Electronic Journal of Differential Equations, Vol. 2006(2006), No. 69, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF WEAK SOLUTIONS FOR NONLINEAR ELLIPTIC SYSTEMS ON $\mathbb{R}^N$

EADA A. EL-ZAHRANI, HASSAN M. SERAG

Abstract. In this paper, we consider the nonlinear elliptic system

$$\begin{gathered} -\Delta_p u = a(x)|u|^{p-2}u - b(x)|u|^{\alpha}|v|^{\beta}v + f, \\ -\Delta_q v = -c(x)|u|^{\alpha}|v|^{\beta}u + d(x)|v|^{q-2}v + g, \\ \lim_{|x|\to\infty} u = \lim_{|x|\to\infty} v = 0 \quad u, v > 0 \end{gathered}$$

on a bounded and unbounded domains of $\mathbb{R}^N$, where Δ_p denotes the p-Laplacian. The existence of weak solutions for these systems is proved using the theory of monotone operators

1. Introduction

The generalized (the so-called weak) formulation of many stationary boundary-value problems for partial differential equations leads to operator equation of type

$$A(u) = f$$

on a Banach space. Indeed, the weak formulation consists in looking for an unknown function u from a Banach space V such that an integral identity containing u holds for each test function v from the space V. Since the identity is linear in v, we can take its sides as values of continuous linear functionals at the element $v \in V$. Denoting the terms containing unknown u as the value of an operator A, we obtain

$$(A(u), v) = (f, v) \quad \forall v \in V,$$

which is equivalent to equality of functionals on V, i.e. the equality of elements of V' (the dual space of V): $A(u) = f$. Functional analysis yields tools for proving existence of generalized (weak) solutions to a relatively wide class of differential equations that appear in mathematical physics and industry.

In our work, we consider nonlinear systems with model A of the form

$$A\{u, v\} = \{-\Delta_p u - a(x)|u|^{p-2}u + b(x)|u|^{\alpha}|v|^{\beta}v, -\Delta_q v + c(x)|u|^{\alpha}|v|^{\beta}u - d(x)|v|^{q-2}v\}$$

These nonlinear systems involving p-Laplacian appear in many problems in pure and applied mathematics e.g. in quasiconformal mappings, non-Newtonian fluids, and nonlinear elasticity [3, 4, 9].

2000 *Mathematics Subject Classification.* 35B45, 35J55.
Key words and phrases. Weak solutions; nonlinear elliptic systems; p-Laplacian; monotone operators.

Submitted February 9, 2006. Published July 6, 2006.

The existence of solutions for such systems was proved, using the method of sub and super solutions in [7, 8, 20]. Here, we use another technique for proving the existence of weak solutions. We use the theory of monotone operators.

First, we consider the following system defined on a bounded domain Ω of $\mathbb{R}^N$ with boundary $\partial\Omega$:

$$
\begin{gathered}
-\Delta_p u = a(x)|u|^{p-2}u - b(x)|u|^{\alpha}|v|^{\beta}v + f(x), \quad \text{in } \Omega \\
-\Delta_q v = d(x)|v|^{q-2}v - c(x)|v|^{\beta}|u|^{\alpha}u + g(x), \quad \text{in } \Omega \\
u = v = 0, \quad \text{on } \partial\Omega\,.
\end{gathered}
$$

Then, we generalize the discussion to system defined on the whole space $\mathbb{R}^N$.

This article is organized as follows: Some technical results and definitions are introduced in section two concerning the theory of nonlinear monotone operators, also, the scalar case is discussed. Section three, is devoted to study the existence of solutions for nonlinear systems defined on a bounded domain. In section four, the existence of solutions for nonlinear systems defined on unbounded domain is proved.

2. Scalar case

First, we introduce some technical results [6, 8, 21].

Definitions. Let $A : V \to V'$ be an operator on a Banach space V. We say that the operator A is:
Coercive if $\lim_{\|u\|\to\infty} \frac{\langle A(u),u\rangle}{\|u\|} = \infty$;
Monotone if $\langle A(u_1) - A(u_2), u_1 - u_2\rangle \geq 0$ for all u_1, u_2;
Strongly continuous if $u_n \overset{w}{\to} u$ implies $A(u_n) \to A(u)$;
Weakly continuous if $u_n \overset{w}{\to} u$ implies $A(u_n) \overset{w}{\to} A(u)$;
Demicontinuous if $u_n \to u$ implies $A(u_n) \overset{w}{\to} A(u)$.
The operator A is said to satisfy the M_o-condition if $u_n \overset{w}{\to} u$, $A(u_n) \overset{w}{\to} f$, and $[\langle A(u_n), u_n\rangle \to \langle f, u\rangle]$ imply $A(u) = f$.

