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ENERGY QUANTIZATION FOR YAMABE'S PROBLEM IN CONFORMAL DIMENSION

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Abstract. Rivière [11] proved an energy quantization for Yang-Mills fields defined on n-dimensional Riemannian manifolds, when n is larger than the critical dimension 4. More precisely, he proved that the defect measure of a weakly converging sequence of Yang-Mills fields is quantized, provided the $W^{2,1}$ norm of their curvature is uniformly bounded. In the present paper, we prove a similar quantization phenomenon for the nonlinear elliptic equation

$$-\Delta u = u|u|^{4/(n-2)},$$

in a subset Ω of $\mathbb{R}^n$.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$ with $n \geq 3$. We consider the equation

$$-\Delta u = u|u|^{4/(n-2)} \qquad \text{in} \quad \Omega \tag{1.1}$$

We will say that u is a weak solution of (1.1) in Ω, if, for all $\Phi \in C^\infty(\Omega)$ with compact support in Ω, we have

$$-\int_\Omega \Delta\Phi(x)u(x)dx = \int_\Omega \Phi(x)u(x)|u(x)|^{4/(n-2)}dx \tag{1.2}$$

If in addition u satisfies

$$\int_\Omega \Big[\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial \Phi^j}{\partial x_i} - \frac{1}{2}|\nabla u|^2\frac{\partial \Phi^i}{\partial x_i} + \frac{n-2}{2n}|u|^{2n/(n-2)}\frac{\partial \Phi^i}{\partial x_i}\Big]dx = 0 \tag{1.3}$$

for any $\Phi = (\Phi^1, \Phi^2 \dots, \Phi^n) \in C^\infty(\Omega)$ with compact support in Ω, we say that u is stationary. In other words, a weak solution u in $\mathbf{H}^1(\Omega) \cap \mathbf{L}^{2n/(n-2)}(\Omega)$ of (1.1) is stationary if the functional E defined by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{n-2}{2n}\int_\Omega |u|^{2n/(n-2)}$$

is stationary with respect to domain variations, i.e.

$$\frac{d}{dt}(E(u_t))|_{t=0} = 0$$

where $u_t(x) = u(x + t\Phi)$. It is easy to verify that a smooth solution is stationary.

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In this paper we prove a monotonicity formula for stationary weak solution u in $\mathbf{H}^1(\Omega) \cap \mathbf{L}^{2n/(n-2)}(\Omega)$ of (1.1) by a similar idea as in [6]. More precisely we have the following result.

Lemma 1.1. *Suppose that $u \in \mathbf{L}^{2n/(n-2)}(\Omega) \cap \mathbf{H}^1(\Omega)$ is a stationary weak solution of (1.1). Consider the function*

$$E_u(x,r) = \int_{B(x,r)} |u|^{2n/(n-2)} dy + \frac{d}{dr}\int_{\partial B(x,r)} u^2 ds + r^{-1}\int_{B(x,r)} u^2 ds.$$

Then $r \mapsto E_u(x,r)$ is positive, nondecreasing and continuous.

This monotonicity formula together with ideas which go back to the work of Schoen [12], allowed to prove the following result.

Theorem 1.2. *There exists $\varepsilon > 0$ and $r_0 > 0$ depend only on n such that, for any smooth solution $u \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{L}^{2n/(n-2)}(\Omega)$ of (1.1), we have: For any $x_0 \in \Omega$, if*

$$\int_{B(x_0,r_0)} |\nabla u|^2 + |u|^{2n/(n-2)} \le \varepsilon,$$

then

$$\|u\|_{\boldsymbol{L}^\infty(B_{\frac{r}{2}}(x_0))} \le \frac{C(\varepsilon)}{r^{(n-2)/2}} \quad \text{for any} \quad r < r_0,$$

where $B_{\frac{r}{2}}(x_0)$ is the ball centered at x_0 with radius $\frac{r}{2}$, and $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Zongming Guo and Jiay Li [5] studied sequences of smooth solutions of (1.1) having uniformly bounded energy, they proved the following result.

Theorem 1.3. *Let u_i be a sequence of smooth solutions of (1.1) such that*

$$\|u_i\|_{H^1(\Omega)} + \|u_i\|_{L^{2n/(n-2)}(\Omega)}$$

is bounded. Let u_∞ be the weak limit of u_i in $\boldsymbol{H}^1(\Omega) \cap \boldsymbol{L}^{2n/(n-2)}(\Omega)$. Then u_∞ is smooth and satisfies equation (1.1) outside a closed singular subset Σ of Ω. Moreover, there exists $r_0 > 0$ and $\varepsilon_0 > 0$ such that

$$\Sigma = \cap_{0<r<r_0}\big\{x \in \Omega : \liminf_{i\to\infty} E_{u_i}(x,r) \ge \varepsilon_0\big\}.$$

We define the sequence of Radon measures

$$\eta_i := (\frac{1}{2}|\nabla u_i|^2 + \frac{n-2}{2n}|u_i|^{2n/(n-2)})\, dx$$

Assumption that the sequence $(\|\nabla u_i\|_{\mathbf{H}^1(\Omega)} + \|u_i\|_{\mathbf{L}^{2n/(n-2)}(\Omega)})_i$ is bounded, and up to a subsequences, we can assume that $\eta_i \rightharpoonup \eta$ in the sense of measures as $i \to \infty$. Namely, for any continuous function ϕ with compact support in Ω

$$\lim_{i\to\infty}\int_\Omega \phi\, d\eta_i = \int_\Omega \phi\, d\eta.$$

Fatou's Lemma then implies that we can decompose

$$\eta = (\frac{1}{2}|\nabla u|^2 + \frac{n-2}{2n}|u|^{\frac{2n}{n-2}})\, dx + \nu$$

where ν is a nonnegative Radon measure. Moreover, we prove that ν satisfies the following lemma.

Lemma 1.4. *Let $\delta > 0$ such that $B_\delta \subset \Omega$. Then we have*

(i) $\Sigma \subset spt(\nu)$
(ii) *There exists a measurable, upper-semi-continuous function* Θ *such that*

$$\nu(x) = \Theta(x)\mathcal{H}^0\lfloor\Sigma, \quad \textit{for } x \in \Sigma.$$

Moreover, there exists some constants c *and* $C > 0$ *(only depending on* n *and* Ω*) such that*

$$c\varepsilon_0 < \Theta(x) < C \quad \mathcal{H}^0 - a.e. \ in \ \Sigma$$

where $H^0\lfloor\Sigma$ *is the restriction to* Σ *of the Hausdorff measure and* Θ *is a measurable function on* Σ.

The main question we would like to address in the present paper concerns the multiplicity Θ of the defect measure which has been defined above. More precisely, we have proved the following theorem.

Theorem 1.5. *Let* ν *be the defect measure of the sequence* $(|\nabla u_i|^2+|u_i|^{2n/(n-2)})dx$ *defined above. Then* ν *is quantized. That is, for a.e* $x \in \Sigma$,

$$\Theta(x) = \sum_{j=1}^{j=N_x} \|\nabla v_{x,j}\|^2_{L^2(\Omega)} + \|v_{x,j}\|^{2n/(n-2)}_{L^{2n/(n-2)}(\Omega)} \tag{1.4}$$

where N_x *is a positive integer and where the functions* $v_{x,j}$ *are solutions of* $\Delta v + v^{\frac{n+2}{n-2}} = 0$ *which are defined on* $\mathbb{R}^n$, *issued from* $(u_{i'})$ *and that concentrate at* x *as* $i \to \infty$.

