ENERGY QUANTIZATION FOR YAMABE’S PROBLEM IN CONFORMAL DIMENSION

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Abstract. Riviè re [11] proved an energy quantization for Yang-Mills fields defined on \( n \)-dimensional Riemannian manifolds, when \( n \) is larger than the critical dimension 4. More precisely, he proved that the defect measure of a weakly converging sequence of Yang-Mills fields is quantized, provided the \( W^{2,1} \) norm of their curvature is uniformly bounded. In the present paper, we prove a similar quantization phenomenon for the nonlinear elliptic equation

\[
-\Delta u = u|u|^{4/(n-2)},
\]

in a subset \( \Omega \) of \( \mathbb{R}^n \).

1. Introduction

Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) with \( n \geq 3 \). We consider the equation

\[
-\Delta u = u|u|^{4/(n-2)} \quad \text{in} \quad \Omega
\] (1.1)

We will say that \( u \) is a weak solution of (1.1) in \( \Omega \), if, for all \( \Phi \in C^\infty(\Omega) \) with compact support in \( \Omega \), we have

\[
-\int_{\Omega} \Delta \Phi(x) u(x) dx = \int_{\Omega} \Phi(x) u(x)|u(x)|^{4/(n-2)} dx
\] (1.2)

If in addition \( u \) satisfies

\[
\int \left[ \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial \Phi^j}{\partial x_i} \right] dx - \frac{1}{2} |\nabla u|^2 \frac{\partial \Phi^i}{\partial x_i} + \frac{n-2}{2n} |u|^{2n/(n-2)} \frac{\partial \Phi^i}{\partial x_i} dx = 0
\] (1.3)

for any \( \Phi = (\Phi^1, \Phi^2, \ldots, \Phi^n) \in C^\infty(\Omega) \) with compact support in \( \Omega \), we say that \( u \) is stationary. In other words, a weak solution \( u \) in \( H^1(\Omega) \cap L^{2n/(n-2)}(\Omega) \) of (1.1) is stationary if the functional \( E \) defined by

\[
E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{n-2}{2n} \int_{\Omega} |u|^{2n/(n-2)}
\]

is stationary with respect to domain variations, i.e.

\[
\frac{d}{dt}(E(u_t))|_{t=0} = 0
\]

where \( u_t(x) = u(x + t\Phi) \). It is easy to verify that a smooth solution is stationary.

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In this paper we prove a monotonicity formula for stationary weak solution $u$ in $H^1(\Omega) \cap L^{2n/(n-2)}(\Omega)$ of (1.1) by a similar idea as in [6]. More precisely we have the following result.

**Lemma 1.1.** Suppose that $u \in L^{2n/(n-2)}(\Omega) \cap H^1(\Omega)$ is a stationary weak solution of (1.1). Consider the function

$$E_u(x, r) = \int_{B(x, r)} |u|^{2n/(n-2)} \, dy + \frac{d}{dr} \int_{\partial B(x, r)} u^2 \, ds + r^{-1} \int_{B(x, r)} u^2 \, ds.$$ 

Then $r \mapsto E_u(x, r)$ is positive, nondecreasing and continuous.

This monotonicity formula together with ideas which go back to the work of Schoen [12], allowed to prove the following result.

**Theorem 1.2.** There exists $\varepsilon > 0$ and $r_0 > 0$ depend only on $n$ such that, for any smooth solution $u \in H^1(\Omega) \cap L^{2n/(n-2)}(\Omega)$ of (1.1), we have: For any $x_0 \in \Omega$, if

$$\int_{B(x_0, r_0)} |\nabla u|^2 + |u|^{2n/(n-2)} \leq \varepsilon,$$

then

$$\|u\|_{L^\infty(B_{r_0/2}(x_0))} \leq \frac{C(\varepsilon)}{r^{(n-2)/2}} \quad \text{for any} \quad r < r_0,$$

where $B_{r_0}(x_0)$ is the ball centered at $x_0$ with radius $r_0/2$, and $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Zongming Guo and Jiay Li [5] studied sequences of smooth solutions of (1.1) having uniformly bounded energy, they proved the following result.

