.2}
\end{aligned}$$

By (2.8) , since $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega)$, we have $A_{1.2} = \varepsilon_n$. Thus

$$A = \int_\Omega [a(\nabla T_k(u_n)) - a(\nabla T_k(u))]\nabla\big(T_k(u_n) - T_k(u)\big)\varphi_n' dx + \varepsilon_n. \tag{2.13}$$

By (2.10), (2.11), (2.12), (2.13) and from (2.9), we obtain

$$\int_\Omega [a\big(\nabla T_k(u_n)\big) - a\big(\nabla T_k(u)\big)]\nabla\big(T_k(u_n) - T_k(u)\big)[\varphi_n' - b_k|\varphi_n|]dx \leq \varepsilon_n.$$

Since $\varphi_n' - b_k|\varphi_n| \geq \frac{1}{2}$ with $a = 1$ and $b = b_k$) and

$$\begin{gathered}
[a(\nabla T_k(u_n)) - a(\nabla T_k(u))]\nabla\big(T_k(u_n) - T_k(u)\big) \geq 0, \\
\int_\Omega [a(\nabla T_k(u_n)) - a(\nabla T_k(u))]\nabla\big(T_k(u_n) - T_k(u)\big)dx = \varepsilon_n;
\end{gathered}$$

therefore,

$$\langle -\Delta_p(T_k(u_n)) + \Delta_p(T_k(u)), T_k(u_n) - T_k(u)\rangle \to 0.$$

Since $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega)$,

$$\begin{gathered}
\langle -\Delta_p(T_k(u)), T_k(u_n) - T_k(u)\rangle \to 0, \\
\langle -\Delta_p(T_k(u_n)), T_k(u_n) - T_k(u)\rangle \to 0.
\end{gathered}$$

Since $-\Delta_p$ belongs to the class (S^+) (see [2]), $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,p}(\Omega)$. $\square$

Lemma 2.7. *The following to limit hold:*

$$\begin{gathered}
\lim_{k\to+\infty} [\sup_{n\in\mathbb{N}} \int_{\{|u_n|\geq k\}} |\nabla u_n|^p dx] = 0, \\
\lim_{k\to+\infty} [\sup_{n\in\mathbb{N}} \int_{\{|u_n|\geq k\}} |f(x, u_n, \nabla u_n)| dx] = 0.
\end{gathered} \tag{2.14}$$

Proof. For the first limit, we define $\psi : \mathbb{R} \to \mathbb{R}^+$ by $\psi(-s) = -\psi(s)$ for all $s \in \mathbb{R}$ and

$$\psi(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq k-1, \\ s - (k-1) & \text{if } k-1 \leq s \leq k, \\ 1 & \text{if } s \geq k, \end{cases}$$

where $k > 1$, so that ψ is continuous, bounded in $\mathbb{R}$ and $\psi(u_n) \in W_0^{1,p}(\Omega)$. We choose $v = \psi(u_n)$, as a test function in (2.4) we have

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\psi(u_n)dx \\
&= \int_\Omega f(x,u_n,\nabla u_n)\psi(u_n)dx + \int_\Omega g_n\psi(u_n)dx + \int_\Omega F\nabla\psi(u_n)dx.
\end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned}
\int_\Omega |\nabla\psi(u_n)|^p dx \le \int_\Omega f(x,u_n,\nabla u_n)\psi(u_n)dx + \int_{\{|u_n|\ge\ k-1\}} |g_n|dx \\
+ c\int_{\{k-1<|u_n|<k\}} |F|^{p'}dx + \frac{1}{2}\int_\Omega |\nabla\psi(u_n)|^p dx.
\end{aligned}$$

So that

$$\begin{aligned}
0 &\le \frac{1}{2}\int_\Omega |\nabla\psi(u_n)|^p dx \\
&\le \int_\Omega f(x,u_n,\nabla u_n)\psi(u_n)dx + \int_{\{|u_n|\ge\ k-1\}} |g_n|dx \\
&\quad + c\int_{\{k-1<|u_n|<k\}} |F|^{p'}dx.
\end{aligned} \tag{2.15}$$