The sentence "issued from $(u_{i'})$ and that concentrate at x as $i \to \infty$" means that there are sequences of conformal maps ψ^i_j , a finite family of balls $(B^l_{i,j})_l$ such that the pulled back function

$$\tilde{u}_{i,j} = (\psi^i_j)^* u_{i'}$$

satisfies

$$\begin{gathered}
\tilde{u}_{i,j} \to v_j \quad \text{strongly in} \quad \mathbf{L}^2(\mathbb{R}^n \setminus \cup_l B^l_{i,j})), \\
\nabla\tilde{u}_{i,j} \to \nabla v_j \quad \text{strongly in} \quad \mathbf{L}^2(\mathbb{R}^n \setminus \cup_l B^l_{i,j}))
\end{gathered}$$

In the context of Yang-Mills fields in dimension $n \geq 4$ a similar concentration result has been proven by Rivière [11]. More precisely, Rivière has shown that, if $(A_i)_i$ is a sequence of Yang-Mills connections such that $(\|\nabla_A\nabla_A F_A\|_{\mathbf{L}^1(B^n_1)})_i$ is bounded, then the corresponding defect measure $\nu = \Theta\mathcal{H}^{n-4}\lfloor\Sigma$ of a sequence of smooth Yang-Mills connections is quantized.

The proof of Theorem 1.5 uses technics introduced by Lin and Rivière in their study of Ginzburg-Landau vortices [10] and also the technics developed by Rivière in [5]. These technics use as an essential tool the Lorentz spaces, more specifically the $\mathbf{L}^{2,\infty}$-$\mathbf{L}^{2,1}$ duality [14].

This paper is organized in the following way: In Section 2 we establish first a monotonicity formula for smooth solutions of problem (1.1) which allows us to prove an ε-regularity Theorem. Then, we prove Theorem 1.2 and Lemma 1.4. While Section 3 is devoted to the proof of our main result, Theorem 1.5.

2. A monotonicity Inequality

In this section, we establish a monotonicity formula for smooth solutions of problem (1.1). Using Pohozaev identity: Multiplying (1.1) by $x_i\frac{\partial u}{\partial x_i}$ (summation over i is understood) and integrating over B(x,r), the ball centered at x of radius r, we obtain

$$-\int_{B(x,r)} x_i\frac{\partial u}{\partial x_i}\Delta u\,dy = -\int_{B(x,r)} x_i\frac{\partial u}{\partial x_i}u|u|^{4/(n-2)}\,dy$$

By Green formula, we get

$$\begin{aligned}
&\frac{n-2}{2}\int_{B(x,r)}|u|^{2n/(n-2)}dy - \frac{n-2}{2}\int_{B(x,r)}|\nabla u|^2dy\\
&-\frac{n-2}{2n}\int_{\partial B(x,r)}|u|^{2n/(n-2)}\,ds + \frac{1}{2}r\int_{\partial B(x,r)}|\nabla u|^2ds \qquad (2.1)\\
&= r\int_{\partial B(x,r)}|\frac{\partial u}{\partial r}|^2dy
\end{aligned}$$

On the other hand, multiplying (1.1) by u and integrating over $B(x,r)$, we get

$$\int_{B(x,r)}|\nabla u|^2dy - \int_{\partial B(x,r)}u\frac{\partial u}{\partial r}ds = \int_{B(x,r)}|u|^{2n/(n-2)}dy \qquad (2.2)$$

Deriving (2.2) with respect to r, we obtain

$$\int_{\partial B(x,r)}|\nabla u|^2dy - \frac{d}{dr}\int_{\partial B(x,r)}u\frac{\partial u}{\partial r}ds = \int_{\partial B(x,r)}|u|^{2n/(n-2)}dy \qquad (2.3)$$

Combining (2.1), (2.2) and (2.3), we get

$$\begin{aligned}
&-\frac{r}{n}\int_{\partial B(x,r)}|u|^{2n/(n-2)}\,ds\\
&=\frac{1}{2}r\frac{d}{dr}\int_{\partial B(x,r)}u\frac{\partial u}{\partial r}\,ds - r\int_{\partial B(x,r)}|\frac{\partial u}{\partial r}|^2\,dy + r^{-1}u\frac{\partial u}{\partial r}\,ds.
\end{aligned}$$

Moreover, we have that

$$\begin{aligned}
\frac{d^2}{dr^2}(\int_{\partial B(x,r)}u^2\,ds) &= \frac{d}{dr}(2\int_{\partial B(x,r)}u\frac{\partial u}{\partial r}\,ds + \frac{n-1}{r}\int_{\partial B(x,r)}u^2\,ds)\\
&= (n-1)\Big[\frac{2}{r}\int_{\partial B(x,r)}u\frac{\partial u}{\partial r}\,ds + (\frac{n-1}{r^2}-\frac{1}{r^2})\int_{\partial B(x,r)}u^2\,ds\Big]\\
&\quad + 2\frac{d}{dr}\int_{\partial B(x,r)}u\frac{\partial u}{\partial r}\,ds\\
&= \frac{n-1}{r}\Big[2\int_{\partial B(x,r)}u\frac{\partial u}{\partial r}\,ds + \frac{n-2}{r}\int_{\partial B(x,r)}u^2\,ds\Big]\\
&\quad + 2\frac{d}{dr}\int_{\partial B(x,r)}u\frac{\partial u}{\partial r}\,ds.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{n}\frac{d}{dr}\int_{B(x,r)}|u|^{2n/(n-2)}\,dy+\frac{1}{n}\frac{d^2}{dr^2}\int_{\partial B(x,r)}u^2\,ds\\
&=\int_{\partial B(x,r)}(|\frac{\partial u}{\partial r}|^2+\frac{2n-3}{2r}u\frac{\partial u}{\partial r}+\frac{(n-1)(n-2)}{4}r^{-2}u^2)\,ds.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\frac{d}{dr}(\frac{1}{r}\int_{\partial B(x,r)}u^2ds)\\
&=-\frac{1}{r^2}\int_{\partial B(x,r)}u^2ds+\frac{2}{r}\int_{\partial B(x,r)}u\frac{\partial u}{\partial r}ds+\frac{n-1}{r^2}\int_{\partial B(x,r)}u^2ds\\
&=\frac{n-2}{r^2}\int_{\partial B(x,r)}u^2ds+\frac{2}{r}\int_{\partial B(x,r)}u\frac{\partial u}{\partial r}ds.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\frac{d}{dr}\Big[\frac{1}{n}\int_{B(x,r)}|u|^{2n/(n-2)}dy+\frac{1}{n}\frac{d}{dr}\int_{\partial B(x,r)}u^2ds-\frac{1}{n}\frac{1}{r}\int_{\partial B(x,r)}u^2ds\Big]\\
&=\int_{\partial B(x,r)}(|\frac{\partial u}{\partial r}|^2+(n-2)r^{-1}u|\frac{\partial u}{\partial r}|+\frac{(n-2)^2}{4}r^{-2}u^2)ds\\
&=\int_{\partial B(x,r)}(\frac{\partial u}{\partial r}+\frac{n-2}{2}r^{-1}u)^2ds\geq 0
\end{aligned}$$