**Theorem 1.3.** Let $u_i$ be a sequence of smooth solutions of (1.1) such that

$$\|u_i\|_{H^1(\Omega)} + \|u_i\|_{L^{2n/(n-2)}(\Omega)}$$

is bounded. Let $u_\infty$ be the weak limit of $u_i$ in $H^1(\Omega) \cap L^{2n/(n-2)}(\Omega)$. Then $u_\infty$ is smooth and satisfies equation (1.1) outside a closed singular subset $\Sigma$ of $\Omega$. Moreover, there exists $r_0 > 0$ and $\varepsilon_0 > 0$ such that

$$\Sigma = \cap_{0 < r < r_0} \{ x \in \Omega : \liminf_{i \to \infty} E_{u_i}(x, r) \geq \varepsilon_0 \}.$$
(i) $\Sigma \subset \text{spt}(\nu)$

(ii) There exists a measurable, upper-semi-continuous function $\Theta$ such that

$$\nu(x) = \Theta(x) \mathcal{H}^0|_{\Sigma}, \quad \text{for } x \in \Sigma.$$ 

Moreover, there exists some constants $c$ and $C > 0$ (only depending on $n$ and $\Omega$) such that

$$c\varepsilon < \Theta(x) < C \mathcal{H}^0 - \text{a.e. in } \Sigma$$

where $H^0|_{\Sigma}$ is the restriction to $\Sigma$ of the Hausdorff measure and $\Theta$ is a measurable function on $\Sigma$.

The main question we would like to address in the present paper concerns the multiplicity $\Theta$ of the defect measure which has been defined above. More precisely, we have proved the following theorem.

**Theorem 1.5.** Let $\nu$ be the defect measure of the sequence $(|\nabla u_i|^2 + |u_i|^{2n/(n-2)})dx$ defined above. Then $\nu$ is quantized. That is, for a.e $x \in \Sigma$,

$$\Theta(x) = \sum_{j=1}^{N_x} \|\nabla v_{x,j}\|_{L^2(\Omega)}^2 + \|v_{x,j}\|_{L^{2n/(n-2)}(\Omega)}^{2n/(n-2)} \quad (1.4)$$

where $N_x$ is a positive integer and where the functions $v_{x,j}$ are solutions of $\Delta v + v^{n+2/\nu} = 0$ which are defined on $\mathbb{R}^n$, issued from $(u_i)$ and that concentrate at $x$ as $i \to \infty$.

The sentence “issued from $(u_i)$ and that concentrate at $x$ as $i \to \infty$” means that there are sequences of conformal maps $\psi^i_j$, a finite family of balls $(B^i_{l,j})_l$ such that the pulled back function

$$\tilde{u}_{i,j} = (\psi^i_j)^* u_i$$

satisfies

$$\tilde{u}_{i,j} \to v_j \quad \text{strongly in } L^2(\mathbb{R}^n \setminus \cup_{l} B^i_{l,j})),$$

$$\nabla \tilde{u}_{i,j} \to \nabla v_j \quad \text{strongly in } L^2(\mathbb{R}^n \setminus \cup_{l} B^i_{l,j}))$$

In the context of Yang-Mills fields in dimension $n \geq 4$ a similar concentration result has been proven by Rivièrè [11]. More precisely, Rivièrè has shown that, if $(A_i)_i$ is a sequence of Yang-Mills connections such that $(\|\nabla A_i \nabla F_{A_i}\|_{L^1(B^i_1)})_i$ is bounded, then the corresponding defect measure $\nu = \Theta \mathcal{H}^{n-4}|_{\Sigma}$ of a sequence of smooth Yang-Mills connections is quantized.

The proof of Theorem 1.5 uses technics introduced by Lin and Rivièrè in their study of Ginzburg-Landau vortices [10] and also the technics developed by Rivièrè in [5]. These technics use as an essential tool the Lorentz spaces, more specifically the $L^{2,\infty} - L^{2,1}$ duality [14].

This paper is organized in the following way: In Section 2 we establish first a monotonicity formula for smooth solutions of problem (1.1) which allows us to prove an $\varepsilon$-regularity Theorem. Then, we prove Theorem 1.2 and Lemma 1.4.

While Section 3 is devoted to the proof of our main result, Theorem 1.5.
2. A monotonicity Inequality

In this section, we establish a monotonicity formula for smooth solutions of problem \([1.1]\). Using Pohozaev identity: Multiplying \([1.1]\) by \(x_i \frac{\partial u}{\partial x_i}\) (summation over \(i\) is understood) and integrating over \(B(x,r)\), the ball centered at \(x\) of radius \(r\), we obtain

\[
- \int_{B(x,r)} x_i \frac{\partial u}{\partial x_i} \Delta u \, dy = - \int_{B(x,r)} x_i \frac{\partial u}{\partial x_i} u |u|^{4/(n-2)} \, dy
\]

By Green formula, we get

\[
\frac{n-2}{2} \int_{B(x,r)} |u|^{2n/(n-2)} \, dy - \frac{n-2}{2} \int_{B(x,r)} |\nabla u|^2 \, dy
\]

\[
- \frac{n-2}{2n} \int_{\partial B(x,r)} |u|^{2n/(n-2)} \, ds + \frac{1}{2r} \int_{\partial B(x,r)} |\nabla u|^2 \, ds
\]

\[
= r \int_{\partial B(x,r)} \left| \frac{\partial u}{\partial r} \right|^2 \, dy
\]