Theorem 2.1. *Let V be a separable reflexive Banach space and $A : V \to V'$ an operator which is: coercive, bounded, demicontinuous, and satisfying M_o condition. Then the equation $A(u) = f$ admits a solution for each $f \in V'$.*

Now, we prove the existence of a weak solution $u \in W_0^{1,p}(\Omega)$ for the scalar case

$$
\begin{gathered}
-\Delta_p u = m(x)|u|^{p-2}u + f(x), \quad x \in \Omega, \\
u = 0 \qquad \text{on } \partial\Omega
\end{gathered}
\tag{2.1}
$$

where $0 < a(x) \in L^\infty(\Omega)$ and Ω is a bounded domain of $\mathbb{R}^N$. In this case, the operator A is $Au = -\Delta_p u - m(x)|u|^{p-2}u$.

It is proved in [2], that if $m(x)$ is a positive function in $L^\infty(\Omega)$, then the first eigenvalue $\lambda_p(m)$ of the Dirichlet p-Laplacian problem

$$
\begin{gathered}
-\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega \\
u(x) = 0 \quad \text{on } \partial\Omega
\end{gathered}
\tag{2.2}
$$

is simple, isolated and it is the unique positive eigenvalue having a nonnegative eigenfunction. Moreover it is characterized by

$$
\int_\Omega |\nabla u|^p \geq \lambda_p(m) \int_\Omega m(x)|u|^p \tag{2.3}
$$

We prove that (2.1) admits a weak solution if $\lambda_p(m) > 1$. First, we prove that A is a bounded operator:

$$(Au, v) = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v - \int_\Omega m(x)|u|^{p-2}uv$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
|(Au, v)| &\le \Big(\int_\Omega |\nabla u|^p\Big)^{\frac{p-1}{p}} \Big(\int_\Omega |\nabla v|^p\Big)^{\frac{1}{p}} + \Big(\int_\Omega m(x)|u|^p\Big)^{\frac{p-1}{p}} \Big(\int_\Omega m(x)|v|^p\Big)^{\frac{1}{p}} \\
&\le \|u\|_{1,p}^{p-1}\|v\|_{1,p}
\end{aligned}$$

To prove that A is continuous, let us assume that $u_n \to u$ in $W_0^{1,p}(\Omega)$. Then $\|u_n - u\|_{1,p} \to 0$ So that

$$\|\nabla u_n - \nabla u\|_p \to 0$$

Applying Dominated convergence theorem, we obtain

$$\||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\|_p \to 0$$

hence

$$\|Au_n - Au\|_p \le \||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\|_p + \||u_n|^{p-2}u_n - |u|^{p-2}u\|_p \to 0$$

Operator A is strictly monotone:

$$\begin{aligned}
(Au_1 - Au_2, u_1 - u_2) &= \int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla u_1 + \int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \nabla u_2 \\
&\quad - \int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla u_2 - \int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \nabla u_1 \\
&\ge \int_\Omega |\nabla u_1|^p + \int_\Omega |\nabla u_2|^p - \Big(\int_\Omega |\nabla u_1|^p\Big)^{p-1/p}\Big(\int_\Omega |\nabla u_2|^p\Big)^{1/p} \\
&\quad - \Big(\int_\Omega |\nabla u_2|^p\Big)^{p-1/p}\Big(\int_\Omega |\nabla u_1|^p\Big)^{1/p} \\
&= \|u_1\|_p^p + \|u_2\|_p^p - \|u_1\|_p^{p-1}\|u_2\|_p - \|u_2\|_p^{p-1}\|u_1\|_p \\
&= \big(\|u_1\|_{1,p}^{p-1} - \|u_2\|_{1,p}^{p-1}\big)\big(\|u_1\|_{1,p} - \|u_2\|_{1,p}\big) > 0\,.
\end{aligned}$$

Also, A is a coercive operator, since from (2.3), we have

$$\begin{aligned}
(Au, u) &= \int_\Omega |\nabla u|^p - \int_\Omega m|u|^p \\
&\ge \int_\Omega |\nabla u|^p - \frac{1}{\lambda_p(m)}\int_\Omega |\nabla u|^p \\
&= \Big(1 - \frac{1}{\lambda_p(m)}\Big)\int_\Omega |\nabla u|^p\,.
\end{aligned}$$

Then

$$\frac{(Au, u)}{\|u\|_p} = \|u\|_{1,p}^{p-1} \to \infty \quad \text{as} \quad \|u\|_{1,p} \to \infty$$

which proves the existence of a weak solution for (2.1).

3. Nonlinear systems on bounded domains

In this section, we consider the system

$$
\begin{gathered}
-\Delta_p u = a(x)|u|^{p-2}u - b(x)|u|^{\alpha}|v|^{\beta}v + f(x), \quad \text{in } \Omega \\
-\Delta_q v = d(x)|v|^{q-2}v - c(x)|v|^{\beta}|u|^{\alpha}u + g(x), \quad \text{in } \Omega \\
u = v = 0, \quad \text{on } \partial\Omega
\end{gathered}
\tag{3.1}
$$

where Ω is a bounded domain of $\mathbb{R}^N$, $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$, $\alpha+\beta+2<N$ and $a(x), b(x), c(x), d(x)$ are positive functions in $L^{\infty}(\Omega)$.