We conclude that

$$E_u(x,r)=\frac{1}{n}\int_{B(x,r)}|u|^{2n/(n-2)}dy+\frac{1}{n}\frac{d}{dr}\int_{B(x,r)}u^2ds+\frac{1}{n}r^{-1}\int_{B(x,r)}u^2ds \quad (2.4)$$

is a nondecreasing function of r. Using the fact that

$$\int_{B(x,r)}|u|^{2n/(n-2)}dy-\int_{\partial B(x,r)}|\nabla u|^2dy=-\int_{\partial B(x,r)}u\frac{\partial u}{\partial r}ds,$$

one can easily get

$$\begin{aligned}
E_u(x,r)&=\frac{1}{n}\int_{B(x,r)}|u|^{2n/(n-2)}dy+\frac{1}{4}\frac{d}{dr}\int_{\partial B(x,r)}u^2ds-\frac{1}{4}r^{-1}\int_{\partial B(x,r)}u^2ds\\
&=\frac{\frac{n}{2}}{n}\int_{B(x,r)}|u|^{2n/(n-2)}dy+\frac{1-\frac{n}{2}}{n}\int_{B(x,r)}|u|^{2n/(n-2)}dy\\
&\quad+\frac{1}{4}\frac{d}{dr}\int_{\partial B(x,r)}u^2ds-\frac{1}{4}r^{-1}\int_{\partial B(x,r)}u^2ds\\
&=\frac{1}{2}\int_{B(x,r)}|\nabla u|^2dy-\frac{1}{2}\int_{\partial B(x,r)}u\frac{\partial u}{\partial r}ds-\frac{n-2}{2n}\int_{B(x,r)}|u|^{2n/(n-2)}dy\\
&\quad+\frac{1}{4}\frac{d}{dr}\int_{\partial B(x,r)}u^2ds-\frac{1}{4}r^{-1}\int_{\partial B(x,r)}u^2ds\\
&=\frac{1}{2}\int_{B(x,r)}(|\nabla u|^2-\frac{n-2}{2n}|u|^{2n/(n-2)})dy+\frac{1}{4}\frac{d}{dr}\int_{\partial B(x,r)}u^2\,ds\\
&\quad-\frac{1}{4}r^{-1}\int_{\partial B(x,r)}u^2ds-\frac{1}{2}\int_{\partial B(x,r)}u\frac{\partial u}{\partial r}ds.
\end{aligned}$$

We obtain an equivalent formulation of $E_u(x,r)$

$$E_u(x,r) = \frac{1}{2}\int_{B(x,r)} (|\nabla u|^2 - \frac{n-2}{2n}|u|^{2n/(n-2)}dy + \frac{n-2}{4}r^{-1}\int_{\partial B(x,r)} u^2 ds \quad (2.5)$$

Moreover, using the fact that

$$\frac{d}{dr}\int_{\partial B(x,r)} u^2 ds = 2\int_{\partial B(x,r)} u\frac{\partial u}{\partial r}ds + \frac{n-1}{r}\int_{\partial B(x,r)} u^2$$

we obtain

$$\begin{aligned}
\frac{1}{r}\int_{\partial B(x,r)} u^2\,ds &= \frac{1}{n-1}\frac{d}{dr}\int_{\partial B(x,r)} u^2\,ds - \frac{2}{n-1}\int_{\partial B(x,r)} u\frac{\partial u}{\partial r}\,ds \\
&= \frac{1}{n-1}\frac{d}{dr}\int_{\partial B(x,r)} u^2\,ds \\
&\quad + \frac{2}{n-1}\Big[\int_{B(x,r)} |u|^{2n/(n-2)}\,dy - \int_{B(x,r)} |\nabla u|^2\,dy\Big]
\end{aligned}$$

Then $E_u(x,r)$ can also be written

$$\begin{aligned}
&E_u(x,r) \\
&= \frac{1}{2(n-1)}\int_{B(x,r)} (|\nabla u|^2 + \frac{n-2}{n}|u|^{2n/(n-2)})\,dy + \frac{n-2}{4(n-1)}\frac{d}{dr}\int_{\partial B(x,r)} u^2\,ds.
\end{aligned}$$

Proof of Lemma 1.1. To prove that $(x,r) \mapsto E_u(x,r)$ is continuous it suffices to prove that

$$(x,r) \mapsto \int_{\partial B(x,r)} u^2 ds$$

is continuous with respect to x and r. We have

$$\int_{\partial B(x,r)} u\frac{\partial u}{\partial r}ds = \int_{B(x,r)} |\nabla u|^2 - \int_{B(x,r)} |u|^{2n/(n-2)}dy$$

Thus $(x,r) \mapsto \int_{\partial B(x,r)} u\frac{\partial u}{\partial r}$ is continuous, and this allows to get the conclusion.

Now, to prove that E_u is positive, we proceed by contradiction. If the result is not true, then there would exists $x \in \Omega$ and $R > 0$ such that $E_u(x,R) < 0$. For almost every y in some neighborhood of x, we have

$$\lim_{r\to 0}\int_{\partial B(x,r)} u\frac{\partial u}{\partial r}\,ds = 0$$

integrating $E_u(x,r)$ over the interval $[0,R]$ and using the fact that r $\mapsto E_u(x,r)$ is increasing, we obtain

$$\begin{aligned}
\int_0^R E_u(y,r)dr &= \frac{1}{2(n-1)}\int_0^R dr\int_{B(y,r)} (|\nabla u|^2 + \frac{n-2}{2n}|u|^{2n/(n-2)})dx \\
&\quad + \frac{n-2}{4(n-1)}\int_{\partial B(y,R)} u^2\,ds \\
&\le RE_u(y,R) < 0
\end{aligned}$$

which is not possible. This proves Lemma 1.1. $\square$

Lemma 2.1. *There exist $r_0 > 0$ and some constant $c > 0$, depending only on n, such that*

$$\int_{B(x,r)} (|\nabla u|^2 + |u|^{2n/(n-2)})\, dy < cE_u(x,r)$$

for any $r < r_0/2$.

Proof. Using the fact that $(x,r) \mapsto E_u(x,r)$ is nondecreasing, we have

$$\begin{aligned}
rE_u(x,r) &\geq \int_0^r E_u(x,s)\, ds \\
&= \frac{1}{2n-2} \int_0^r ds \int_{B(x,s)} (|\nabla u|^2 + \frac{n-2}{n}|u|^{2n/(n-2)})\, dy \\
&\quad + \frac{n-2}{4(n-1)} \int_0^r ds \int_{\partial B(x,s)} u^2\, d\sigma \\
&\geq \frac{1}{2(n-1)} \frac{n-2}{n} \int_{\frac{r}{2}}^r ds \int_{B(x,s)} (|\nabla u|^2 + |u|^{2n/(n-2)}) dy \\
&\geq C(n) \frac{r}{2} \int_{B(x,\frac{r}{2})} (|\nabla u|^2 + |u|^{2n/(n-2)})\, dy
\end{aligned}$$

where $C(n)$ is a positive constant depending only on n. This gives the desired result. $\square$

As a consequence of Lemma 2.1, we have the following result.

