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WEAK SOLUTIONS FOR QUASILINEAR DEGENERATE PARABOLIC SYSTEMS

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Abstract. This paper concerns the initial Dirichlet boundary-value problem for a class of quasilinear degenerate parabolic systems. Due to the degeneracies, the problem does not have classical solutions in general. Combining the special form of the system, a proper concept of a weak solution is presented, then the existence and uniqueness of weak solutions are proved. Moreover, the asymptotic behavior of weak solutions will also be discussed.

1. Introduction and Results

This paper concerns the initial Dirichlet boundary-value problem for the quasilinear degenerate parabolic system

$$\begin{gathered} u_t = a(u)(\Delta u + \alpha v) \quad \text{in } \Omega_\infty, \\ v_t = b(v)(\Delta v + \beta u) \quad \text{in } \Omega_\infty, \\ u(x,t) = 0, \quad v(x,t) = 0, \quad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $\Omega_\infty = \Omega \times (0,+\infty), \Omega$ is a bounded domain in $\mathbb{R}^N$ with approximately smooth boundary $\partial\Omega$, α and β are positive constants, $a(\cdot), b(\cdot) \in \mathcal{K} = \{y(s) \in C^1[0,\infty); y(0) = 0, y'(s) > 0, \forall s > 0\}$, and

$$u_0, v_0 \in \mathcal{S} = \{y(x) \in C(\overline{\Omega}) \cap H^1(\Omega); y(x) \geq 0 \text{ on } \overline{\Omega}, y(x) = 0 \text{ on } \partial\Omega\}.$$

This system can be used to describe the development of two groups in the dynamics of biological groups where u and v are the densities of the different groups. Similar systems can be found in [4, 7, 8, 10, 11, 14].

The system has been studied in a series of papers, see [3, 15, 5] and references therein. For instance, it was proved in [3] that under the following assumption conditions:

(H1) $u_0, v_0 \in C^1(\overline{\Omega})$, $u_0 > 0$, $v_0 > 0$ in Ω, $u_0 = v_0 = 0$ on $\partial\Omega$;
(H2) $\frac{\partial u_0}{\partial \nu} < 0$, $\frac{\partial v_0}{\partial \nu} < 0$ on $\partial\Omega$, where ν denotes the outward normal to $\partial\Omega$;

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(H3) $a, b \in C[0,\infty) \cap C^1(0,\infty)$ such that $a, b > 0$ in $(0,\infty)$ and $a', b' > 0$ in $(0,\infty)$;

(H4) Either $\liminf_{s\to\infty} \frac{a(s)}{b(s)} > 0$ or $\liminf_{s\to\infty} \frac{b(s)}{a(s)} > 0$ holds,

the positive solution of (1.1) blows up in finite time if and only if $\lambda_1^2 < \alpha\beta$, $\int_0^\infty ds/(sa(s)) < \infty$ and $\int_0^\infty ds/(sb(s)) < \infty$, where λ_1 denotes the first eigenvalue of $-\Delta$ in Ω with the homogeneous Dirichlet boundary condition. In [15], the author discussed a special case of (1.1): $a(u) = u^p, b(u) = u^q$ with $p, q \geq 1$, and proved that under the conditions (H1)-(H2), the positive solutions of (1.1) exist globally if and only if $\lambda_1^2 \geq \alpha\beta$. For single equation $(a(s) = b(s), \alpha = \beta, u_0 = v_0), u_t = a(u)(\Delta u + \alpha u)$, we refer to [2, 6, 12] and references therein. In [2, 12], for instance, the authors studied the equation with $a(s) = s$ and obtained some interesting results.

Since the system is degenerate at points where $u = 0$ or $v = 0$, problem (1.1) does not always have classical solutions, and we have to consider weak solutions. Moreover, we are only interested in the nonnegative weak solutions.

We remark that, as usual, one may easily define a weak solution (which, for instance, means a function satisfying the condition (a), (b) and (d) of the following Definition 1.1). However, because of the special form of this system, such weak solutions may not be uniquely determined by the initial data. In fact, for single equation some examples showing the non-uniqueness had been constructed, see [2, 12]. So, it is natural to ask how to define a weak solution to guarantee both uniqueness and existence. One of purposes of this paper is to give a positive answer to the question. Moreover, the asymptotic behavior of solutions will also be discussed. This is the only work concerning the study of weak solutions to the system, as far as we know.

Before giving a proper concept of weak solutions, we first define the support of a nonnegative measurable function $w : \Omega \to [0,\infty)$:

$$\operatorname{supp} w = \overline{\Big\{x \in G; \lim_{\rho\to 0^+} \frac{\mu(G \cap B_\rho(x))}{\mu(B_\rho(x))} > 0\Big\}},$$

where $G = \{x \in \Omega; w(x) > 0\}$, $B_\rho(x) = \{y \in \Omega; |x - y| < \rho\}$, and $\mu(E)$ denotes the Lebesgue measure of a set E in $\mathbb{R}^N$. It is easy to see that if $w \in C(\Omega)$, then $\operatorname{supp} w = \overline{G}$.

For $T > 0, \rho > 0$ and $w \in \mathcal{S}$, denote $\Omega_T, \Omega(w)$ and $\Omega^\rho(w)$ by

$$\begin{gathered} \Omega_T = \Omega \times (0,T), \\ \Omega(w) = \{x \in \Omega; w(x) > 0\}, \\ \Omega^\rho(w) = \{x \in \Omega(w); \operatorname{dist}(x, \partial\Omega(w)) > \rho\}. \end{gathered}$$

Definition 1.1. (u, v) is called a weak solution of (1.1), if for any $T > 0$ the following conditions hold:

(a) $u, v \geq 0$ a.e. in Ω_T, $u, v \in L^\infty(\Omega_T) \cap L^2(0,T; H_0^1(\Omega))$, $u_t, v_t \in L^2(\Omega_T)$;

(b) For any $\varphi, \psi \in C_0^\infty(\Omega_T)$, there holds

$$\begin{aligned} &\iint_{\Omega_T} \Big(- u\varphi_t + a(u)\nabla u \nabla\varphi + a'(u)|\nabla u|^2\varphi - \alpha a(u) v\varphi\Big)\, dx\, dt \\ &+ \iint_{\Omega_T} \Big(- v\psi_t + b(v)\nabla v \nabla\psi + b'(v)|\nabla v|^2\psi - \beta b(v) u\psi\Big)\, dx\, dt = 0; \end{aligned}$$

(c) $\operatorname{supp} u(t) = \operatorname{supp} u_0$, $\operatorname{supp} v(t) = \operatorname{supp} v_0$, a.e. in $(0,T)$, and for all $\rho > 0$ there exist positive constants $c_1 = c_1(\rho)$ and $c_2 = c_2(\rho)$ such that

$$u \geq c_1 \quad \text{a.e. in } \Omega^\rho(u_0) \times (0,T),$$
$$v \geq c_2 \quad \text{a.e. in } \Omega^\rho(v_0) \times (0,T);$$

(d) $(u(t), v(t)) \to (u_0, v_0)$ in $[L^1(\Omega)]^2$, as $t \to 0^+$.

The purposes of this paper are to prove the following theorems.

Theorem 1.2. *Let $a, b \in \mathcal{K}, \alpha, \beta > 0$, and assume $u_0, v_0 \in \mathcal{S}$. If $\max\{\alpha,\beta\} < \lambda_1$, where λ_1 is the same as before, then* (1.1) *admits a unique weak solution.*

Theorem 1.3. *Suppose $a, b \in \mathcal{K}$, and there exist positive constants $\sigma_2, \rho_2, \rho_1, \rho_0$ and a nonnegative constant σ_1 such that $\sigma_2 \geq 1, \sigma_2 > \sigma_1 \geq 0, \rho_2 \geq \rho_1$, and*

$$\lim_{s\to 0^+} \frac{a(s)}{s^{\sigma_2}} = \rho_2, \quad \lim_{s\to +\infty} \frac{a(s)}{s^{\sigma_2}} = \rho_1, \tag{1.2}$$

$$a(s) \geq \sigma_2^{-1} s a'(s), \quad \rho_0 b(s) s^{\sigma_1} \geq a(s), \quad \forall s \geq 0, \tag{1.3}$$

and let $\alpha, \beta > 0, u_0, v_0 \in \mathcal{S}$, and $A_0 \equiv \int_\Omega \Big(\frac{u_0^2}{2} + \int_0^{v_0} \frac{sa(s)}{b(s)} ds\Big) dx > 0$. If $\max\{\alpha,\beta\} < \frac{4}{(2+\sigma_2)^2}\lambda_1$, then there exists a positive constant C depending only on the known data such that

$$\int_\Omega \Big(\frac{u^2(x,t)}{2} + \int_0^{v(x,t)} \frac{sa(s)}{b(s)} ds\Big) dx \leq \Big[\frac{1}{Ct + A_0^{-\sigma_2/2}}\Big]^{2/\sigma_2},$$

where (u,v) is the unique weak solution of (1.1).