Using (1.5) and that $\psi(s)$ has the same sign as s, and that is zero if $|s| \le k-1$, we get

$$\begin{aligned}
\int_\Omega f(x,u_n,\nabla u_n)\psi(u_n)dx &= \int_{\{|u_n|>k-1\}} f(x,u_n,\nabla u_n)\psi(u_n)dx \\
&\le \int_{\{|u_n|>k-1\}} [-\rho|\nabla u_n|^p|\psi(u_n)| + \alpha'|u_n|^{p-1}|\psi(u_n)| \\
&\quad + \beta'|\nabla u_n|^{p-1}|\psi(u_n)| + a_1(x)|\psi(u_n)|]dx.
\end{aligned}$$

From (2.15), we have

$$\begin{aligned}
&\rho\int_{\{|u_n|>k-1\}} |\nabla u_n|^p|\psi(u_n)|dx \\
&\le \int_{\{|u_n|\ge k-1\}} |g_n|dx + c\int_{\{k-1<|u_n|<k\}} |F|^{p'}dx + \alpha'\int_{\{|u_n|>k-1\}} |u_n|^{p-1}|\psi(u_n)|dx \\
&\quad + \beta'\int_{\{|u_n|>k-1\}} |\nabla u_n|^{p-1}|\psi(u_n)|dx + \int_{\{|u_n|>k-1\}} a_1(x)|\psi(u_n)|dx.
\end{aligned} \tag{2.16}$$

Since $u_n \to u$ in $L^p(\Omega)$, there exists $v \in L^p(\Omega)$ such that $|u_n| \le |v|$. Since $|g_n| \le |\widetilde{g}|$, $|\widetilde{g}| \in L^1(\Omega)$ and $|\psi(s)| \le 1$, we have

$$\begin{aligned}
&\rho\int_{\{|u_n|>k-1\}} |\nabla u_n|^p|\psi(u_n)|dx \\
&\le \int_\Omega [|\widetilde{g}| + c|F|^{p'} + \alpha'|v|^{p-1} + a_1(x)]\chi_{\{|v|\ge k-1\}}dx + \beta'\int_{\{|v|>k-1\}} |\nabla u_n|^{p-1}dx \\
&\le \int_\Omega r(x)\chi_{\{|v|\ge k-1\}}dx + \beta'\|u_n\|_{1,p}^{p-1}(\int_\Omega \chi_{\{|v|\ge k-1\}}dx)^{1/p},
\end{aligned}$$

where $r(x) = |\widetilde{g}| + c|F|^{p'} + \alpha'|v|^{p-1} + a_1(x)$. We have $r \in L^1(\Omega)$ and $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$, so that

$$\lim_{k\to+\infty}[\sup_{n\in\mathbb{N}}\int_{\{|u_n|>k-1\}}|\nabla u_n|^p|\psi(u_n)|dx] = 0.$$

Since

$$\begin{aligned}\int_{\{|u_n|\geq k\}}|\nabla u_n|^p dx &= \int_{\{|u_n|\geq k\}}|\nabla u_n|^p|\psi(u_n)|dx\\ &\leq \int_{\{|u_n|>k-1\}}|\nabla u_n|^p|\psi(u_n)|dx,\end{aligned}$$

it follows that

$$\lim_{k\to+\infty}[\sup_{n\in\mathbb{N}}\int_{\{|u_n|\geq k\}}|\nabla u_n|^p dx] = 0.$$

For the second limit, we let $l : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ defined by

$$l(x,s,\xi) = f(x,s,\xi) - \alpha|s|^{p-1}\operatorname{sgn}(s) - \beta|\xi|^{p-2}\xi - \big(|s|^{p-1} + |\xi|^{p-1} + a_1(x)\big)\operatorname{sgn}(s).$$

From (1.4), we get $l(x,s,\xi)s \leq -\rho|\xi|^p|s|$ for almost every $x\in\Omega$, and for all $(\xi,s)\in\mathbb{R}^N\times\mathbb{R}$.