Lemma 2.2. *Assume that there exist x_0 and $r_0 > 0$ such that $E_u(x_0, r_0) \leq \varepsilon$ then*

$$\int_{B(x,r)} (|\nabla u|^2 + \frac{n-2}{n}|u|^{2n/(n-2)})\, dy \leq C\varepsilon \quad \forall \quad 0 < r < 2r_0$$

where C is a positive constant depending only on n.

Proof. Let x_0 and r_0 be such that $E_u(x_0, r_0) \leq \varepsilon$ and let $0 < r < r_0$, then for all $x \in B(x_0, \frac{r}{2})$ we have

$$B(x, \frac{r}{2}) \subset B(x_0, r) \subset B(x_0, r_0)$$

Thus

$$\begin{aligned}
E_u(x_0, r_0) &\geq \frac{n-2}{2n(n-1)} \int_{B(x,\frac{r}{2})} |u|^{2n/(n-2)}\, dy \\
&\quad + \frac{1}{2(n-1)} \int_{B(x,\frac{r}{2})} |\nabla u|^2 dy + \frac{n-2}{4(n-1)} \frac{d}{dr} \int_{\partial B(x_0,r)} u^2\, ds \\
&\geq \frac{1}{2(n-1)} \int_{B(x,\frac{r}{2})} \Big(|u|^{2n/(n-2)} + \nabla u|^2\Big)\, dy + \frac{n-2}{4(n-1)} \frac{d}{dr} \int_{\partial B(x_0,r)} u^2\, ds
\end{aligned}$$

Integrating between 0 and r, we obtain

$$\begin{aligned}
&rE_u(x_0, r_0)\\
&\geq \frac{1}{2(n-1)} \int_0^r ds \int_{B(x,\frac{s}{2})} (|u|^{2n/(n-2)} + \nabla u|^2)\, dy + \frac{n-2}{4(n-1)} \int_{\partial B(x_0,r)} u^2\, ds\\
&\geq \frac{1}{2(n-1)} \int_0^r ds \int_{B(x,\frac{s}{2})} (|\nabla u|^2 + |u|^{2n/(n-2)})\, dy\\
&\geq \frac{1}{2(n-1)} \int_{\frac{r}{2}}^r ds \int_{B(x,\frac{s}{2})} (|\nabla u|^2 + |u|^{2n/(n-2)})\, dy\\
&\geq \frac{1}{2(n-1)} \frac{r}{2} \int_{B(x,\frac{r}{2})} (|\nabla u|^2 + |u|^{2n/(n-2)})\, dy.
\end{aligned}$$

Then

$$E_u(x_0, r_0) \geq \frac{1}{4(n-1)} \int_{B(x,\frac{r}{2})} (|\nabla u|^2 + |u|^{2n/(n-2)})\, dy,$$

thus

$$\int_{B(x,r)} (|\nabla u|^2 + |u|^{2n/(n-2)})\, dy \leq C\varepsilon \quad \forall r < 2r_0.$$

This proves the desired result. $\square$

Proof of Theorem 1.2. Without loss of generality, we can assume that $x_0 = 0$ and we denote by B_{r_0} the ball of radius r_0 centered at $x_0 = 0$.

We use the idea of Schoen [12]. For $r < r_0$, we define

$$F(y) = (\frac{r}{2} - |y|)^{(n-2)/2} u(y)$$

Clearly F is continuous over $B_{\frac{r}{2}}$, then there exist $y_0 \in B_{\frac{r}{2}}$ such that

$$F(y_0) = \max_{y \in B_{\frac{r}{2}}} (\frac{r}{2} - |y|)^{(n-2)/2} u(y) = (\frac{r}{2} - |y_0|)^{(n-2)/2} u(y_0)$$

Let $0 < \sigma < \frac{r}{2}$, for all $y \in B_\sigma$, we have

$$u(y) \leq \frac{(\frac{r}{2} - |y_0|)^{(n-2)/2}}{(\frac{r}{2} - |y|)^{(n-2)/2}} u(y_0)$$

Then

$$\sup_{y \in B_\sigma} u(y) \leq \frac{(\frac{r}{2} - |y_0|)^{(n-2)/2}}{(\frac{r}{2} - |y|)^{(n-2)/2}} \sup_{y \in B_{\sigma_0}} u(y)$$

where $\sigma_0 = |y_0|$. Let $y_1 \in B_{\sigma_0}$ be such that

$$u(y_1) = \sup_{y \in B_{\sigma_0}} u(y)$$

We claim that

$$u(y_1) \leq \frac{2^{(n-2)/2}}{(\frac{r}{2} - |y_0|)^{(n-2)/2}}.$$

Indeed, on the contrary case, we get

$$(u(y_1))^{-2/(n-2)} \leq \frac{1}{2}(\frac{r}{2} - |y_0|)$$

Let $\mu = (u(y_1))^{-2/(n-2)}$. We have

$$B_\mu(y_1) \subset B_{\frac{\sigma_0 + \frac{r}{2}}{2}}$$

($|z-y_1|<\mu$ take $|z|<\frac{\frac{r}{2}+|y_0|}{2}$). Hence

$$\sup_{y\in B_\mu(y_1)} u(y) \le \frac{(\frac{r}{2}-|y_0|)^{(n-2)/2}}{(\frac{\frac{r}{2}-|y_0|}{2})^{(n-2)/2}} u(y_1) = 2^{(n-2)/2}u(y_1)$$

Let $v(x)=\mu^{(n-2)/2}u(\mu x+y_1)$. Easy computations shows that v satisfies

$$\begin{aligned}\Delta v^{2n/(n-2)} &= \frac{2n}{n-2}\Big[\frac{n+2}{n-2}v^{4/(n-2)}|\nabla v|^2 + v^{\frac{n+2}{n-2}}\triangle v\Big]\\ &\ge \frac{2n}{n-2}v^{\frac{n+2}{n-2}}\triangle v = -\frac{2n}{n-2}v^{2\frac{n+2}{n-2}}\end{aligned}$$

On the other hand

$$v^{2n/(n-2)}(0) = \mu^{\frac{n-2}{2}\frac{2n}{n-2}}u^{\frac{2n}{n-2}}(y_1) = 1.$$

Moreover, we have

$$\begin{aligned}\sup_{B_1} v(x) &= \mu^{(n-2)/2}\sup_{B_1} u(\mu x+y_1)\\ &= \mu^{(n-2)/2}\sup_{B_\mu(y_1)} u(x)\\ &\le \mu^{(n-2)/2}2^{(n-2)/2}u(y_1) = 2^{(n-2)/2}.\end{aligned}$$

Then $\sup_{B_1} v^{2n/(n-2)} \le 2^n$. Therefore,

$$-\Delta v^{2n/(n-2)} \le C(n)v^{2n/(n-2)}.$$

We conclude that

$$1 = v^{2n/(n-2)}(0) \le C\int_{B_1} v^{2n/(n-2)}(x)dx = C\mu^n\int_{B_\mu} u^{2n/(n-2)}(x)dx \le C\varepsilon.$$

For ϵ sufficiently small, we derive a contradiction. It follows that

$$\sup_{B_{\frac{r}{2}}} u(y) \le \frac{(\frac{r}{2}-|y_0|)^{(n-2)/2}}{(\frac{r}{2}-|y|)^{(n-2)/2}}\cdot\frac{2^{(n-2)/2}}{(\frac{r}{2}-|y_0|)^{(n-2)/2}} = \frac{2^{(n-2)/2}}{(\frac{r}{2}-|y|)^{(n-2)/2}}.$$