This paper is organized as follows: in next section, we prove Theorem 1.2. Section 3 is devoted to the proof of Theorem 1.3.

2. Proof of Theorem 1.2

2.1. **Proof of existence.** To establish the existence, we use the method of regularization. For this purpose, we consider for $T > 0$,

$$\begin{gathered} u_{\varepsilon t} = a_\varepsilon(u_\varepsilon)(\Delta u_\varepsilon + \alpha v_\varepsilon) \quad \text{in } \Omega_T, \\ v_{\varepsilon t} = b_\varepsilon(v_\varepsilon)(\Delta v_\varepsilon + \beta u_\varepsilon) \quad \text{in } \Omega_T, \\ u_\varepsilon(x,t) = v_\varepsilon(x,t) = \varepsilon \quad \text{on } \partial\Omega \times (0,T), \\ u_\varepsilon(x,0) = u_0(x) + \varepsilon, \quad v_\varepsilon(x,0) = v_0(x) + \varepsilon \quad \text{in } \Omega, \end{gathered} \tag{2.1}$$

where $\varepsilon \in (0,1)$, $a_\varepsilon, b_\varepsilon \in C^1(\mathbb{R})$ and

$$a_\varepsilon(s) = \begin{cases} a(s), & s \geq \varepsilon, \\ \frac{a^2(s)+a^2(\varepsilon)}{2a(\varepsilon)}, & s < \varepsilon. \end{cases} \qquad b_\varepsilon(s) = \begin{cases} b(s), & s \geq \varepsilon, \\ \frac{b^2(s)+a^2(\varepsilon)}{2b(\varepsilon)}, & s < \varepsilon. \end{cases}$$

Lemma 2.1. *Let $\max\{\alpha,\beta\} < \lambda_1$. If $(u_\varepsilon, v_\varepsilon)$ is a classical solution of* (2.1)*, then there exists a positive constant C independent of ε such that*

$$\varepsilon \leq u_\varepsilon, v_\varepsilon \leq C \quad \text{in } \Omega_T.$$

Proof. First, it is easy to see that the maximal principle implies the left-hand side of the above claim. It suffices to show it's right-hand side.

It is well known that for $\tilde{\lambda} = (\max\{\alpha, \beta\} + \lambda_1)/2 < \lambda_1$, there exists a bounded domain $\tilde{\Omega}$ such that $\tilde{\Omega} \supset \overline{\Omega}$ and $\tilde{\lambda}$ is the first eigenvalue of $-\Delta$ in $\tilde{\Omega}$ with homogeneous Dirichlet boundary condition [1]. Denote by $\tilde{\phi}$ the associated eigenfunction. Then $\tilde{\phi} \in C^2(\Omega) \cap C(\overline{\Omega}), \tilde{\phi} > 0$ in $\tilde{\Omega}$, and hence there exists a positive constant δ such that $\tilde{\phi} \geq \delta$ on $\overline{\Omega}$. Now choosing a positive constant $\tilde{k}$ such that

$$\tilde{k}\tilde{\phi} \geq \max_{\overline{\Omega}} u_0 + 1 \quad \text{on } \overline{\Omega}.$$

Let $w = \tilde{k}\tilde{\phi}$. Then we have

$$\begin{gathered}
w_t - a_\varepsilon(w)(\Delta w + \alpha w)a_\varepsilon(w)[\tilde{\lambda} - \alpha]w > 0 \quad \text{in } \Omega_T, \\
w_t - b_\varepsilon(w)(\Delta w + \beta w) = b_\varepsilon(w)[\tilde{\lambda} - \beta]w > 0 \quad \text{in } \Omega_T,
\end{gathered}$$

hence it follows from Nagumo's lemma [13, pp. 4697] that

$$u_\varepsilon, v_\varepsilon \leq w \quad \text{in } \Omega_T.$$

The proof is complete. □

By the standard theory of parabolic equations [9, pp. 596], (2.1) admits a unique classical solution $(u_\varepsilon, v_\varepsilon)$ satisfying the inequalities of Lemma 2.1. Moreover, the maximal principle implies

$$u_{\varepsilon_2} \geq u_{\varepsilon_1}, \quad v_{\varepsilon_2} \geq v_{\varepsilon_1}, \quad \text{in } \Omega, \quad \text{for } \varepsilon_2 > \varepsilon_1. \tag{2.2}$$

Thus, by Lemma 2.1, $(u_\varepsilon, v_\varepsilon)$ solves the problem

$$\begin{gathered}
u_{\varepsilon t} = a(u_\varepsilon)(\Delta u_\varepsilon + \alpha v_\varepsilon) \quad \text{in } \Omega_T, \\
v_{\varepsilon t} = b(v_\varepsilon)(\Delta v_\varepsilon + \beta u_\varepsilon) \quad \text{in } \Omega_T, \\
u_\varepsilon(x,t) = v_\varepsilon(x,t) = \varepsilon \quad \text{on } \partial\Omega \times (0,T), \\
u_\varepsilon(x,0) = u_0(x) + \varepsilon, \quad v_\varepsilon(x,0) = v_0(x) + \varepsilon \quad \text{in } \Omega,
\end{gathered} \tag{2.3}$$

In view of Lemma 2.1 and (2.2), one can derive that there exist nonnegative functions $u, v \in L^\infty(\Omega)$ such that

$$(u_\varepsilon, v_\varepsilon) \to (u, v) \quad \text{a.e. in } \Omega_T, \text{ as } \varepsilon \to 0. \tag{2.4}$$

Next, we shall show that (u, v) is a weak solution of (1.1). For this, it suffices to prove that (u, v) satisfies the conditions (a)-(d) in Definition 1.1. Let us first check the condition (a). To do this, it needs to establish some basic estimates on u_ε and v_ε.

Lemma 2.2. *For all $\tau \in (0,T)$ and $\varepsilon \in (0,1)$, we have*

(1)
$$\iint_{\Omega_\tau} \frac{u_{\varepsilon t}^2}{a(u_\varepsilon)}\,dx\,dt + \int_\Omega |\nabla u_\varepsilon(x,\tau)|^2 dx \leq C$$

(2)
$$\iint_{\Omega_\tau} \frac{v_{\varepsilon t}^2}{b(u_\varepsilon)}\,dx\,dt + \int_\Omega |\nabla v_\varepsilon(x,\tau)|^2 dx \leq C.$$

Here C are positive constants independent of ε.