Theorem 3.1. *For $(f,\, g) \in L^p(\Omega)\times L^q(\Omega)$, there exists a weak solution $(u,\, v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ for system* (3.1) *if the following condition is satisfied:*

$$
\lambda_p(a) > 1, \quad \textit{and} \quad \lambda_q(d) > 1\,. \tag{3.2}
$$

Proof. We transform the weak formulation of the system (3.1) to the operator form

$$
A(u,v) - B(u,v) = F
$$

where, A, B and F are operators defined on $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ by

$$
(A(u,v),(\Phi_1,\Phi_2)) = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\Phi_1 + \int_\Omega |\nabla v|^{q-2}\nabla v\nabla\Phi_2,
$$

$$
\begin{aligned}
(B(u,v),(\Phi_1,\Phi_2)) &= \int_\Omega a(x)|u|^{p-2}u\Phi_1 + \int_\Omega d(x)|v|^{q-2}v\Phi_2 \\
&\quad - \int_\Omega b(x)|u|^{\alpha}|v|^{\beta}v\Phi_1 - \int_\Omega c(x)|v|^{\beta}|u|^{\alpha}u\Phi_2
\end{aligned}
$$

and

$$
(F,\Phi) = ((f_1,f_2),(\Phi_1,\Phi_2)) = \int_\Omega f_1\Phi_1 + \int_\Omega f_2\Phi_2
$$

We can write the operator $A(u,v)$ as the sum of the two operators $J_2(v), J_1(u)$, where

$$
(J_2(v),(\Phi_2)) = \int_\Omega |\nabla v|^{q-2}\nabla v\nabla\Phi_2 \quad \text{and} \quad (J_1(u),(\Phi_1)) = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\Phi_1\,.
$$

Operators J_1 and J_2 are bounded, continuous, and strictly monotone; so their sum, the operator A, will be the same. For the operator $B(u,v)$,

$$
B(u,v): W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega) \to L^p(\Omega)\times L^q(\Omega) \subset W_0^{-1,p'}(\Omega)\times W_0^{-1,q'}(\Omega),
$$

using Dominated convergence theorem and compact imbedding property [1] for the space $W_0^{1,p}(\Omega)$ inside the space $L^p(\Omega)$ and the space $W_0^{1,q}(\Omega)$ inside $L^q(\Omega)$, when Ω is a bounded domain of $\mathbb{R}^N$, we can prove that it is a strongly continuous operator. To prove that let us assume that $v_n \xrightarrow{w} v$ in $W_0^{1,q}(\Omega)$ and $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$. Then $(u_n,v_n)\to(u,v)$ in $L^p(\Omega)\times L^q(\Omega)$. Also, $(\nabla u_n,\nabla v_n)\to(\nabla u,\nabla v)$ in $L^p(\Omega)\times L^q(\Omega)$. By the Dominated Convergence Theorem, we have:

$$
\begin{gathered}
a(x)|u_n|^{p-2}u_n \to a(x)|u|^{p-2}u \qquad \text{in } L^p(\Omega) \\
d(x)|v_n|^{q-2}v_n \to d(x)|v|^{q-2}v \qquad \text{in } L^q(\Omega) \\
-b(x)|u_n|^{\alpha}|v_n|^{\beta}v_n \to -b(x)|u|^{\alpha}|v|^{\beta}v \qquad \text{in } L^p(\Omega) \\
-c(x)|v_n|^{\beta}|u_n|^{\alpha}u_n \to -c(x)|v|^{\beta}|u|^{\alpha}u \qquad \text{in } L^q(\Omega)\,.
\end{gathered}
$$

Since

$$\begin{aligned}&(B(u_n,v_n)-B(u,v),(w_1,w_2))\\&=\int_\Omega a(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)w_1+\int_\Omega d(x)(|v_n|^{q-2}v_n-|v|^{q-2}v)w_2\\&-\int b(x)(|u_n|^{\alpha}|v_n|^{\beta}v_n-|u|^{\alpha}|v|^{\beta}v)w_1-\int_\Omega c(x)(|v_n|^{\beta}|u_n|^{\alpha}u_n-|v|^{\beta}|u|^{\alpha}u)w_2,\end{aligned}$$

it follows that

$$\begin{aligned}&\|B(u_n,v_n)-B(u,v)\|\\&\le\|a(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)\|_p+\|d(x)(|v_n|^{q-2}v_n-|v|^{q-2}v)\|_q\\&+\|b(x)(|u_n|^{\alpha}|v_n|^{\beta+1}-|u|^{\alpha}|v|^{\beta+1})\|_p+\|c(x)(|u_n|^{\alpha+1}|v_n|^{\beta}-|u|^{\alpha+1}|v|^{\beta})\|_q\to 0\,.\end{aligned}$$