For $|y|<r/4$, we have

$$\sup_{B_{\frac{r}{4}}} u(y) \le C(n)/r^{(n-2)/2}$$

This in turns proves the Theorem 1.3. ☐

Proof of Lemma 1.4. We keep the above notations. To show (i), suppose $x_0\in B_1\setminus\Sigma$, then there exists $r_1>0$ such that

$$\liminf_{i\to\infty} E_{u_i}(x_0,r_1) < \varepsilon_0.$$

Then, we may find a sequence $n_j\to\infty$ as $j\to\infty$ such that

$$\sup_{n_j} E_{u_{n_j}}(x_0,r_1) < \varepsilon_0.$$

We deduce from the ε-regularity Theorem (Theorem 1.2) that

$$\sup_{n_j}\sup_{x\in B_{\frac{r_1}{16}}(x_0)} |u_{n_j}| \le \frac{C}{r_1^{(n-2)/2}}.$$

for some constant C depending only on n. Then

$$u_{n_j}\to u \quad \text{in } C^1(B_{\frac{r_1}{16}}(x_0))$$

a similar argument allows to show that

$$\nabla u_{n_j} \to \nabla u \quad \text{in } C^1(B_{\frac{r_1}{16}}(x_0))$$

Then

$$\mu_{n_j} := \Big(\frac{1}{2}|\nabla u_{n_j}|^2 + \frac{n-2}{2n} u_{n_j}^{2n/(n-2)}\Big)\, dx \to \Big(\frac{1}{2}|\nabla u|^2 + \frac{n-2}{2n} u^{2n/(n-2)}\Big)\, dx$$

as radon measure. Hence $\nu = 0$ on $B_{\frac{r_1}{16}}(x_0)$ i.e $x_0 \notin \operatorname{supp}(\nu)$ and then we deduce that $\operatorname{supp}(\nu) \subset \Sigma$.

To show (ii), let us first recall some properties of the function $E_u(x,r)$ that has been defined above:

- For all $x \in \Omega$, there exists $r_0 > 0$ and a constant $C > 0$ such that

$$\int_{B(x,r)} (\frac{1}{2}|\nabla u|^2 + \frac{n-2}{2n}|u|^{2n/(n-2)}) < C E_u(x, r_0) \quad \forall r < \frac{r_0}{2}.$$

This is explained in the proof of Lemma 1.1.

- Using the fact that $E_u(x,.)$ is increasing on r together with the fact that

$$\lim_{r\searrow 0} E_u(x,r) = 0 \quad \mathcal{H}^0 - \text{a.e. } x \in \Omega$$

we deduce that for $\mathcal{H}^0$-a.e. $x \in \Sigma$, $\lim_{r\searrow 0} \int_{B(x,r)} \nu$ exists. and the density $\Theta(\eta,.)$ defined by

$$\Theta(\eta, x) := \lim_{r\searrow 0} \eta(B_r(x)) \tag{2.6}$$

exists for every $x \in \Omega$. Moreover, for $\mathcal{H}^0$-a.e. $x \in \Omega$, $\Theta_u(x) = 0$, where

$$\Theta_u(x) := \lim_{r\searrow 0} \int_{B(x,r)} (\frac{1}{2}|\nabla u|^2 + \frac{n-2}{2n}|u|^{\frac{2n}{n-2}})\, dy. \tag{2.7}$$

Now, for r sufficiently small and i sufficiently large

$$\int_{B(x,r)} \frac{1}{2}|\nabla u_i|^2 + \frac{n-2}{2n} u_i^{2n/(n-2)} \le C E_{u_i}(x,r) \le C(\Lambda, \Omega) \tag{2.8}$$

where Λ is given above and $C(\Lambda,\Omega)$ is a constant depending only on Λ and Ω. Hence

$$\eta(B(x,r)) \le C(\Lambda,\Omega) \quad \text{for } x \in B_1^n \tag{2.9}$$

In particular, this implies that $\eta\lfloor\Sigma$ is absolutely continuous with respect to $\mathcal{H}^0\lfloor\Sigma$. Applying Radon-Nikodym's Theorem [4], we conclude that

$$\eta\lfloor\Sigma = \Theta(x)\mathcal{H}^0\lfloor\Sigma \quad \text{for } \mathcal{H}^0\text{-a.e. } x \in \Sigma \tag{2.10}$$

Using 2.8 we conclude that

$$\nu(x) = \Theta(x)\mathcal{H}^0\lfloor\Sigma \tag{2.11}$$

for a $\mathcal{H}^0$-a.e. $x \in \Sigma$ (recall that $\eta = (\frac{1}{2}|\nabla u|^2 + \frac{n-2}{2n}|u|^{\frac{2n}{n-2}})\, dx + \nu$ and $\operatorname{supp}(\nu) \subset \Sigma$). The estimate on Θ follows from 2.9. $\square$

For any $y \in B_1^n$ and any sufficiently small $\lambda > 0$, we define the scaled measure $\eta_{y,\lambda}$ by

$$\eta_{y,\lambda}(x) := \eta(y + \lambda x) \tag{2.12}$$

We have the following lemma.

Lemma 2.3. *Assume that* $(\lambda_j)_j$ *satisfies* $\lim_{j\to\infty}\lambda_j = 0$. *Then, there exist a subsequence* $(\lambda_{j'})_{j'}$ *and a Radon measure* χ *defined on* Ω*, such that* $\eta_{y,\lambda_{j'}} \rightharpoonup \chi$ *in the sense of measures.*

Proof. For each $i \in \mathbb{N}$, we define the scaled function $u_{i,y,\lambda}$ by

$$u_{i,y,\lambda}(x) := \lambda^{\frac{n-2}{2}} u_i(\lambda x + y) \quad \text{for } y \in B_1^n. \tag{2.13}$$

Then $u_{i,y,\lambda}$ is a solution of

$$-\Delta u = u|u|^{4/(n-2)} \quad \text{on } B_1^n.$$

In addition, for any $r > 0$ sufficiently small, we have

$$\begin{aligned} &\int_{B_r(0)} \Big(\frac{1}{2}|\nabla u_{i,y,\lambda}|^2 + \frac{n-2}{2n}|u_{i,y,\lambda}|^{\frac{2k}{k-2}}\Big)\, dx \\ &= \int_{B_{\lambda r}(y)} \Big(\frac{1}{2}|\nabla u_i|^2 + \frac{n-2}{2n}|u_i|^{\frac{2n}{n-2}}\Big)\, dx \le C(\Lambda,\Omega). \end{aligned} \tag{2.14}$$