Proof. Since the proof is exactly the same for (1) and (2), we will show the validity of (1). Multiplying the first equation of (2.3) by $u_{\varepsilon t}/a(u_\varepsilon)$ and integrating over Ω_τ and noticing $u_{\varepsilon t} = 0$ on $\partial\Omega \times (0,T)$, we obtain

$$
\begin{aligned}
&\iint_{\Omega_\tau} \frac{u_{\varepsilon t}^2}{a(u_\varepsilon)}\,dx\,dt \\
&= \iint_{\Omega_\tau} (\Delta u_\varepsilon + \alpha v_\varepsilon) u_{\varepsilon t}\,dx\,dt \\
&= -\iint_{\Omega_\tau} \nabla u_\varepsilon \nabla u_{\varepsilon t}\,dx\,dt + \iint_{\Omega_\tau} \alpha v_\varepsilon u_{\varepsilon t}\,dx\,dt \\
&= -\iint_{\Omega_\tau} \frac{\partial}{\partial t}\Big(\frac{|\nabla u_\varepsilon|^2}{2}\Big)\,dx\,dt + \iint_{\Omega_\tau} \Big[\alpha v_\varepsilon a(u_\varepsilon)^{1/2}\Big]\Big[\frac{u_{\varepsilon t}}{a(u_\varepsilon)^{1/2}}\Big]\,dx\,dt \\
&\le \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx + \iint_{\Omega_\tau} \Big[\alpha v_\varepsilon a(u_\varepsilon)^{1/2}\Big]\Big[\frac{u_{\varepsilon t}}{a(u_\varepsilon)^{1/2}}\Big]\,dx\,dt.
\end{aligned}
$$

By the inequality $ab \le \frac{1}{2}(a^2 + b^2)$, we have,

$$
\iint_{\Omega_\tau} \frac{u_{\varepsilon t}^2}{a(u_\varepsilon)} dx\,dt \le \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx + \frac{\alpha^2}{2}\iint_{\Omega_\tau} v_\varepsilon^2 a(u_\varepsilon)\,dx\,dt + \frac{1}{2}\iint_{\Omega_\tau} \frac{u_{\varepsilon t}^2}{a(u_\varepsilon)} dx\,dt,
$$

i.e.

$$
\iint_{\Omega_\tau} \frac{u_{\varepsilon t}^2}{a(u_\varepsilon)}\,dx\,dt \le \int_\Omega |\nabla u_0|^2 dx + \alpha^2 \iint_{\Omega_\tau} v_\varepsilon^2 a(u_\varepsilon)\,dx\,dt,
$$

and then, by Lemma 2.1, (1) is proved. This completes the proof. □

From Lemma 2.1, (2.4) and Lemma 2.2, one may derive that

$$
(u_\varepsilon, v_\varepsilon) \rightharpoonup (u, v) \quad \text{in } [H^1(\Omega_T)]^2, \quad \text{as } \varepsilon \to 0, \tag{2.5}
$$

where $\rightharpoonup$ denotes the weak convergence, and

$$
0 \le u, v \in L^\infty(\Omega_T) \cap L^2(0,T; H_0^1(\Omega)); \quad u_t, v_t \in L^2(\Omega_T).
$$

Thus the condition (a) is satisfied. Next, let us check the condition (b). For this, the following estimates are required.

Lemma 2.3. *For any $\theta \in (0,1)$, there exist positive constants C_1 and C_2 independent of ε such that*

(1)
$$
\iint_{\Omega_T} \frac{a'(u_\varepsilon)|\nabla u_\varepsilon|^2}{a(u_\varepsilon)^\theta}\,dx\,dt \le C_1
$$
provided $a'(0) > 0$

(2)
$$
\iint_{\Omega_T} \frac{b'(v_\varepsilon)|\nabla v_\varepsilon|^2}{b(v_\varepsilon)^\theta}\,dx\,dt \le C_2
$$
provided $b'(0) > 0$.

Proof. Since the proof is exactly the same for (1) and (2), we shall show the validity of (1). Given that $a'(0) > 0$ and $\theta \in (0,1)$, we claim that for any $l > 0$, $1/a(s)^\theta$ is integrable on $[0,l]$. Indeed, since $a'(s) > 0$ for $s \ge 0$, we have $M = \min_{s\in[0,l]} a'(s) >$

0, and hence for any $s \in (0, l]$, by mean value theorem and noticing $a(0) = 0$, there exists $\xi_s \in [0, s]$ such that $a(s) = a'(\xi_s)s \geq Ms$. Therefore, for $\theta \in (0, 1)$, we have

$$\int_0^l \frac{1}{a(s)^\theta} ds \leq \int_0^l \frac{1}{M^\theta s^\theta} ds \leq \frac{l^{1-\theta}}{M^\theta(1-\theta)}.$$

This proves the above claim. Now multiplying the first equation of (2.3) by $a(u_\varepsilon)^{-\theta}$ and integrating Ω_T and noticing $\frac{\partial u_\varepsilon}{\partial \nu} \leq 0$ on $\partial\Omega \times (0, T)$, where ν denotes the unit outward normal to $\partial\Omega \times (0, T)$, we have

$$\begin{aligned}
&\iint_{\Omega_T} \frac{u_{\varepsilon t}}{a(u_\varepsilon)^\theta}\, dx\, dt \\
&= \int_\Omega \int_0^{u_\varepsilon(x,T)} \frac{1}{a(s)^\theta}\, ds\, dx - \int_\Omega \int_0^{u_0(x)+\varepsilon} \frac{1}{a(s)^\theta}\, ds\, dx \\
&= \iint_{\Omega_T} a(u_\varepsilon)^{1-\theta}(\Delta u_\varepsilon + \alpha v_\varepsilon)\, dx\, dt \\
&= \iint_{\Omega_T} \Big[\operatorname{div}(a(u_\varepsilon)^{1-\theta}\nabla u_\varepsilon) - (1-\theta)\frac{a'(u_\varepsilon)|\nabla u_\varepsilon|^2}{a(u_\varepsilon)^\theta} + \alpha v_\varepsilon a(u_\varepsilon)^{1-\theta} \Big]\, dx\, dt \\
&= \int_0^T \int_{\partial\Omega} a(u_\varepsilon)^{1-\theta} \frac{\partial u_\varepsilon}{\partial \nu}\, d\sigma\, dt - (1-\theta) \iint_{\Omega_T} \frac{a'(u_\varepsilon)|\nabla u_\varepsilon|^2}{a(u_\varepsilon)^\theta}\, dx\, dt \\
&\quad + \alpha \iint_{\Omega_T} v_\varepsilon a(u_\varepsilon)^{1-\theta}\, dx\, dt \\
&\leq -(1-\theta) \iint_{\Omega_T} \frac{a'(u_\varepsilon)|\nabla u_\varepsilon|^2}{a(u_\varepsilon)^\theta}\, dx\, dt + \alpha \iint_{\Omega_T} v_\varepsilon a(u_\varepsilon)^{1-\theta}\, dx\, dt,
\end{aligned}$$

and hence

$$\begin{aligned}
&\iint_{\Omega_T} \frac{a'(u_\varepsilon)|\nabla u_\varepsilon|^2}{a(u_\varepsilon)^\theta}\, dx\, dt \\
&\leq \frac{1}{1-\theta} \int_\Omega \int_0^{u_0(x)+\varepsilon} \frac{1}{a(s)^\theta}\, ds\, dx + \frac{\alpha}{1-\theta} \iint_{\Omega_T} v_\varepsilon a(u_\varepsilon)^{1-\theta}\, dx\, dt,
\end{aligned}$$

and then, by Lemma 2.1, (1) is proved. This completes the proof. □

Denote

$$\phi_a(s) = \int_0^s a(y)dy, \quad \phi_b(s) = \int_0^s b(y)dy, \quad \forall s \geq 0.$$

Lemma 2.4. *As $\varepsilon \to 0$, we have*

(1) $\iint_{\Omega_T} |\nabla\phi_a(u_\varepsilon) - \nabla\phi_a(u)|^2\, dx\, dt \to 0$;
(2) $\iint_{\Omega_T} |\nabla\phi_b(v_\varepsilon) - \nabla\phi_b(v)|^2\, dx\, dt \to 0$;
(3) $\iint_{Q_{u_\varepsilon}(c)} |\nabla u_\varepsilon - \nabla u|^2\, dx\, dt \to 0$;
(4) $\iint_{Q_{v_\varepsilon}(c)} |\nabla v_\varepsilon - \nabla v|^2\, dx\, dt \to 0$;

where $Q_{u_\varepsilon}(c) = \{(x,t) \in \Omega_T; u_\varepsilon \geq c > 0\}$ and $Q_{v_\varepsilon}(c) = \{(x,t) \in \Omega_T; v_\varepsilon \geq c > 0\}$.