This proves that $-B(u,v)$ is a strongly continuous operator. So $A(u,v)-B(u,v)$ will be an operator satisfying the M_o-condition. Now, it remains to prove that $A(u,v)-B(u,v)$ is a coercive operator:

$$\begin{aligned}&|(A(u,v)-B(u,v),(u,v))|\\&=\int_\Omega|\nabla u|^p+\int|\nabla v|^q-\int_\Omega a(x)|u|^p-\int d(x)|v|^q\\&\quad+\int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1}+\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}\\&\ge\int_\Omega|\nabla u|^p+\int_\Omega|\nabla v|^q-\frac{1}{\lambda_p(a)}\int_\Omega|\nabla u|^p-\frac{1}{\lambda_q(d)}\int_\Omega|\nabla v|^q\\&=\Big(1-\frac{1}{\lambda_p(a)}\Big)\int_\Omega|\nabla u|^p+\Big(1-\frac{1}{\lambda_q(d)}\Big)\int_\Omega|\nabla v|^q\end{aligned}$$

From (3.2), we deduce

$$(A(u,v)-B(u,v),(u,v))\ge c(\|u\|_{1,p}^p+\|v\|_{1,q}^q)=c|(u,v)\|_{W_0^{1,p}\times W_0^{1,q}}$$

So that

$$\langle A(u,v)-B(u,v),(u,v)\rangle\to\infty\quad\text{as}\quad\|(u,v)\|_{W_0^{1,p}\times W_0^{1,q}}\to\infty\,.$$

This proves the coercive condition and so, the existence of a weak solution for system (3.1). □

4. Nonlinear systems defined on $\mathbb{R}^n$

We consider the nonlinear system

$$\begin{gathered}-\Delta_p u=a(x)|u|^{p-2}u-b(x)|u|^{\alpha}|v|^{\beta}v+f,\\-\Delta_q v=-c(x)|u|^{\alpha}|v|^{\beta}u+d(x)|v|^{q-2}v+g,\\\lim_{|x|\to\infty}u=\lim_{|x|\to\infty}v=0\quad u,v>0\end{gathered}\tag{4.1}$$

which is defined on $\mathbb{R}^N$. We assume that $1\le\frac{2N}{N+1}<p,q<N$ and the coefficients $a(x),b(x),c(x),d(x)$ are smooth positive functions such that

$$\begin{gathered}a(x),d(x)\in L^{p/N}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n),\\\frac{\alpha+1}{p}+\frac{\beta+1}{q}=1,\quad\alpha+\beta+2<N,\end{gathered}\tag{4.2}$$

and

$$
\begin{aligned}
b(x) &< (a(x))^{\alpha+1/p}(d(x))^{\beta+1/q}\\
c(x) &< (a(x))^{\alpha+1/p}(d(x))^{\beta+1/q}
\end{aligned}
\tag{4.3}
$$

To prove our theorem, we need the following results which are studied in [12] and that we recall briefly: Let us introduce the Sobolev space $D^{1,p}(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$
\|u\|_{D^{1,p}} = \Big(\int_{\mathbb{R}^N} |\nabla u|^p\Big)^{1/p}.
$$

It can be shown that

$$
D^{1,p}(\mathbb{R}^N) = \big\{u \in L^{\frac{Np}{N-p}}(\mathbb{R}^N) : \nabla u \in (L^p(\mathbb{R}^N))^N\big\}
$$

and that there exists $k > 0$ such that for all $u \in D^{1,p}(\mathbb{R}^N)$,

$$
\|u\|_{L^{Np/(N-p)}} \le K\|u\|_{D^{1,p}(\mathbb{R}^N)}\,. \tag{4.4}
$$

Clearly, the space $D^{1,p}(\mathbb{R}^N)$ is a reflexive Banach space embedded continuously in the space $L^{Np/(N-p)}(\mathbb{R}^N)$.