Finally for fixed λ,

$$\begin{aligned} &\Big(\frac{1}{2}|\nabla u_{i,y,\lambda}|^2 + \frac{n-2}{2n}|u_{i,y,\lambda}|^{2n/(n-2)}\Big)(x)\, dx \\ &= \lambda^n \Big(\frac{1}{2}|\nabla u_i|^2 - \frac{n-2}{2n}|u_i|^{2n/(n-2)}\Big)(\lambda x + y)\, dx \\ &\rightharpoonup \eta(\lambda x + y) = \eta_{y,\lambda}(x) \end{aligned}$$

in the sense of measures as $i \to \infty$. On the other hand letting i tends to infinity in (2.14), we conclude that for any $r > 0$

$$\eta_{y,\lambda}(B_r(0)) \le C(\Omega,\Lambda). \tag{2.15}$$

Hence, we may find a subsequence $\{\lambda_j'\}$ of $\{\lambda_j\}$ and a Radon measure χ such that $\eta_{y,\lambda_j'}$ converge weakly to χ as Radon measure on Ω. Then

$$\lim_{j\to\infty}\lim_{i\to\infty}\Big(\frac{1}{2}|\nabla u_{i,y,\lambda_j'}|^2 + \frac{n-2}{2n}|u_{i,y,\lambda_j'}|^{\frac{2n}{n-2}}\Big)\, dx = \lim_{j\to\infty}\eta_{y,\lambda_j'}(x) = \chi$$

Using a diagonal subsequence argument, we may find a subsequence $i_j \to \infty$, such that

$$\lim_{j\to\infty}\Big(\frac{1}{2}|\nabla u_{i_j,y,\lambda_j'}|^2 + \frac{n-2}{2n}|u_{i_j,y,\lambda_j'}|^{\frac{2n}{n-2}}\Big)\, dx = \chi$$

This proves the Lemma. □

Remark 2.4. Observe that

$$\chi(B_r(0)) = \lim_{j\to\infty}\eta_{y,\lambda_j'}(B_r(0)) = \lim_{j\to\infty}\eta(B_{\lambda_j' r}(y)) = \Theta(\eta, y)$$

In particular, we deduce that $\chi(B_r(0))$ is independent of r.

3. Proof of Theorem 1.5

The idea of the proof comes from Rivière [11] in the context of Yang-Mills Fields. To simplify notation and since the result is local, we assume that Ω is the unit ball B^n of $\mathbb{R}^n$. Let (u_k) be a sequence of smooth solutions of (1.1) such that

$$\Big(\|u_k\|_{\mathbf{H}^1(\Omega)} + \|u_k\|_{\mathbf{L}^{2n/(n-2)}(\Omega)}\Big)$$

is bounded and let ν be the defect measure defined above. We claim that for $\delta > 0$, we have

$$\lim_{k\to\infty} \sup_{y\in B_1(x_0)} \int_{B_\delta(y_0)} \Big(|u_k|^{2n/(n-2)} + |\nabla u_k|^2\Big) \geq \varepsilon(n) \tag{3.1}$$

where $\varepsilon(n)$ is given by Theorem 1.5. Indeed if (3.1) would not hold, we have for $\delta > 0$ and $k \in \mathbb{N}$ large enough

$$\sup_{y\in B_1(x_0)} \int_{B_\delta(y_0)} \Big(|u_k|^{2n/(n-2)} + |\nabla u_k|^2\Big) \leq \varepsilon(n)$$

and by Theorem 1.2 we have

$$\|\nabla u_k\|_{\mathbf{L}^\infty(B_{\frac{\delta}{2}}(y))} \leq C(\epsilon)/r^{n/2}$$

This contradict the concentration phenomenon and the claim is proved. We then conclude that there exists sequences $\delta_k \to 0$ as $k \to \infty$ and $(y_k) \subset B_1(x_0)$ such that

$$\begin{aligned}\int_{B_{\delta_k}(y_0)} \Big(|u_k|^{2n/(n-2)} + |\nabla u_k|^2\Big)\,dx &= \sup_{y\in B_1(x_0)} \int_{B_{\delta_k(y_0)}} \Big(|u_k|^{2n/(n-2)} + |\nabla u_k|^2\Big)\,dx \\ &= \frac{\varepsilon(n)}{2}.\end{aligned} \tag{3.2}$$

In other words, y_k is located at a bubble of characteristic size δ_k. More precisely, if one introduces the function

$$\widetilde{u}_k(x) = \delta_k^{(n-2)/2} u_k(\delta_k x + y_k);$$

we have, up to a subsequence, that

$$\begin{gathered}\widetilde{u}_k \to u_\infty \quad \text{in } \mathbf{C}^\infty_{\rm loc}(\mathbb{R}^n) \quad \text{as } k\to\infty,\\ \nabla\widetilde{u}_k \to \nabla u_\infty \quad \text{in } \mathbf{C}^\infty_{\rm loc}(\mathbb{R}^n) \quad \text{as } k\to\infty\,.\end{gathered}$$

Therefore,

$$-\Delta u_\infty = u_\infty |u_\infty|^{4/(n-2)} \quad \text{in } \mathbb{R}^n.$$

This is the first bubble we detect. On the other hand, we have clearly that

$$\int_{\mathbb{R}^n} \Big(|u_\infty|^{2n/(n-2)} + |\nabla u_\infty|^2\Big)\,dx = \lim_{R\to\infty}\lim_{k\to\infty} \int_{B_{R\delta_k}(y_k)} \Big(|u_k|^{2n/(n-2)} + |\nabla u_k|^2\Big)\,dx. \tag{3.3}$$

Indeed:

$$
\begin{aligned}
&\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{R\delta_k}(y_k)}\Big(|u_k|^{2n/(n-2)}+|\nabla u_k|^2\Big)\,dx\\
&=\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_R(0)}\Big(|u_k|^{2n/(n-2)}+|\nabla(u_k)|^2\Big)\,(\delta_k x+y_k)\,\delta_k^n\,dx\\
&=\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_R(0)}\Big(|\delta_k{}^{\frac{2-n}{2}}\widetilde{u}_k(x)|^{2n/(n-2)}+|\delta_k{}^{\frac{2-n}{2}}\delta_k{}^{-1}\nabla\widetilde{u}_k(x)|^2\Big)\,\delta_k^n\,dx\\
&=\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_R(0)}\Big(|\widetilde{u}_k(x)|^{2n/(n-2)}+|\nabla\widetilde{u}_k(x)|^2\Big)\,dx\\
&=\lim_{R\to\infty}\int_{B_R(0)}\Big(|u_\infty(x)|^{2n/(n-2)}+|\nabla u_\infty(x)|^2\Big)\,dx\\
&=\int_{\mathbb{R}^n}\Big(|u_\infty(x)|^{2n/(n-2)}+|\nabla u_\infty(x)|^2\Big)\,dx\,.
\end{aligned}
$$

Assume first that we have only one bubble of characteristic δ_k. We have shown that

$$
\Theta=\lim_{k\to\infty}\int_{B_1^n(0)}\Big(|\nabla u_k|^2+|u_k|^{2n/(n-2)}\Big)\,dx=\int_{\mathbb{R}^n}\Big(|\nabla u_\infty|^2+|u_\infty|^{2n/(n-2)}\Big)\,dx, \tag{3.4}
$$

where Θ is defined above. It suffices to prove that

$$
\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_1^n(0)\setminus B_{R\delta_k(y_k)}}\Big(|u_k(x)|^{2n/(n-2)}+|\nabla u_k(x)|^2\Big)\,dx=0\,. \tag{3.5}
$$

In other words there is no "neck" of energy which is quantized.