Proof. Let us first prove (1). Multiplying the first equation of (2.3) by $[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]$ and integrating over Ω_T, we obtain

$$\begin{aligned}
0 &= \iint_{\Omega_T} u_{\varepsilon t}[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt \\
&\quad + \iint_{\Omega_T} a(u_\varepsilon)\nabla u_\varepsilon \nabla[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt \\
&\quad + \iint_{\Omega_T} a'(u_\varepsilon)|\nabla u_\varepsilon|^2[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt \\
&\quad - \alpha \iint_{\Omega_T} a(u_\varepsilon)v_\varepsilon[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt \\
&= \iint_{\Omega_T} u_{\varepsilon t}[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt \\
&\quad + \iint_{\Omega_T} \nabla\phi_a(u_\varepsilon)\nabla[\phi_a(u_\varepsilon) - \phi_a(u)]\,dx\,dt \\
&\quad + \iint_{\Omega_T} a'(u_\varepsilon)|\nabla u_\varepsilon|^2[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt \\
&\quad - \alpha \iint_{\Omega_T} a(u_\varepsilon)v_\varepsilon[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt\,.
\end{aligned}$$

Note that $u_\varepsilon \geq u$, $\phi_a'(s) \geq 0$, $a'(s) \geq 0$ for all $s \geq 0$, so the above expression is greater than or equal to

$$\begin{aligned}
&\iint_{\Omega_T} u_{\varepsilon t}[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt + \iint_{\Omega_T} \nabla\phi_a(u_\varepsilon)\nabla[\phi_a(u_\varepsilon) - \phi_a(u)]\,dx\,dt \\
&- \phi_a(\varepsilon)\iint_{\Omega_T} a'(u_\varepsilon)|\nabla u_\varepsilon|^2\,dx\,dt - \alpha\iint_{\Omega_T} a(u_\varepsilon)v_\varepsilon[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt \\
&= \iint_{\Omega_T} u_{\varepsilon t}[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt + \iint_{\Omega_T} |\nabla[\phi_a(u_\varepsilon) - \phi_a(u)]|^2\,dx\,dt \\
&\quad + \iint_{\Omega_T} \nabla\phi_a(u)\nabla[\phi_a(u_\varepsilon) - \phi_a(u)]\,dx\,dt - \phi_a(\varepsilon)\iint_{\Omega_T} a'(u_\varepsilon)|\nabla u_\varepsilon|^2\,dx\,dt \\
&\quad - \alpha\iint_{\Omega_T} a(u_\varepsilon)v_\varepsilon[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt.
\end{aligned}$$

Using (2.4), (2.5), Lemma 2.1 and Lemma 2.2 and noticing $\phi_a(0) = 0$, we have

$$\begin{aligned}
&\iint_{\Omega_T} |\nabla[\phi_a(u_\varepsilon) - \phi_a(u)]|^2\,dx\,dt \\
&\leq -\iint_{\Omega_T} u_{\varepsilon t}[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt - \iint_{\Omega_T} \nabla\phi_a(u)\nabla[\phi_a(u_\varepsilon) - \phi_a(u)]\,dx\,dt \\
&\quad + \phi_a(\varepsilon)\iint_{\Omega_T} a'(u_\varepsilon)|\nabla u_\varepsilon|^2\,dx\,dt + \alpha\iint_{\Omega_T} a(u_\varepsilon)v_\varepsilon[\phi_a(u_\varepsilon) - \phi_a(u) - \phi_a(\varepsilon)]\,dx\,dt \\
&\to 0 \quad (\text{as } \varepsilon \to 0).
\end{aligned}$$

Thus (1) is proved. Similarly (2) can be proved. We shall show (3). Using the equality

$$a(u_\varepsilon)(\nabla u_\varepsilon - \nabla u) = [\nabla\phi_a(u_\varepsilon) - \nabla\phi_a(u)] - [a(u_\varepsilon) - a(u)]\nabla u,$$

the inequality $(a+b)^2 \le 2(a^2+b^2)$, (2.4) and (1), we obtain

$$\begin{aligned}
&\iint_{\Omega_T} a(u_\varepsilon)^2|\nabla u_\varepsilon - \nabla u|^2\,dx\,dt\\
&\le 2\iint_{\Omega_T} |\nabla\phi_a(u_\varepsilon) - \nabla\phi_a(u)|^2\,dx\,dt + 2\iint_{\Omega_T} |a(u_\varepsilon) - a(u)|^2|\nabla u|^2\,dx\,dt\\
&\to 0 \quad (\text{as } \varepsilon \to 0),
\end{aligned}$$

and then, by $a'(s) \ge 0$ for all $s \ge 0$, so that (3) is proved. Similarly (4) can be obtained. The proof is complete. □

Lemma 2.5. *As $\varepsilon \to 0$, we have*

(1) $\iint_{\Omega_T} |a'(u_\varepsilon)|\nabla u_\varepsilon|^2 - a'(u)|\nabla u|^2|\,dx\,dt \to 0$;
(2) $\iint_{\Omega_T} |b'(v_\varepsilon)|\nabla v_\varepsilon|^2 - b'(v)|\nabla v|^2|\,dx\,dt \to 0$.

Proof. Since the proof is exactly the same for (1) and (2), we will show the validity of (1). For $\rho > 0$, let $\chi_\varepsilon^{(\rho)}$ and $\chi^{(\rho)}$ be the characteristic functions of $\{(x,t) \in \Omega_T; u_\varepsilon \le \rho\}$ and $\{(x,t) \in \Omega_T; u \le \rho\}$, respectively. Then

$$\begin{aligned}
&\iint_{\Omega_T} |a'(u_\varepsilon)|\nabla u_\varepsilon|^2 - a'(u)|\nabla u|^2|\,dx\,dt\\
&\le \iint_{\Omega_T} a'(u_\varepsilon)||\nabla u_\varepsilon|^2 - |\nabla u|^2|\,dx\,dt + \iint_{\Omega_T} |a'(u_\varepsilon) - a'(u)||\nabla u|^2\,dx\,dt\\
&\le \iint_{\Omega_T} \chi_\varepsilon^{(\rho)} a'(u_\varepsilon)|\nabla u_\varepsilon|^2\,dx\,dt + \iint_{\Omega_T} \chi_\varepsilon^{(\rho)} a'(u_\varepsilon)|\nabla u|^2\,dx\,dt\\
&\quad + \iint_{\Omega_T} (1-\chi_\varepsilon^{(\rho)})a'(u_\varepsilon)||\nabla u_\varepsilon|^2 - |\nabla u|^2|\,dx\,dt + \iint_{\Omega_T} |a'(u_\varepsilon) - a'(u)||\nabla u|^2\,dx\,dt\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Clearly, $I_4 \to 0$ as $\varepsilon \to 0$. Since $u_\varepsilon \ge u$ a.e. in Ω, $\chi_\varepsilon^{(\rho)} \le \chi^{(\rho)}$ a.e. in Ω. Therefore,

$$I_2 \le C\iint_{\Omega_T} \chi^{(\rho)}|\nabla u|^2\,dx\,dt \to 0 \quad (\rho \to 0).$$

Next, we estimate I_1. If $a'(0) = 0$, then

$$I_1 \le \max_{s\in[0,\rho]} a'(s)\iint_{\Omega_T} |\nabla u_\varepsilon|^2\,dx\,dt \le C\max_{s\in[0,\rho]} a'(s) \to 0 \quad (\rho \to 0).$$

If $a'(0) > 0$, taking $\theta = 1/2$ in Lemma 2.3 and noticing $a'(s) > 0$ for $s \ge 0$ and $a(0) = 0$, we have

$$\begin{aligned}
I_1 &= \iint_{\Omega_T} \chi_\varepsilon^{(\rho)} a(u_\varepsilon)^{1/2}\frac{a'(u_\varepsilon)|\nabla u_\varepsilon|^2}{a(u_\varepsilon)^{1/2}}\,dx\,dt\\
&\le a(\rho)^{1/2}\iint_{\Omega_T} \frac{a'(u_\varepsilon)|\nabla u_\varepsilon|^2}{a(u_\varepsilon)^{1/2}}\,dx\,dt\\
&\le Ca(\rho)^{1/2} \to 0 \quad (\text{as } \rho \to 0).
\end{aligned}$$

In any case, we obtain

$$I_1 \to 0 \quad (\text{as } \rho \to 0), \text{ uniformly in } \varepsilon.$$

Hence, for any $\delta > 0$, we can find a $\rho > 0$ sufficiently small such that $I_1 + I_2 < \delta/2$. For fixed $\rho > 0$, it follows from Lemma 2.4 that

$$I_3 \le C \iint_{\Omega_T} (1 - \chi_\varepsilon^{(\rho)}) ||\nabla u_\varepsilon|^2 - |\nabla u|^2| \,dx\,dt \to 0 \quad (\text{as } \varepsilon \to 0).$$

Therefore, there exists $\varepsilon_1 \in (0, 1)$ such that $I_3 < \delta/2$ as $\varepsilon < \varepsilon_1$. Consequently, we obtain

$$I_1 + I_2 + I_3 < \delta, \quad \forall \varepsilon < \varepsilon_1.$$

Thus (1) holds. The proof of Lemma 2.5 is complete. □

From Lemma 2.5 it is easy to check that (u, v) satisfies the condition (b) in Definition 1.1. Finally, we shall show that (u, v) satisfies the condition (c). The proof can be completed by combining the following two lemmas.