Lemma 4.1. *The eigenvalue problem*

$$
\begin{gathered}
-\Delta_p u = \lambda a(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N\\
u(x) \to 0 \quad \text{as } |x| \to \infty
\end{gathered}
\tag{4.5}
$$

admits a positive principal eigenvalue $\Lambda_a(p)$ which is associated with a positive eigenfunction $\phi \in D^{1,p}(\mathbb{R}^N)$; moreover $\Lambda_a(p)$ is characterized by

$$
\Lambda_a(p)\int_{R^N} a(x)|u|^p \le \int_{R^N} |\nabla u|^p, \quad \forall u \in D^{1,p}(\mathbb{R}^N) \tag{4.6}
$$

Theorem 4.2. *For $(f,g) \in L^{\frac{Np}{N(p-1)+p}}(\mathbb{R}^N) \times L^{\frac{Nq}{N(q-1)+q}}(\mathbb{R}^N)$, there exists a weak solution $(u,\, v) \in D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$ for system (4.1) if the following conditions are satisfied:*

$$
\Lambda_p(a) > 1, \quad \text{and} \quad \Lambda_q(d) > 1\,. \tag{4.7}
$$

Proof. By transforming the weak formulation for the system to the operator formulation, we will get the bounded operators A, B, F on the space $D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)$ which take the same previous definitions in Theorem 3.1. To distinguish that: let us assume that (Φ_1, Φ_2) in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$, then applying Hölder inequality, we get

$$
\begin{aligned}
&|(A(u,v),(\Phi_1,\Phi_2))|\\
&\le \int_{R^N} |\nabla u|^{p-1}|\nabla\Phi_1| + \int_{R^N} |\nabla v|^{q-1}|\nabla\Phi_2|\\
&\le \Big(\int_{R^N} |\nabla u|^p\Big)^{\frac{p-1}{p}}\Big(\int_{R^N} |\nabla\Phi_1|^p\Big)^{\frac{1}{p}} + \Big(\int_{R^N} |\nabla v|^q\Big)^{\frac{q-1}{q}}\Big(\int_{R^N} |\nabla\Phi_2|^q\Big)^{\frac{1}{q}}\\
&= \|u\|^{p-1}_{D^{1,p}}\|\Phi_1\|_{D^{1,p}} + \|v\|^{q-1}_{D^{1,q}}\|\Phi_2\|_{D^{1,q}}\\
&\le (\|u\|^{p-1}_{D^{1,p}} + \|v\|^{q-1}_{D^{1,q}})(\|\Phi_1\|_{D^{1,p}} + \|\Phi_2\|_{D^{1,q}})\\
&= \big(\|u\|^{p-1}_{D^{1,p}} + \|v\|^{q-1}_{D^{1,q}}\big)\|(\Phi_1,\Phi_2)\|_{D^{1,p}\times D^{1,q}}
\end{aligned}
$$

For the operator $B(u,v)$, we have

$$
\begin{aligned}
&|(B(u,v),(\Phi_1,\Phi_2))|\\
&\le \Big(\int (a(x))^{\frac{N}{p}}\Big)^{\frac{p}{N}}\Big(\int_{R^N}|u(x)|^{\frac{Np}{N-p}}\Big)^{\frac{(p-1)(N-p)}{Np}}\Big(\int_{R^N}|\Phi_1|^{\frac{Np}{N-p}}\Big)^{\frac{N-p}{Np}}\\
&\quad+\Big(\int_{R^N}(d(x))^{\frac{N}{q}}\Big)^{\frac{q}{N}}\Big(\int_{R^N}|v|^{\frac{Nq}{N-q}}\Big)^{\frac{(q-1)(N-q)}{Nq}}\Big(\int_{R^N}|\Phi_2|^{\frac{Nq}{N-q}}\Big)^{\frac{N-q}{Nq}}\\
&\quad+\Big(\int_{R^N}(b(x))^{\frac{N}{\alpha+\beta+2}}\Big)^{\frac{\alpha+\beta+2}{N}}\Big(\int_{R^N}|u|^{\frac{Np}{N-p}}\Big)^{\frac{\alpha(N-p)}{Np}}\Big(\int_{R^N}|v|^{\frac{Nq}{N-q}}\Big)^{\frac{(\beta+1)(N-q)}{Nq}}\\
&\quad\times\Big(\int_{R^N}|\Phi_1|^{\frac{Np}{N-p}}\Big)^{\frac{N-p}{Np}}+\Big(\int_{R^N}(c(x))^{\frac{N}{\alpha+\beta+2}}\Big)^{\frac{\alpha+\beta+2}{N}}\Big(\int_{R^N}|u|^{\frac{Np}{N-p}}\Big)^{\frac{(\alpha+1)(N-p)}{Np}}\\
&\quad\times\Big(\int_{R^N}|v|^{\frac{Nq}{N-q}}\Big)^{\frac{\beta(N-q)}{Nq}}\Big(\int_{R^N}|\Phi_2|^{\frac{Np}{N-p}}\Big)^{\frac{N-q}{Nq}}\\
&\le k_1\|u\|^{p-1}_{D^{1,p}}\|\Phi_1\|_{D^{1,p}}+k_2\|v\|^{q-1}_{D^{1,q}}\|\Phi_2\|_{D^{1,q}}\\
&\quad+k_3\|u\|^{\alpha}_{D^{1,p}}\|v\|^{\beta+1}_{D^{1,p}}\|\Phi_1\|_{D^{1,p}}+k_4\|u\|^{\alpha+1}_{D^{1,p}}\|v\|^{\beta}_{D^{1,q}}\|\Phi_2\|_{D^{1,q}}\\
&\le \Big(k_1\|u\|^{p-1}_{D^{1,p}}+k_2\|v\|^{q-1}_{D^{1,q}}+k_3\|u\|^{\alpha}_{D^{1,p}}\|v\|^{\beta+1}_{D^{1,p}}+k_4\|u\|^{\alpha+1}_{D^{1,p}}\|v\|^{\beta}_{D^{1,q}}\Big)\\
&\quad\times\|(\Phi_1,\Phi_2)\|_{D^{1,p}\times D^{1,q}},
\end{aligned}
$$