By (2.15) and using that $\psi(s)$ has the same sign as s and that it is zero if $|s|\leq k-1$, we have

$$\begin{aligned}0 \leq &\int_{\{|u_n|\geq\ k-1\}}|g_n|dx\\ &+ c\int_{\{k-1<|u_n|<k\}}|F|^{p'}dx + \int_\Omega l(x,u_n,\nabla u_n)\psi(u_n)dx\\ &+ \int_\Omega[\alpha'|u_n|^{p-1} + \beta'|\nabla u_n|^{p-1} + a_1(x)]|\psi(u_n)|dx.\end{aligned}$$

Since $l(x,u_n,\nabla u_n)\psi(u_n) \leq -|l(x,u_n,\nabla u_n)|\chi_{\{|u_n|\geq\ k\}}$, we have

$$\begin{aligned}\int_{\{|u_n|\geq\ k\}}|l(x,u_n,\nabla u_n)|dx &\leq \int_{\{|u_n|\geq\ k-1\}}|g_n|dx + c\int_{\{k-1<|u_n|<k\}}|F|^{p'}dx\\ &+ \int_{\{|u_n|\geq\ k-1\}}\alpha'|u_n|^{p-1}|\psi(u_n)|dx\\ &+ \int_{\{|u_n|\geq\ k-1\}}\beta'|\nabla u_n|^{p-1}|\psi(u_n)|dx\\ &+ \int_{\{|u_n|\geq\ k-1\}}a_1(x)|\psi(u_n)|dx.\end{aligned}$$

In the same way as in the first limit, we prove that

$$\lim_{k\to+\infty}[\sup_{n\in\mathbb{N}}\int_{\{|u_n|\geq k\}}|l(x,u_n,\nabla u_n)|dx] = 0.$$

Also

$$|f(x,u_n,\nabla u_n)| \leq |l(x,u_n,\nabla u_n)| + \alpha'|u_n|^{p-1} + \beta'|\nabla u_n|^{p-1} + a_1(x),$$
$$\lim_{k\to+\infty}[\sup_{n\in\mathbb{N}}\int_{\{|u_n|\geq k\}}|f(x,u_n,\nabla u_n)|dx] = 0.$$

$\square$

Lemma 2.8. *The sequence $(u_n)_n$ converges strongly to u in $W_0^{1,p}(\Omega)$.*

Proof. We begin by proving that the sequence $\{|\nabla u_n|^p\}$ is equi-integrable in $L^1(\Omega)$. Let $\varepsilon > 0$ be fixed. Let now E be a measurable subset of Ω, we have

$$\int_E |\nabla u_n|^p dx = \int_{E\cap\{|u_n|\le k\}} |\nabla u_n|^p dx + \int_{E\cap\{|u_n|>k\}} |\nabla u_n|^p dx.$$

By lemma 2.7 there exists $k > 0$ such that for all $n \in \mathbb{N}$,

$$\int_{\{|u_n|>k\}} |\nabla u_n|^p dx \le \frac{\varepsilon}{2}.$$

For k fixed, we have

$$\int_{E\cap\{|u_n|\le k\}} |\nabla u_n|^p dx \le \int_E |\nabla T_k(u_n)|^p dx.$$

Since $T_k(u_n)$ converges strongly to $T_k(u)$ in $W_0^{1,p}(\Omega)$, there exists $\gamma > 0$ such that

$$\text{meas}(E) < \gamma \Rightarrow \forall n \in \mathbb{N} \quad \int_E |\nabla T_k(u_n)|^p dx \le \frac{\varepsilon}{2},$$

so that

$$\forall n \in \mathbb{N} \quad \int_{E\cap\{|u_n|\le k\}} |\nabla u_n|^p dx \le \frac{\varepsilon}{2}.$$