Lemma 2.6. *(1) For any $\rho > 0$ sufficiently small, there exist positive constants $c_1 = c_1(\rho)$ and $c_2 = c_2(\rho)$ such that*

$$u \ge c_1 \quad a.e.\ in\ \Omega^\rho(u_0) \times (0, T),$$
$$v \ge c_2 \quad a.e.\ in\ \Omega^\rho(v_0) \times (0, T);$$

(2) $\operatorname{supp} u(t) \supseteq \operatorname{supp} u_0$, $\operatorname{supp} v(t) \supseteq \operatorname{supp} v_0$, *a.e. in* $(0, T)$.

Proof. Note that, in view of the definition of support of a nonnegative function, the conclusion (1) implies (2). Since the proof is exactly the same for the first conclusion and the second conclusion of (1), we will show the validity of the former. It is easy to see that there exists a positive constant $c = c(\rho)$ such that

$$u_0 \ge c > 0 \text{ in } \Omega^\rho(u_0).$$

Denote by λ_ρ the first eigenvalue of $-\Delta$ in $\Omega^{\rho/2}(u_0)$ with the homogeneous Dirichlet boundary condition and ϕ_ρ the associated eigenfunction with $\max_{\overline{\Omega^{\rho/2}(u_0)}} \phi_\rho = c$. Let

$$\underline{u} = e^{-\kappa t}\phi_\rho, \quad (x, t) \in \Omega^{\rho/2}(u_0) \times (0, T),$$

where $\kappa = a(\sup_{0<\varepsilon<1} |u_\varepsilon|_\infty)\lambda_\rho + 1$.Then

$$\underline{u}_t - a(u_\varepsilon)\Delta\underline{u} = e^{-\kappa t}\phi_\rho(-\kappa + a(u_\varepsilon)\lambda_\rho) \le 0 \text{ in } \Omega^{\rho/2}(u_0) \times (0, T).$$

Hence, u_ε and $\underline{u}$ are the classical sup-solution and sub-solution of the equation

$$w_t - a(u_\varepsilon)\Delta w = 0 \quad \text{in } \Omega^{\rho/2}(u_0) \times (0, T).$$

On the other hand, obviously we have

$$u_\varepsilon \ge \underline{u} \quad \text{on } \partial\Omega^{\rho/2}(u_0) \times (0, T), \quad u_\varepsilon(x, 0) \ge \underline{u}(x, 0) \quad \text{in } \Omega^{\rho/2}(u_0).$$

By the comparison principle, we obtain

$$u_\varepsilon \ge e^{-\kappa t}\phi_\rho \ge e^{-\kappa T} \min_{\Omega^\rho(u_0)} \phi_\rho \equiv c_1(\rho) > 0 \quad \text{in } \Omega^\rho(u_0) \times (0, T).$$

Passing to the limit as $\varepsilon \to 0$, we have

$$u \ge c_1(\rho) > 0 \quad \text{a.e. in } \Omega^\rho(u_0) \times (0, T).$$

This completes the proof. □

Lemma 2.7. $\operatorname{supp} u(t) \subseteq \operatorname{supp} u_0$, $\operatorname{supp} v(t) \subseteq \operatorname{supp} v_0$, *a.e. in* $(0, T)$.

Proof. Since the proof is exactly the same for the former and the latter, we will show the validity of the former. Without loss of generality, we may assume $\operatorname{supp} u_0 \subsetneq \overline{\Omega}$. For any $\delta > 0$, let $\psi(x) = \psi_\delta(x) = \inf\big\{\frac{d(x)}{\delta}, 1\big\}$, where $d(x) = \operatorname{dist}(x, \partial\Omega \cup \operatorname{supp} u_0)$. It is well known that the distance functions $d(x)$ is Lipschitz with the constant 1, and hence it follows from Rademacher's theorem [16, pp. 49-51] that $d(x)$ is differentiable almost everywhere. Multiplying the first equation of (2.3) by $\varphi = \frac{\psi}{a(u_\varepsilon)}$ and integrating over Ω_t, we have

$$\int_0^t \int_\Omega \Big(\frac{u_{\varepsilon t}\psi}{a(u_\varepsilon)} + \nabla u_\varepsilon \nabla \psi - \alpha v_\varepsilon \psi \Big)\, dx\, d\tau = 0.$$

By Lemma 2.1 and Lemma 2.2, there exists a positive constant C independent of ε such that

$$\int_0^t \int_\Omega \frac{u_{\varepsilon t}\psi}{a(u_\varepsilon)} dx\, d\tau \le C,$$

hence

$$\int_\Omega \Big(\int_\varepsilon^{u_\varepsilon(x,t)} \frac{1}{a(s)} ds - \int_\varepsilon^{u_0(x)+\varepsilon} \frac{1}{a(s)} ds \Big) \psi(x) dx \le C.$$

Noticing $\psi u_0 = 0$ in Ω, we have

$$\int_\Omega \Big(\int_\varepsilon^{u_0(x)+\varepsilon} \frac{1}{a(s)} ds \Big) \psi(x) dx = 0;$$

therefore,

$$\int_\Omega \Big(\int_\varepsilon^{u_\varepsilon(x,t)} \frac{1}{a(s)} ds \Big) \psi(x) dx \le C.$$

By virtue of $u_\varepsilon \ge u$ a.e. in Ω_T, we obtain

$$\int_{\{x\in\Omega;\psi(x)=1\}} \int_\varepsilon^{u(x,t)} \frac{1}{a(s)} ds dx \le C.$$

Hence for any $\sigma \in (0,1)$ and $\varepsilon \in (0,\sigma)$ and a.e. $t \in (0,T)$, we have

$$\mu(\{x \in \{x \in \Omega; \psi = 1\}; u(x,t) > \sigma\}) \int_\varepsilon^\sigma \frac{1}{a(s)} ds \le C;$$

therefore,

$$\mu(\{x \in \{x \in \Omega; \psi = 1\}; u(x,t) > \sigma\}) \le C\Big[\int_\varepsilon^\sigma \frac{1}{a(s)} ds\Big]^{-1},$$

where C is a positive constant independent of ε. We claim that

$$\lim_{\varepsilon\to 0} \int_\varepsilon^\sigma \frac{1}{a(s)} ds = +\infty.$$

Indeed, by the mean value theorem and noticing $a(0) = 0$, we derive that for any $s \in [0,\sigma]$ there exists $\xi_s \in [0,s]$ such that $a(s) = a'(\xi_s)s \le Ms$, where $M = \max_{s\in[0,\sigma]} a'(s) > 0$. Thus

$$\int_\varepsilon^\sigma \frac{1}{a(s)} ds \ge \int_\varepsilon^\sigma \frac{1}{Ms} ds = \frac{1}{M}[\ln(\sigma) - \ln(\varepsilon)],$$

then passing to the limit as $\varepsilon \to 0$, we prove the above claim and obtain

$$\mu(\{x \in \{x \in \Omega; \psi = 1\}; u(x,t) > \sigma\}) = 0 \quad \text{a.e. in } (0,T),$$

so that, since $\sigma \in (0,1)$ is arbitrary, we obtain

$$\mu(\{x \in \{x \in \Omega; \psi = 1\}; u(x,t) > 0\}) = 0 \quad \text{a.e. in } (0,T).$$

Since $\delta > 0$ is arbitrary, we conclude that

$$u(x,t) = 0 \quad \text{a.e. in } (\Omega \setminus \operatorname{supp} u_0) \times (0,T).$$

This completes the proof of Lemma 2.7. Thus the proof of the existence is complete. □

2.2. **Proof of uniqueness.** Let (u_2, v_2) and (u_1, v_1) be two weak solutions of (1.1), and $E = \operatorname{supp} u_0 \cap \operatorname{supp} v_0$. It suffices to prove that for any $T > 0$, $u_2 = u_1, v_2 = v_1$ a.e. in Ω_T.