this proves the boundedness of the operator $B(u,v)$. For F, we have

$$
\begin{aligned}
|(F,\Phi)| &= |((f_1,f_2),(\Phi_1,\Phi_2))|\\
&\le \Big(\int_{R^N}(|f_1|)^{\frac{Np}{N(p-1)+p}}\Big)^{\frac{N(p-1)+p}{Np}}\Big(\int_{R^N}|\Phi_1|^{\frac{Np}{N-p}}\Big)^{\frac{N-p}{Np}}\\
&\quad+\Big(\int_{R^N}(|f_2|)^{\frac{Nq}{N(q-1)+q}}\Big)^{\frac{N(q-1)+q}{Nq}}\Big(\int_{R^N}|\Phi_2|^{\frac{Nq}{N-q}}\Big)^{\frac{N-q}{Nq}}\\
&\le \Big(\|f_1\|_{\frac{Np}{N(p-1)+p}}+\|f_2\|_{\frac{Nq}{N(q-1)+q}}\Big)\|(\Phi_1,\Phi_2)\|_{D^{1,p}\times D^{1,q}}.
\end{aligned}
$$

Now, the operator $A(u,v)=J_1(u)+J_2(v)$ is continuous and strictly monotone on $D^{1,p}\times D^{1,q}$, since

$$
\begin{gathered}
(J_1(u_1)-J_1(u_2),u_1-u_2)\ge(\|u_1\|^{p-1}_{D^{1,p}}-\|u_2\|^{p-1}_{D^{1,p}})(\|u_1\|_{D^{1,p}}-\|u_2\|_{D^{1,q}})>0,\\
(J_2(u_1)-J_2(u_2),u_1-u_2)\ge(\|u_1\|^{q-1}_{D^{1,q}}-\|u_2\|^{q-1}_{D^{1,q}})(\|u_1\|_{D^{1,q}}-\|u_2\|_{D^{1,q}})>0
\end{gathered}
$$

For the operator $B(u,v)$, we can prove that it is a strongly continuous operator by using Dominated convergence theorem and continuous imbedding property for the space $D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)$ into $L^{\frac{Np}{N-p}}(\mathbb{R}^N)\times L^{\frac{Nq}{N-q}}(\mathbb{R}^N)$: let us assume that $v_n\xrightarrow{w}v$ in $D^{1,q}(\mathbb{R}^N)$ and $u_n\xrightarrow{w}u$ in $D^{1,p}(\mathbb{R}^N)$. Then $(u_n,v_n)\to(u,v)$ in $L^p(\mathbb{R}^N)\times L^q(\mathbb{R}^N)$ and $(\nabla u_n,\nabla v_n)\to(\nabla u,\nabla v)$ in $L^p(\mathbb{R}^N)\times L^q(\mathbb{R}^N)$. Now, the sequence (u_n) is bounded in $D^{1,p}(\mathbb{R}^N)$, then it is containing a subsequence again denoted by (u_n) converges strongly to u in $L^{\frac{Np}{N-p}}(B_{r_0})$ for any bounded ball $B_{r_0}=\{x\in\mathbb{R}^N:\|x\|\le r_0\}$. Similarly (v_n) converges strongly to v in $L^{\frac{Nq}{N-q}}(B_{r_0})$. Since $u_n,u\in L^{\frac{Np}{N-p}}(B_{r_0})$

and $v_n, v \in L^{\frac{Nq}{N-q}}(B_{r_0})$. Then using the dominated convergence theorem, we have

$$\|a(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)\|_{\frac{Np}{N(p-1)+p}} \to 0, \tag{4.8}$$

$$\|d(x)(|v_n|^{q-2}v_n - |v|^{q-2}v)\|_{\frac{Nq}{N(q-1)+q}} \to 0, \tag{4.9}$$

$$\|b(x)(|u_n|^{\alpha-1}|v_n|^{\beta+1}u_n - |u|^{\alpha-1}|v|^{\beta+1}u)\|_{\frac{Np}{N(p-1)+p}} \to 0, \tag{4.10}$$

$$\|c(x)(|u_n|^{\alpha+1}|v_n|^{\beta-1}u_n - |u|^{\alpha+1}|v|^{\beta-1}u)\|_{\frac{Nq}{N(q-1)+q}} \to 0\,. \tag{4.11}$$