Then, there exists $\gamma > 0$ such that

$$\text{meas}(E) < \gamma \Rightarrow \forall n \in \mathbb{N} \int_E |\nabla u_n|^p dx \le \varepsilon.$$

Therefore, the sequence $\{|\nabla u_n|^p\}$ is equi-integrable in $L^1(\Omega)$. By Lemma 2.6 we have $\nabla u_n \to \nabla u$ for almost every $x \in \Omega$, so, $|\nabla u_n|^p \to |\nabla u|^p$ strongly in $L^1(\Omega)$, thus the sequence $(u_n)_n$ converges strongly to u in $W_0^{1,p}(\Omega)$. $\square$

Lemma 2.9. *The sequence $(f(x, u_n, \nabla u_n))_n$ converges to $f(x, u, \nabla u)$ in $L^1(\Omega)$.*

Proof. We begin by proving that the sequence $\{|f(x, u_n, \nabla u_n)|\}$ is equi-integrable in $L^1(\Omega)$. Let $\varepsilon > 0$ be fixed. Let now E be a measurable subset of Ω, we have

$$\begin{aligned}&\int_E |f(x, u_n, \nabla u_n)|dx \\ &= \int_{E\cap\{|u_n|\le k\}} |f(x, u_n, \nabla u_n)|dx + \int_{E\cap\{|u_n|>k\}} |f(x, u_n, \nabla u_n)|dx.\end{aligned}$$

By Lemma 2.7, there exists $k > 0$ such that

$$\forall n \in \mathbb{N}, \int_{E\cap\{|u_n|>k\}} |f(x, u_n, \nabla u_n)|dx \le \frac{\varepsilon}{2}.$$

When k is fixed, by (1.3) we have

$$\int_{E\cap\{|u_n|\le k\}} |f(x, u_n, \nabla u_n)|dx \le \int_E [b_k|\nabla T_k(u_n)|^p + \phi_k(x)]dx.$$

Since $\phi_k \in L^1(\Omega)$ and $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,p}(\Omega)$, there exists $\gamma > 0$ such that

$$\text{meas}(E) < \gamma \Rightarrow \forall n \in \mathbb{N} \int_E [b_k|\nabla T_k(u_n)|^p + \phi_k(x)]dx \le \frac{\varepsilon}{2},$$

so that

$$\forall n \in \mathbb{N} \int_{E\cap\{|u_n|\le k\}} |f(x,u_n,\nabla u_n)|dx \le \frac{\varepsilon}{2}.$$

Therefore, the sequence $\{|f(x,u_n,\nabla u_n)|\}_n$ is equi-integrable in $L^1(\Omega)$. Since $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, we have $f(x,u_n,\nabla u_n) \to f(x,u,\nabla u)$ for almost every $x \in \Omega$. so $f(x,u_n,\nabla u_n) \to f(x,u,\nabla u)$ strongly in $L^1(\Omega)$. $\square$

Going back to the the proof of Theorem 1.1, by (2.4) we have that for every $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$,

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v\,dx = \int_\Omega f(x,u_n,\nabla u_n)v\,dx + \int_\Omega g_n v + \int_\Omega F\nabla v.$$

As n approaches infinity, we get that for every $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx = \int_\Omega f(x,u,\nabla u)v\,dx + \int_\Omega g v + \int_\Omega F\nabla v.$$

Thus the problem

$$\begin{gathered}-\Delta_p u = f(x,u,\nabla u) + \mu \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega\end{gathered}$$

admits a solution $u \in W_0^{1,p}(\Omega)$ in the sense that $f(x,u,\nabla u) \in L^1(\Omega)$, and for every $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx = \int_\Omega f(x,u,\nabla u)v\,dx + \int_\Omega v\,d\mu.$$

$\square$

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