First Case: $\mu(E) = 0$. Without loss of generality, we may assume that $\operatorname{supp} u_0 \neq \emptyset$. From the definition of weak solutions it follows that $v_2 = v_1 = 0$ a.e. on $\operatorname{supp} u_0$. Denote by λ_ρ the first eigenvalue of $-\Delta$ in $\Omega^\rho(u_0)$ with homogeneous Dirichlet boundary condition and $\phi_\rho(x)$ the associated eigenfunction. Substituting

$$\varphi = \frac{\phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)}{a(u_1)}, \quad \psi = 0,$$

and

$$\varphi = \frac{\phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)}{a(u_2)}, \quad \psi = 0,$$

in the definition of weak solutions, respectively, where $\operatorname{sign}_\delta(z) = \operatorname{sign}(z) \inf\left\{\frac{|z|}{\delta}, 1\right\}$ for $\delta > 0$, we have for any $t \in (0,T)$

$$\begin{aligned}
&\int_0^t \int_\Omega \Big[\frac{u_{1t}\phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)}{a(u_1)} \\
&+ \nabla u_1 \nabla (u_1 - u_2)_+ \phi_\rho \operatorname{sign}'_\delta((u_1 - u_2)_+) + \nabla u_1 \nabla \phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)\Big] dx\, d\tau = 0, \\
&\int_0^t \int_\Omega \Big[\frac{u_{2t}\phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)}{a(u_2)} \\
&+ \nabla u_2 \nabla (u_1 - u_2)_+ \phi_\rho \operatorname{sign}'_\delta((u_1 - u_2)_+) + \nabla u_2 \nabla \phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)\Big] dx\, d\tau = 0,
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_0^t \int_\Omega \Big[(f_a(u_1) - f_a(u_2))_t \phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+) \\
&+ |\nabla (u_1 - u_2)_+|^2 \phi_\rho \operatorname{sign}'_\delta((u_1 - u_2)_+) \\
&+ \nabla (u_1 - u_2) \nabla \phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)\Big] dx\, d\tau = 0,
\end{aligned}$$

where

$$f_a(s) = \int_{c_1}^s \frac{1}{a(y)} dy, \qquad \forall s > 0,$$

where c_1 is the same as that of Definition 1.1 (note that c_1 corresponding to u_1 may be different from that corresponding to u_2. Here c_1 is minimal between them).

Noticing $\operatorname{sgn}'_\delta(z) \geq 0$, we obtain

$$\begin{aligned}&\int_0^t \int_\Omega \Big[(f_a(u_1) - f_a(u_2))_t \phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)\\ &+ \nabla(u_1 - u_2)\nabla\phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)\Big] dx\, d\tau \leq 0,\end{aligned}$$

Passing to the limit as $\delta \to 0$, we have

$$\int_0^t \int_\Omega \Big[(f_a(u_1) - f_a(u_2))_t \phi_\rho \operatorname{sign}((u_1 - u_2)_+) + \nabla(u_1 - u_2)_+ \nabla\phi_\rho\Big] dx\, d\tau \leq 0.$$

Integrating by parts for the second term of the above integral, we obtain

$$\int_0^t \int_\Omega \Big[(f_a(u_1) - f_a(u_2))_t \phi_\rho \operatorname{sign}((u_1 - u_2)_+) + \lambda_\rho (u_1 - u_2)_+ \phi_\rho\Big] dx\, d\tau \leq 0,$$

and then it follows from $\lambda_\rho > 0$ and $\phi_\rho \geq 0$ that

$$\int_0^t \int_\Omega (f_a(u_1) - f_a(u_2))_t \phi_\rho \operatorname{sign}((u_1 - u_2)_+) dx\, d\tau \leq 0.$$

Since $\operatorname{sign}((u_1 - u_2)_+) = \operatorname{sign}(f_a(u_1) - f_a(u_2))_+$ a.e. in $\Omega^\rho(u_0) \times (0, T)$, we have

$$\int_\Omega (f_a(u_1) - f_a(u_2))_+(x, t)\phi_\rho(x) dx \leq 0,$$

which implies $(f_a(u_1) - f_a(u_2))_+ = 0$ a.e. in $\Omega^\rho(u_0) \times (0, T)$, and hence $u_1 \leq u_2$ a.e. in $\Omega^\rho(u_0) \times (0, T)$, and therefore $u_1 \leq u_2$ a.e. in $\operatorname{supp} u_0 \times (0, T)$. By the condition (c) in Definition 1.1, we derive that $u_1 \leq u_2$ a.e. in Ω_T. Similarly, $u_1 \geq u_2$ a.e. in Ω_T. Thus, $u_1 = u_2$ a.e. in Ω_T.

The same reasoning as those given above shows that $v_1 = v_2$ a.e. in Ω_T. This prove the first case.

General Case: $\mu(E) > 0$. Denote by λ_ρ the first eigenvalue of $-\Delta$ in E_ρ with the homogeneous Dirichlet boundary condition and $\phi_\rho(x)$ the associated eigenfunction, where $E_\rho = \{x \in E; \operatorname{dist}(x, \partial E) > \rho > 0\}$. Substituting

$$\varphi = \frac{\phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)}{a(u_1)}, \quad \psi = \frac{\phi_\rho \operatorname{sign}_\delta((v_1 - v_2)_+)}{b(v_1)},$$

and

$$\varphi = \frac{\phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)}{a(u_2)}, \quad \psi = \frac{\phi_\rho \operatorname{sign}_\delta((v_1 - v_2)_+)}{b(v_2)},$$

in the definition of weak solutions, respectively, we have

$$\begin{aligned}&\int_0^t \int_\Omega \Big[\frac{u_{1t}\phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)}{a(u_1)} + \nabla u_1 \nabla(u_1 - u_2)_+ \phi_\rho \operatorname{sign}'_\delta((u_1 - u_2)_+)\\ &+ \nabla u_1 \nabla\phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+) - \alpha v_1 \phi_\rho \operatorname{sign}_\delta((u_1 - u_2)_+)\Big] dx\, d\tau\\ &+ \int_0^t \int_\Omega \Big[\frac{v_{1t}\phi_\rho \operatorname{sign}_\delta((v_1 - v_2)_+)}{a(v_1)} + \nabla v_1 \nabla(v_1 - v_2)_+ \phi_\rho \operatorname{sign}'_\delta((v_1 - v_2)_+)\\ &+ \nabla v_1 \nabla\phi_\rho \operatorname{sign}_\delta((v_1 - v_2)_+) - \beta u_1 \phi_\rho \operatorname{sign}_\delta((v_1 - v_2)_+)\Big] dx\, d\tau = 0,\end{aligned}$$

$$\begin{aligned}&\int_0^t\int_\Omega\Big[\frac{u_{2t}\phi_\rho\operatorname{sign}_\delta((u_1-u_2)_+)}{a(u_2)}+\nabla u_2\nabla(u_1-u_2)_+\phi_\rho\operatorname{sign}'_\delta((u_1-u_2)_+)\\&+\nabla u_2\nabla\phi_\rho\operatorname{sign}_\delta((u_1-u_2)_+)-\alpha v_2\phi_\rho\operatorname{sign}_\delta((u_1-u_2)_+)\Big]dx\,d\tau\\&+\int_0^t\int_\Omega\Big[\frac{v_{2t}\phi_\rho\operatorname{sign}_\delta((v_1-v_2)_+)}{a(v_2)}+\nabla v_2\nabla(v_1-v_2)_+\phi_\rho\operatorname{sign}'_\delta((v_1-v_2)_+)\\&+\nabla v_2\nabla\phi_\rho\operatorname{sign}_\delta((v_1-v_2)_+)-\beta u_2\phi_\rho\operatorname{sign}_\delta((v_1-v_2)_+)\Big]dx\,d\tau=0,\end{aligned}$$