Then

$$\begin{aligned}
&\|B(u_n, v_n) - B(u, v)\|_{D^{1,p}(B_{r_0})\times D^{1,q}(B_{r_0})} \\
&\leq \|a(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)\|_{\frac{Np}{N(p-1+p)}} + \|d(x)(|v_n|^{q-2}v_n - |v|^{q-2}v)\|_{\frac{Nq}{N(q-1+q)}} \\
&\quad + \|b(x)(|u_n|^{\alpha}|v_n|^{\beta+1}u_n - |u|^{\alpha}|v|^{\beta+1}v)\|_{\frac{Np}{N(p-1)+p}} \\
&\quad + \|c(x)(|u_n|^{\alpha+1}|v_n|^{\beta-1}u_n - |u|^{\alpha+1}|v|^{\beta-1}v)\|_{\frac{Nq}{N(q-1)+q}} \to 0\,.
\end{aligned}$$

It remains to study the norm

$$\|B(u_n, v_n) - B(u, v)\|_{D^{1,p}(\mathbb{R}^N - B_{r_0})\times D^{1,q}(\mathbb{R}^N - B_{r_0})}$$

It is sufficient to study the norms in the inequalities (4.8)–(4.11) and try to make it as small as possible. We will study the norm in (4.8) only because the others will be the same.

Since, (u_n) converges weakly in the space $D^{1,p}(\mathbb{R}^N)$, using Sobelev inequality, (u_n) will be bounded in the space $L^{\frac{Np}{N-p}}(\mathbb{R}^N)$, so $|u_n|^{p-1}$ will be bounded in $L^{\frac{Np}{N(p-1)+p}}(\mathbb{R}^N - B_{r_0})$ and $(|u_n|^{p-2}u_n - |u|^{p-2}u)$ is bounded in $L^{\frac{Np}{N(p-1)+p}}(\mathbb{R}^N - B_{r_0})$.

Since, $a(x) \in L^{\frac{N}{p}}(\mathbb{R}^N)$, we can make the integral $\int_{(\mathbb{R}^N - B_{r_0})} |a(x)|^{\frac{N}{p}}$ as small as possible by choosing r_0 big as possible, this means that there exists $r_0 > 0$ such that

$$\|a(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)\|_{L^{\frac{Np}{N(p-1)+p}}(\mathbb{R}^N - B_{r_0})} < \frac{\epsilon}{4}.M = \frac{\epsilon}{4}$$

for all $n \geq N_0$, $r \geq r_0$. Since

$$\begin{aligned}
&\|a(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)\|_{L^{\frac{Np}{N(p-1)+p}}(\mathbb{R}^N)} \\
&= \|a(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)\|_{L^{\frac{Np}{N(p-1)+p}}(B_{r_0})} \\
&\quad + \|a(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)\|_{L^{\frac{Np}{N(p-1)+p}}(\mathbb{R}^N - B_{r_0})},
\end{aligned}$$

it follows that

$$\|a(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)\|_{L^{\frac{Np}{N(p-1)+p}}(\mathbb{R}^N)} \to 0\,.$$

By repeating the previous steps on the remaining terms in

$$\|B(u_n, v_n) - B(u, v)\|_{D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)},$$

we can prove that this norm tending strongly to zero and then the operator $B(u, v)$ is strongly continuous. It remains to justify that the operator $A(u, v) - B(u, v)$ is a coercive operator. From (4.3), (4.6) and (4.7), we obtain

$$\begin{aligned}
&(A(u,v)-B(u,v),(u,v))\\
&=\int_{\mathbb{R}^N}|\nabla u|^p+\int_{\mathbb{R}^N}|\nabla v|^q-\int_{\mathbb{R}^N}a(x)|u|^p-\int_{\mathbb{R}^N}d(x)|v|^q\\
&\quad+\int b(x)|u|^{\alpha+1}|v|^{\beta+1}+\int b(x)|u|^{\alpha+1}|v|^{\beta+1}\\
&\ge\int_{\mathbb{R}^N}|\nabla u|^p+\int_{\mathbb{R}^N}|\nabla v|^q-\frac{1}{\Lambda_a(p)}\int_{\mathbb{R}^N}|\nabla u|^p-\frac{1}{\Lambda_d(q)}\int_{\mathbb{R}^N}|\nabla v|^q\\
&=\Big(1-\frac{1}{\Lambda_a(p)}\Big)\int_{\mathbb{R}^N}|\nabla u|^p+\Big(1-\frac{1}{\Lambda_d(q)}\Big)\int_{\mathbb{R}^N}|\nabla v|^q\\
&>c\Big(\|u\|^p_{D^{1,p}}+\|v\|^q_{D^{1,q}}\Big)\,.
\end{aligned}$$

So that

$$(A(u,v)-B(u,v),(u,v))\to\infty\quad\text{as}\quad\|(u,v)\|_{D^{1,p}\times D^{1,q}}\to\infty$$

The coercive condition for the operator completes the proof of the existence of a weak solution for system (4.1). $\square$

References

[1] R. A. Adams; *Sobolev Spaces*, Academic Press, New York, 1975.