To simplify notation, we assume that $y_k=0$. We claim that for any $\varepsilon>0$ small enough, there exists $R>0$ and $k_0\in\mathbb{N}$ such that for any $k\geq k_0$ and $R\delta_k\leq r\leq\frac{1}{2}$, we have

$$
\int_{B_{2r}^n(0)\setminus B_{r(0)}}\Big(|u_k(x)|^{2n/(n-2)}+|\nabla u_k(x)|^2\Big)\,dx\leq\varepsilon \tag{3.6}
$$

Indeed, if is not the case, we may find $\varepsilon_0>0$, a subsequence $k'\to\infty$ (Still denoted k) and a sequence r_k such that

$$
\begin{gathered}
\int_{B_{2r}^n(0)\setminus B_r(0)}\Big(|u_k(x)|^{2n/(n-2)}+|\nabla u_k(x)|^2\Big)\,dx>\varepsilon_0,\\
\frac{r_k}{\delta_k}\to\infty\quad\text{as}\quad k\to\infty
\end{gathered} \tag{3.7}
$$

Let $\alpha_k\to 0$ such that $r_k/\alpha_k=o(1)$ and $\alpha_k r_k/\delta_k\to\infty$ and let

$$
v_k(x)=r_k^{(n-2)/2}u_k(r_k x)
$$

clearly v_k satisfies

$$
-\Delta v_k=v_k|v_k|^{4/(n-2)}\quad\text{in } B_{2\alpha_k}\setminus B_{\alpha_k}
$$

Therefore,

$$
\int_{B_2^n(0)\setminus B_1(0)}\Big(|v_k(x)|^{2n/(n-2)}+|\nabla v_k(x)|^2\Big)\,dx>\varepsilon(n)
$$

and then we have a second bubble. This contradict our assumption.

We deduce from (3.7) and Theorem 1.2 that for any $\varepsilon < \varepsilon(n)$, there exist $R > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $|x| \geq R\delta_k$

$$|\nabla u_k|(x) \leq C(\epsilon)/|x|^{n/2}$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then

$$|\nabla u_k|^2(x) \leq C(\varepsilon)/|x|^n. \tag{3.8}$$

We define E_λ^k by

$$E_\lambda^k = \text{meas}\,\{x \in \mathbb{R}^n : |\nabla u_k|(x) \geq \lambda\}$$

We have $E_\lambda^k \leq C(\varepsilon)/\lambda^2$; indeed

$$\{x \in \mathbb{R}^n : |\nabla u_k|(x) \geq \lambda\} \subset \{x \in \mathbb{R}^n : |x|^n \leq \frac{C(\varepsilon)}{\lambda^2}\}$$

and

$$\text{meas}\left\{x \in \mathbb{R}^n : |x|^n \leq \frac{C(\varepsilon)}{\lambda^2}\right\} \leq \frac{C(\varepsilon)}{\lambda^2}$$

We deduce from (3.8) that

$$\|\nabla u_k\|_{\mathbf{L}^{2,\infty}(C_{B_{R\delta_k}})} \leq C(\varepsilon) \tag{3.9}$$

where $\mathbf{L}^{2,\infty}$ is the Lorentz space defined in [14], the weak $\mathbf{L}^2$ space, and $\|\cdot\|_{\mathbf{L}^{2,\infty}}$ is the weak norm defined by

$$\|f\|_{\mathbf{L}^{2,\infty}} = \sup_{0<t<\infty} t^{1/2} f^*(t)$$

where f^* is the nonincreasing rearrangement of $|f|$. Indeed

$$\|\nabla u_k\|_{\mathbf{L}^{2,\infty}(C_{B_{R\delta_k}})} = \sup_{0<t<\infty} t^{1/2} (\nabla u_k)^*(t)$$

by definition,

$$(\nabla u_k)^*(t) = inf\{\lambda > 0/E_\lambda^k \leq t\}$$

For all $t > 0$ such that $\frac{C(\varepsilon)}{\lambda^2} \leq t$, we have $E_\lambda^k \leq t$. Then

$$\begin{aligned}
\inf\left\{\lambda > 0 : E_\lambda^k \leq t\right\} &\leq \inf\left\{\lambda > 0 : \frac{C(\varepsilon)}{\lambda^2} \leq t\right\} \\
&\leq \inf\left\{\lambda > 0 : \lambda \geq \frac{(C(\varepsilon))^{1/2}}{t^{1/2}}\right\} \\
&= \frac{(C(\varepsilon))^{1/2}}{t^{1/2}}
\end{aligned}$$

Hence $t^{1/2}(\nabla u_k)^*(t) \leq C(\varepsilon)$ and so

$$\|\nabla u_k\|_{\mathbf{L}^{2,\infty}(C_{B_{R\delta_k}})} \leq C(\varepsilon) \tag{3.10}$$

We claim that the sequence (∇u_k) is uniformly bounded in the Lorentz space $\mathbf{L}^{2,1}(B_1^n)$ (see [14] for the definition). We prove this claim using an iteration proceeding; Indeed, the sequence (u_k) is bounded in $\mathbf{L}^{\frac{2n}{n-2}}(B_1^n)$. Then

$$\Delta u_k = -u_k|u_k|^{4/(n-2)}$$

is bounded in $\mathbf{L}^{\frac{2n}{n+2}}(B_1^n)$ which implies by the elliptic regularity Theorem that the sequence (u_k) is bounded in $\mathbf{W}^{2,\frac{2n}{n+2}}(B_1^n)$. Using the imbedding Theorem for Sobolev spaces

$$\mathbf{W}^{m,p}(B_1^n) \subset \mathbf{W}^{r,s}(B_1^n) \quad \text{if } m \geq r,\ p \geq s \text{ and } m - \frac{n}{p} = r - \frac{n}{s}.$$

In particular, $\mathbf{W}^{2,\frac{2n}{n+2}}(B_1^n)$ is continuously imbedded in $\mathbf{W}^{1,2}(B_1^n)$. On the other hand by Proposition 4 in [14], we have

$$\mathbf{W}^{1,2}(B_1^n) \hookrightarrow \mathbf{L}^{2^*,2}(B_1^n) = \mathbf{L}^{\frac{2n}{n-2},2}(B_1^n)$$

continuously. We then deduce that

$$\Delta u_k = -u_k|u_k|^{4/(n-2)}$$

is bounded in $\mathbf{L}^{\frac{2n}{n+2},\frac{2(n-2)}{n+2}}(B_1^n)$. Here, we have used the following lemma.

Lemma 3.1. *If $f \in \boldsymbol{L}^{p,q}(B_1^n)$ and $\alpha \in \mathbb{Q}^+$, then $f^\alpha \in \boldsymbol{L}^{\frac{p}{\alpha},\frac{q}{\alpha}}(B_1^n)$.*

Proof. In the case where $\alpha \in \mathbb{N}$, the result follows from the fact that

$$f \in \mathbf{L}^{a,b}(B_1^n) \text{ and } g \in \mathbf{L}^{c,d}(B_1^n) \Rightarrow f.g \in \mathbf{L}^{q,r}(B_1^n),$$

where $\frac{1}{q} = \frac{1}{a} + \frac{1}{b}$ and $\frac{1}{r} = \frac{1}{c} + \frac{1}{b}$ (see [2]). The general case is a consequence of the fact that the increasing rearrangement of the function $|f|^\beta$ is equal to the puissance β of the increasing rearrangement of $|f|$ since $(f^\beta)^*$ is the only one function verifying

$$\operatorname{meas}\{x \in \mathbb{R}^n : f^\beta(x) \geq \lambda\} = \operatorname{meas}\{t > 0 : (f^\beta)^*(x) \geq \lambda\}$$