and hence

$$\begin{aligned}&\int_0^t\int_\Omega\Big[(f_a(u_1)-f_a(u_2))_t\phi_\rho\operatorname{sign}_\delta((u_1-u_2)_+)\\&+|\nabla(u_1-u_2)_+|^2\phi_\rho\operatorname{sign}'_\delta((u_1-u_2)_+)+\nabla(u_1-u_2)\nabla\phi_\rho\operatorname{sign}_\delta((u_1-u_2)_+)\\&-\alpha(v_1-v_2)\phi_\rho\operatorname{sign}_\delta((u_1-u_2)_+)\Big]dx\,d\tau\\&+\int_0^t\int_\Omega\Big[(f_b(v_1)-f_b(v_2))_t\phi_\rho\operatorname{sign}_\delta((v_1-v_2)_+)\\&+|\nabla(v_1-v_2)_+|^2\phi_\rho\operatorname{sign}'_\delta((u_1-u_2)_+)+\nabla(v_1-v_2)\nabla\phi_\rho\operatorname{sign}_\delta((v_1-v_2)_+)\\&-\beta(u_1-u_2)\phi_\rho\operatorname{sign}_\delta((v_1-v_2)_+)\Big]dx\,d\tau=0,\end{aligned}$$

where f_a is the same as before and f_b is defined by

$$f_b(s)=\int_{c_2}^s\frac{1}{b(y)}dy,\quad\forall s>0,$$

and c_2 is the same as that of Definition 1.1. Noticing $\operatorname{sign}'_\delta(z)\geq 0$, we obtain from the above equality

$$\begin{aligned}&\int_0^t\int_\Omega\Big[(f_a(u_1)-f_a(u_2))_t\phi_\rho\operatorname{sign}_\delta((u_1-u_2)_+)\\&+\nabla(u_1-u_2)\nabla\phi_\rho\operatorname{sign}_\delta((u_1-u_2)_+)-\alpha(v_1-v_2)\phi_\rho\operatorname{sign}_\delta((u_1-u_2)_+)\Big]dx\,d\tau\\&+\int_0^t\int_\Omega\Big[(f_b(v_1)-f_b(v_2))_t\phi_\rho\operatorname{sign}_\delta((v_1-v_2)_+)\\&+\nabla(v_1-v_2)\nabla\phi_\rho\operatorname{sign}_\delta((v_1-v_2)_+)-\beta(u_1-u_2)\phi_\rho\operatorname{sign}_\delta((v_1-v_2)_+)\Big]dx\,d\tau\leq 0.\end{aligned}$$

Passing to the limit as $\delta\to 0$, we have

$$\begin{aligned}&\int_0^t\int_\Omega\Big[(f_a(u_1)-f_a(u_2))_t\phi_\rho\operatorname{sign}((u_1-u_2)_+)+\nabla(u_1-u_2)_+\nabla\phi_\rho\\&-\alpha(v_1-v_2)\phi_\rho\operatorname{sign}((u_1-u_2)_+)\Big]dx\,d\tau\\&+\int_0^t\int_\Omega\Big[(f_b(v_1)-f_b(v_2))_t\phi_\rho\operatorname{sign}((v_1-v_2)_+)+\nabla(v_1-v_2)_+\nabla\phi_\rho\\&-\beta(u_1-u_2)\phi_\rho\operatorname{sign}((v_1-v_2)_+)\Big]dx\,d\tau\leq 0,\end{aligned}$$

and hence

$$\begin{aligned}&\int_0^t\int_\Omega\Big[(f_a(u_1)-f_a(u_2))_t\phi_\rho\operatorname{sign}((u_1-u_2)_+)+\lambda_\rho(u_1-u_2)_+\phi_\rho\\&-\alpha(v_1-v_2)\phi_\rho\operatorname{sign}((u_1-u_2)_+)\Big]dx\,d\tau\\&+\int_0^t\int_\Omega\Big[(f_b(v_1)-f_b(v_2))_t\phi_\rho\operatorname{sign}((v_1-v_2)_+)+\lambda_\rho(v_1-v_2)_+\phi_\rho\\&-\beta(u_1-u_2)\phi_\rho\operatorname{sign}((v_1-v_2)_+)\Big]dx\,d\tau\le 0.\end{aligned}$$

This implies

$$\begin{aligned}&\int_\Omega[(f_a(u_1)-f_a(u_2))_+(x,t)+(f_b(v_1)-f_b(v_2))_+(x,t)]\phi_\rho(x)dx\\&\le C\int_0^t\int_\Omega(|v_1-v_2|+|u_1-u_2|)\phi_\rho dx\,d\tau.\end{aligned}$$

By the same arguments as the above, we obtain

$$\begin{aligned}&\int_\Omega[(f_a(u_2)-f_a(u_1))_+(x,t)+(f_b(v_2)-f_b(v_1))_+(x,t)]\phi_\rho(x)dx\\&\le C\int_0^t\int_\Omega(|v_1-v_2|+|u_1-u_2|)\phi_\rho dx\,d\tau.\end{aligned}$$

Thus we have

$$\begin{aligned}&\int_\Omega[|(f_a(u_2)-f_a(u_1))(x,t)|+|(f_b(v_2)-f_b(v_1))(x,t)|]\phi_\rho(x)dx\\&\le C\int_0^t\int_\Omega(|v_1-v_2|+|u_1-u_2|)\phi_\rho dx\,d\tau.\end{aligned}\tag{2.6}$$

On the other hand, it follows from $a'(s),b'(s)\ge 0$ for $s\ge 0$ that

$$\begin{gathered}|u_1-u_2|\le a(|u_1+u_2|_{L^\infty(\Omega_T)})|f_a(u_2)-f_a(u_1)|\quad\text{a.e. in } E_\rho\times(0,T),\\|v_1-v_2|\le b(|v_1+v_2|_{L^\infty(\Omega_T)})|f_b(v_2)-f_b(v_1)|\quad\text{a.e. in } E_\rho\times(0,T).\end{gathered}$$

Combining this with (2.6), we have

$$\begin{aligned}&\int_\Omega[|(f_a(u_2)-f_a(u_1))(x,t)|+|(f_b(v_2)-f_b(v_1))(x,t)|]\phi_\rho(x)dx\\&\le C\int_0^t\int_\Omega|f_a(u_2)-f_a(u_1)|+|f_b(v_2)-f_b(v_1)|)\phi_\rho dx\,d\tau,\end{aligned}$$

and then, by Gronwall's theorem, we obtain

$$|f_a(u_2)-f_a(u_1)|+|f_b(v_2)-f_b(v_1)|=0\quad\text{a.e. in } E_\rho\times(0,T).$$

This shows that

$$u_2=u_1,\quad v_2=v_1,\quad\text{a.e. in } E_\rho\times(0,T),$$

and hence $u_2=u_1$, $v_2=v_1$, a.e. in $E\times(0,T)$. Similar to the proof of the first case, it is not difficult to prove that

$$\begin{aligned}u_2=u_1&\quad\text{a.e. in } (\operatorname{supp}u_0-E)\times(0,T),\\v_2=v_1&\quad\text{a.e. in } (\operatorname{supp}v_0-E)\times(0,T).\end{aligned}$$

Combining the above results, we obtain

$$\begin{aligned} u_2 &= u_1 \quad \text{a.e. in } \operatorname{supp} u_0 \times (0,T), \\ v_2 &= v_1 \quad \text{a.e. in } \operatorname{supp} v_0 \times (0,T). \end{aligned}$$

This proves the general case and ends the proof of uniqueness.

3. Proof of Theorem 1.1 1.3

First, we claim that the following inequalities hold:

$$a(u)u^2 + a(v)v^2 \geq (a(u)+a(v))uv, \tag{3.1}$$

$$\rho_2 s^{\sigma_2} \geq a(s) \geq \rho_1 s^{\sigma_2} \quad \text{for } s \geq 0, \tag{3.2}$$

$$\int_0^s a(y)^{1/2} dy \geq \frac{2}{2+\sigma_2} s a(s)^{1/2} \quad \text{for } s \geq 0, \tag{3.3}$$

$$\int_0^s a(y)^{1/2} dy \geq \frac{2\rho_1^{1/2}}{2+\sigma_2} s^{1+\sigma_2/2} \quad \text{for } s \geq 0. \tag{3.4}$$

We shall prove (3.1). It follows from $a'(s) \geq 0$ for all $s \geq 0$ that

$$[sa(s)]' = a(s) + sa'(s) \geq 0 \quad \text{for all } s \geq 0.$$

This shows that $[ua(u) - va(v)][u-v] \geq 0$, which implies (3.1). Let us turn to the proof of (3.2). By virtue of (1.2), it suffices to show that $\big[\frac{a(s)}{s^{\sigma_2}}\big]' \leq 0$ for all $s > 0$. By (1.3), we immediately obtain

$$\Big[\frac{a(s)}{s^{\sigma_2}}\Big]' = \frac{sa'(s) - \sigma_2 a(s)}{s^{\sigma_2+1}} \leq 0 \quad \text{for all} \quad s > 0,$$

as asserted. (3.4) is an immediate consequence of (3.2). Finally we shall show (3.3). Let

$$H(s) = \int_0^s a(y)^{1/2} dy - \frac{2}{2+\sigma_2} s a(s)^{1/2}, \quad s \geq 0.$$

Simple calculation shows, by virtue of (1.3), that

$$\begin{aligned} H'(s) &= a(s)^{1/2} - \frac{2}{2+\sigma_2}\Big[\frac{1}{2} s a'(s) a(s)^{-1/2} + a(s)^{1/2}\Big] \\ &= \frac{a(s)^{-1/2}[\sigma_2 a(s) - a'(s)s]}{2+\sigma_2} \geq 0 \end{aligned}$$

for all $s > 0$. Since $H(0) = 0$, we see that $H(s) \geq 0$ for all $s \geq 0$. This proves (3.3). Thus the above claims hold.