[2] A. Anane; *Siomlicite et Isolution de la Premiere Valeur Propre du p-Laplaian aves Poids*, Comptes Rendus Acad. Sc. Paris, vol. 305 (1987), 725-728.

[3] C. Atkinson, K. El-Ali; *Some boundeary value problems for the Bingham model* , J. non-Newtonian Fluid Mech., vol. 41 (1992), 339-363.

[4] C. Atkinson, Champion, C. R.; *On some boundary-value problems for the equation* $\nabla.(F|\nabla w|\nabla w)=0$, Proc. R. Soc. London A, vol. 448 (1995), 269-279.

[5] G. Barles; *Remarks on Uniqueness Results of the First Eigenvalue of the p-Laplacian* , Ann. Fac. Sc. Toulouse t., vol. 9 (1988), 65-75.

[6] M. Berger; *Nonlinear and Functional Analysis*, Academic Press, New York 1977.

[7] L. Boccardo, J. Fleckinger, F. De Thelin; *Existence of Solutions for Some Non-Linear Co-operative Systems and Some Applications*, Diff. and Int. Eqn., vol. 7 no. 3 (1994), 689-698.

[8] M. Bouchekif, H. Serag, F. and De Thelin; *On Maximum Principle and Existence of Solutions for Some Nonlinear Elliptic Systems*, Rev. Mat. Apl., vol. 16 (1995), 1-16.

[9] J. I. Diaz; *Nonlinear Partial Differential Equations and free Boundaries*, Pitman, Program (1985).

[10] P. Drabek, A. Kufner, F. Nicolosi; *Quasilinear Elliptic Equations with Degenerations and Singularities*, Walter de Gruyter, New York, 1997.

[11] J. Fleckinger, J. Hernandes, P. Takac, F. De Thelin; *On Uniqueness and positivity for Solutions of equations with the p-Laplacian*, J. Diff. and Int. Eqns., vol. 8 (1996), 69-85.

[12] J. Fleckinger, R. Manasevich, N. Stavrakakies, F. De Thelin; *Principal Eigenvalues for some Quasilinear Elliptic Equations on* $\mathbb{R}^n$, Advances in Diff. Eqns., vol. 2 no. 6 (1997), 981-1003.

[13] J. Fleckinger, H. Serag; *Semilinear Cooperative Elliptic Systems on* $\mathbb{R}^n$, Rend. di Mat., vol. Seri VII, 15 Roma (1995), 89-108.

[14] J. Fleckinger, P. Takac; *Uniqueness of Positive Solutions for Non-Linear Cooperative Systems with the p-Laplacian*, Indian Univ. Math. J., vol. 34 no. 4 (1994), 1227-1253.

[15] J. Fleckinger, R. F. Mansevich, N. M. Stravrakakis, F. de Thelin; *Principal Eigenvalues for some Quasilinear Elliptic systems on* R^N Advances in Diff.Eq., vol. 2, no. 6 (1997), p981-1003

[16] J. Francu; *Solvability of Operator Equations, Survey Directed to Differential Equations*, Lecture Notes of IMAMM 94, Proc. of the Seminar "Industrial Mathematics and Mathematical Modelling", Rybnik, Univ. West Bohemia in Pilsen, Faculty of Applied Sciences, Dept. of Math., July, 4-8, 1994.

[17] S. Fucik, A. Kufner; *Nonlinear Differential Equations*, Czeeh edition SNTL, Prague, (1978), English translation, Elsevier, Amsterdam 1980.

[18] S. Fucik, J. Necas, J. Soucek, V. Soucek; *Spectral Analysis of Nonlinear Operators*, Springer-Verlag, New York, 1973.
[19] D. Pascali, S. Sburlan; *Nonlinear mappings of monotone type*, Editura Academiei, Bucuresti, Romania, 1978.
[20] H. M. Serag, E. A. El-Zahrani; *Maximum Principle and Existence of Positive Solution for Nonlinear Systems on* $\mathbb{R}^N$, Electron. J. Diff. Eqns., vol. 2005 (2005) no. 85, 1-12.
[21] E. Zeidler; *Nonlinear Functional Analysis and its Applications I, II*, Springer Verlag, New York, 1986.

Eada A. El-Zahrani
Mathematics Department, Faculty of Science for Girls,, Dammam, P. O. Box 838, Pincode 31113, Saudi Arabia

Hassan M. Serag
Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt
E-mail address: `serraghm@yahoo.com`