This in turns proves Lemma 3.1. □

Now, using in [14, Theorem 8], we deduce from (3.7) that (∇u_k) is uniformly bounded in the space $\mathbf{L}^{(\frac{2n}{n+2})^*,\frac{2(n-2)}{n+2}}(B_1^n) = \mathbf{L}^{2,\frac{2(n-2)}{n+2}}(B_1^n)$. Hence (u_k) is bounded in $\mathbf{L}^{2^*,\frac{2(n-2)}{n+2}}(B_1^n)$. Then

$$\Delta u_k = -u_k|u_k|^{4/(n-2)}$$

is bounded in $\mathbf{L}^{\frac{2n}{n+2},\frac{2(n-2)^2}{(n+2)^2}}(B_1^n)$. Hence, again by [14, Theorem 8], the sequence (∇u_k) is bounded in $\mathbf{L}^{2,\frac{2(n-2)^2}{(n+2)^2}}(B_1^n)$ and by elliptic regularity Theorem

$$\Delta u_k = -u_k|u_k|^{4/(n-2)}$$

is bounded in $\mathbf{L}^{\frac{2n}{n+2},\frac{2(n-2)^3}{(n+2)^3}}(B_1^n)$. We obtain after p iterations that

$$\Delta u_k = -u_k|u_k|^{4/(n-2)}$$

is bounded in $\mathbf{L}^{\frac{2n}{n+2},\frac{2(n-2)^p}{(n+2)^p}}(B_1^n)$. We choose $p > 0$ such that $6p > n$, we have in particular $\frac{2(n-2)^p}{(n+2)^p} < 1$ which gives

$$\Delta u_k = -u_k|u_k|^{4/(n-2)}$$

is bounded in $\mathbf{L}^{\frac{2n}{n+2},1}(B_1^n)$. Here we have used the fact that

$$\mathbf{L}^{p,q_1}(B_1^n) \subset \mathbf{L}^{p,q_2}(B_1^n) \quad \text{if } q_1 < q_2$$

We use also [14, Theorem 8] to deduce that (∇u_k) is bounded in $\mathbf{L}^{(\frac{2n}{n+2})^*,1}(B_1^n) = \mathbf{L}^{2,1}(B_1^n)$. In particular, there exist a constant $C > 0$ depending only on n such that

$$\|\nabla u_k\|_{\mathbf{L}^{2,1}(B_1^n)} \leq C \tag{3.11}$$

We deduce from (3.10), (3.11) together with the $\mathbf{L}^{2,1} - \mathbf{L}^{2,\infty}$ duality that

$$\|\nabla u_k\|_{\mathbf{L}^2(B_1^n \setminus B_{R\delta_k})} \leq \|\nabla u_k\|_{\mathbf{L}^{2,1}(B_1^n \setminus B_{R\delta_k})} \|\nabla u_k\|_{\mathbf{L}^{2,\infty}(B_1^n \setminus B_{R\delta_k})} \leq C(\epsilon)$$

for a constant $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now, we use the embedding $\mathbf{H}^1 \hookrightarrow \mathbf{L}^{2n/(n-2)}$ continuously, we obtain

$$\begin{aligned} \|u_k\|_{\mathbf{L}^{2n/(n-2)}(B_1^n \setminus B_{R\delta_k})} &\leq C \|\nabla u_k\|_{\mathbf{L}^2(B_1^n \setminus B_{R\delta_k})} \\ &\leq C(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

We deduce that

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{B_1^n(0)\setminus B_{R\delta_k(y_k)}} (|u_k|^{2n/(n-2)} + |\nabla u_k|^2)(x)\, dx = 0$$

This proves Theorem 1.5 in the case of one bubble.

The case of more than one bubble can be handled in a very similar way and we just give few details for $m = 2$. The proof starts the same until (3.4) which cannot hold any more otherwise we would have had one bubble only as it is (3.4) holds. It remains to show that: for any $\varepsilon \geq 0$, there are sufficiently large R > 0 and a sequence $r_i \to 0$ such that for any $R\delta_i \leq r_i \leq 1/2$,

$$\begin{gathered} \lim_{R\to\infty} \lim_{i\to\infty} \int_{\{0\}\times B_{r_i}^n \setminus B_{R\delta_i}^n(0)} (\frac{1}{2}|\nabla v_i|^2 + \frac{n-2}{2n}|v_i|^{2n/(n-2)})\, dx = 0\,, \\ \lim_{i\to\infty} \int_{\{0\}\times B_{1/2}^n \setminus B_{r_i}^n(0)} (\frac{1}{2}|\nabla v_i|^2 + \frac{n-2}{2n}|v_i|^{2n/(n-2)})\, dx = 0 \end{gathered} \tag{3.12}$$

where v_i is defined by $v_i(y) = {r_i}^{(n-2)/2}\ u_i(r_i y)\ ,\ y \in \mathbb{R}^n$.

The proof of (3.12) can be done exactly as the proof of (3.4), the case of 2 bubbles is then proved. To prove the general case, for any number $m \geq 2$, one can follow exactly the same strategy.

References

[1] W. Allard, *An integrity Theorem a regularity Theorem for surfaces whose first variation with respect to a parametric elliptic integrand is controlled*, Proc. Symp. Pure Math., 44, (1986), 1-28.

[2] H. Brezis and S. Wainger, *A note on limiting cases of Sobolev embedding and convolution inequalities*, Comm. P.D.E, 5, (1980), 773-789.

[3] S. K. Donaldson and R. P. Thomas, *Gauge theory in higher dimensions* in The geometric Universe (Oxford, 1996), Oxford Univ. Press, 1998, 31-47.

[4] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[5] Z. Guo and Jiayu-Li, *The blow-up locus of semilinear elliptic equations with subcritical exponent*, Calc. Var. Partial Differential Equations 15 (2002), no. 2, 133-153.

[6] F. Pacard, *Partial regulatity for weak solutions of a nonlinear elliptic equation*, Manuscripta Math. 79 (1993), 161-172.

[7] T. Parker, *Bubble tree convergence for harmonic maps*, J. Diff. Geom. , 44, (1996), 545-633.

[8] J. Peetre, *Espaces d'interpolations et théorème de Sobolev*, Ann. Instit. Fourier , Grenoble, 16, (1966), 279-317.

[9] F. G. Lin, *Gradient estimates and blow-up analysis for stationary harmonic maps*, Ann. Math. , 149, (1999), 785-829.
[10] F. G. Lin and T. Rivière, *A Quantization Property for Static Ginzburg-Landau vortices*, Comm. Pure Appl. Math. 54 (2001), no. 2, 206–228.
[11] T. Rivière, *Interpolation Spaces and Energy Quantization for Yang-Mills Fields*, Comm. Anal. Geom. 10 (2002), no. 4, 683–708.
[12] R. Schoen, *Analytic aspects for the harmonic map problem*, Math. Sci. Res. Insti. Publi. 2, Springer, Berlin (1984), 312-358.
[13] L. Simon, *Lectures on Geometric Measure Theory*, Proc. of Math. Anal.3, Australian National Univ. (1983).
[14] L. Tartar, *Imbedding Theorems of Sobolev Spaces into Lorentz Spaces*, Boll. U.M.I. 1, B, (1998) 479-500.

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