Now taking $\varphi = u$ and $\psi = \psi_\varrho = \frac{va(v)}{b(v)+\varrho}$ for $\varrho > 0$ as test functions in Definition 1.1, we obtain

$$\begin{aligned} &\int_\Omega \Big(\frac{u^2(x,t)}{2} + \int_0^{v(x,t)} \frac{sa(s)}{b(s)+\varrho} ds\Big) dx - \int_\Omega \Big(\frac{u_0^2}{2} + \int_0^{v_0} \frac{sa(s)}{b(s)+\varrho} ds\Big) dx \\ &= -\int_0^t \int_\Omega \Big[(a(u) + a'(u)u)|\nabla u|^2 + \frac{b(v)(a(v)+va'(v))}{b(v)+\varrho} |\nabla v|^2)\Big] dx\, d\tau \\ &\quad - \int_0^t \int_\Omega \frac{\varrho v a(v) b'(v)}{(b(v)+\varrho)^2} |\nabla v|^2 dx\, d\tau + \int_0^t \int_\Omega \Big(\alpha a(u) + \frac{\beta a(v) b(v)}{b(v)+\varrho}\Big) uv\, dx\, d\tau. \end{aligned}$$

For $\varrho > 0$, let

$$\Phi_\varrho(t) = \int_\Omega \Big(\frac{u^2(x,t)}{2} + \int_0^{v(x,t)} \frac{sa(s)}{b(s)+\varrho} ds\Big) dx,$$
$$\Phi(t) = \int_\Omega \Big(\frac{u^2(x,t)}{2} + \int_0^{v(x,t)} \frac{sa(s)}{b(s)} ds\Big) dx.$$

Then it is easy to see that $\Phi_\varrho(t) \to \Phi(t)$, $\Phi'_\varrho(t) \to \Phi'(t)$, in $(0,\infty)$, as $\varrho \to 0^+$, and

$$\begin{aligned}
\Phi'_\varrho &= -\int_\Omega \Big[(a(u)+a'(u)u)|\nabla u|^2 + \frac{b(v)(a(v)+va'(v))}{b(v)+\varrho}|\nabla v|^2\Big] dx \\
&\quad - \int_\Omega \frac{\varrho v a(v) b'(v)}{(b(v)+\varrho)^2} |\nabla v|^2 dx + \int_\Omega \Big(\alpha a(u) + \frac{\beta a(v) b(v)}{b(v)+\varrho}\Big) uv dx \\
&\le - \int_\Omega \Big[a(u)|\nabla u|^2 + \frac{b(v)a(v)}{b(v)+\varrho}|\nabla v|^2\Big] dx + \int_\Omega \Big(\alpha a(u) + \frac{\beta a(v)b(v)}{b(v)+\varrho}\Big) uv dx.
\end{aligned}$$

Passing to the limit as $\varrho \to 0$ and using (3.1), we have

$$\begin{aligned}
\Phi' &\le -\int_\Omega (a(u)|\nabla u|^2 + a(v)|\nabla v|^2) dx + \max\{\alpha,\beta\} \int_\Omega (a(u)u^2 + a(v)v^2) dx \\
&= -\int_\Omega \Big(|\nabla \int_0^u a(s)^{1/2} ds|^2 + |\nabla \int_0^v a(s)^{1/2} ds|^2\Big) dx \\
&\quad + \max\{\alpha,\beta\} \int_\Omega (a(u)u^2 + a(v)v^2) dx.
\end{aligned} \tag{3.5}$$

In view of (3.3), we obtain

$$\int_\Omega |\nabla \int_0^u a(s)^{1/2} ds|^2 dx \ge \lambda_1 \int_\Omega \Big(\int_0^u a(s)^{1/2} ds\Big)^2 dx \ge \frac{4\lambda_1}{(2+\sigma_2)^2} \int_\Omega a(u)u^2 dx,$$
$$\int_\Omega |\nabla \int_0^v a(s)^{1/2} ds|^2 dx \ge \lambda_1 \int_\Omega \Big(\int_0^v a(s)^{1/2} ds\Big)^2 dx \ge \frac{4\lambda_1}{(2+\sigma_2)^2} \int_\Omega a(v)v^2 dx.$$

Combining this with (3.5), we have

$$\Phi' \le -\overline{\lambda} \int_\Omega (a(u)u^2 + a(v)v^2) dx, \tag{3.6}$$

where $\overline{\lambda} = \frac{4\lambda_1}{(2+\sigma_2)^2} - \max\{\alpha,\beta\} > 0$, which, in particular, implies that $\Phi'(t) \le 0, \forall t > 0$, and hence

$$\int_\Omega \int_0^{v(x,t)} \frac{sa(s)}{b(s)} ds\, dx \le A_0, \quad \forall t > 0. \tag{3.7}$$

By Hölder's inequality and (3.2), we obtain

$$\int_\Omega u^2 dx \le C\Big(\int_\Omega u^{2+\sigma_2} dx\Big)^{2/(2+\sigma_2)} \le C\Big(\int_\Omega a(u)u^2 dx\Big)^{2/(2+\sigma_2)}. \tag{3.8}$$

By (3.2) and noticing $\sigma_2 > \sigma_1$ and using Hölder's inequality, we have

$$\begin{aligned}\int_\Omega \int_0^v \frac{sa(s)}{b(s)} ds\, dx &\leq C \int_\Omega v^{2+\sigma_1} dx \\ &\leq C\Big(\int_\Omega v^{2+\sigma_2} dx\Big)^{(2+\sigma_1)/(2+\sigma_2)} \\ &\leq C\Big(\int_\Omega a(v) v^2 dx\Big)^{(2+\sigma_1)/(2+\sigma_2)}.\end{aligned} \tag{3.9}$$

Combining (3.6) with (3.7), (3.8) and (3.9) and using the inequality $a^r + b^r \geq 2^{1-r}(a+b)^r$ for $a, b \geq 0, r \geq 1$, we obtain

$$\begin{aligned}\Phi'(t) \leq &-C\Big[\Big(\int_\Omega u^2(x,t)dx\Big)^{(2+\sigma_2)/2} + \Big(\int_0^{v(x,t)} \frac{sa(s)}{b(s)} dsdx\Big)^{(2+\sigma_2)/(2+\sigma_1)}\Big] \\ \leq &-C\Big[\Big(\int_\Omega u^2(x,t)dx\Big)^{(2+\sigma_2)/2} + \Big(\int_\Omega \int_0^{v(x,t)} \frac{sa(s)}{b(s)} dsdx\Big)^{(2+\sigma_2)/2}\Big] \\ \leq &-C\Big(\int_\Omega \Big(\frac{u^2(x,t)}{2} + \int_0^{v(x,t)} \frac{sa(s)}{b(s)} ds\Big)dx\Big)^{(2+\sigma_2)/2} \\ \leq &-C\Phi(t)^{(2+\sigma_2)/2},\end{aligned}$$

which gives

$$\Phi(t) \leq \Big(\frac{1}{Ct + A_0^{-\sigma_2/2}}\Big)^{2/\sigma_2}, \quad \forall t > 0.$$

This completes the proof.

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