On the other hand, multiplying \([1.1]\) by \(u\) and integrating over \(B(x,r)\), we get

\[
\int_{B(x,r)} |\nabla u|^2 \, dy - \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} \, ds = \int_{B(x,r)} |u|^{2n/(n-2)} \, dy
\]

Deriving \([2.2]\) with respect to \(r\), we obtain

\[
\int_{\partial B(x,r)} |\nabla u|^2 \, dy - \frac{d}{dr} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} \, ds = \int_{\partial B(x,r)} |u|^{2n/(n-2)} \, dy
\]

Combining \([2.1]\), \([2.2]\) and \([2.3]\), we get

\[
- \frac{r}{n} \int_{\partial B(x,r)} |u|^{2n/(n-2)} \, ds
\]

\[
= \frac{1}{2} \frac{d}{dr} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} \, ds - r \int_{\partial B(x,r)} \left| \frac{\partial u}{\partial r} \right|^2 \, dy + r^{-1} u \frac{\partial u}{\partial r} \, ds.
\]

Moreover, we have that

\[
\frac{d^2}{dr^2} \left( \int_{\partial B(x,r)} u^2 \, ds \right) = \frac{d}{dr} \left( 2 \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} \, ds + \frac{n-1}{r} \int_{\partial B(x,r)} u^2 \, ds \right)
\]

\[
= (n-1) \left[ \frac{2}{r} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} \, ds + \left( \frac{n-1}{r^2} - \frac{1}{r^2} \right) \int_{\partial B(x,r)} u^2 \, ds \right]
\]

\[
+ 2 \frac{d}{dr} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} \, ds
\]

\[
= \frac{n-1}{r} \left[ 2 \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} \, ds + \frac{n-2}{r} \int_{\partial B(x,r)} u^2 \, ds \right]
\]

\[
+ 2 \frac{d}{dr} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} \, ds.
\]
Hence
\[
\frac{1}{n} \frac{d}{dr} \int_{B(x,r)} |u|^{2n/(n-2)} dy + \frac{1}{n} \frac{d^2}{dr^2} \int_{\partial B(x,r)} u^2 ds = \int_{\partial B(x,r)} \left( \left| \frac{\partial u}{\partial r} \right|^2 + \frac{2n-3}{2r} u \frac{\partial u}{\partial r} + \frac{(n-1)(n-2)}{4} r^{-2} u^2 \right) ds.
\]

Moreover
\[
\frac{d}{dr} \left( \frac{1}{r} \int_{\partial B(x,r)} u^2 ds \right)
= -\frac{1}{r^2} \int_{\partial B(x,r)} u^2 ds + \frac{2}{r} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds + \frac{n-1}{r^2} \int_{\partial B(x,r)} u^2 ds
= \frac{n-2}{r^2} \int_{\partial B(x,r)} u^2 ds + \frac{2}{r} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds.
\]

We obtain
\[
\frac{d}{dr} \left[ \frac{1}{n} \int_{B(x,r)} |u|^{2n/(n-2)} dy + \frac{1}{n} \frac{d}{dr} \int_{B(x,r)} u^2 ds - \frac{1}{n} \frac{1}{r} \int_{\partial B(x,r)} u^2 ds \right]
= \int_{\partial B(x,r)} \left( \left| \frac{\partial u}{\partial r} \right|^2 + (n-2)r^{-1} u \frac{\partial u}{\partial r} + \frac{(n-2)^2}{4} r^{-2} u^2 \right) ds
= \int_{\partial B(x,r)} \left( \frac{\partial u}{\partial r} + \frac{n-2}{2} r^{-1} u \right)^2 ds \geq 0
\]

We conclude that
\[
E_u(x,r) = \frac{1}{n} \int_{B(x,r)} |u|^{2n/(n-2)} dy + \frac{1}{n} \frac{d}{dr} \int_{B(x,r)} u^2 ds + \frac{1}{n} \frac{1}{r} \int_{B(x,r)} u^2 ds
\]
is a nondecreasing function of \( r \). Using the fact that
\[
\int_{B(x,r)} |u|^{2n/(n-2)} dy - \int_{\partial B(x,r)} |\nabla u|^2 dy = -\int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds,
\]
one can easily get
\[
E_u(x,r) = \frac{1}{n} \int_{B(x,r)} |u|^{2n/(n-2)} dy + \frac{1}{n} \frac{d}{dr} \int_{B(x,r)} u^2 ds - \frac{1}{4} \int_{\partial B(x,r)} u^2 ds
= \frac{n}{2} \int_{B(x,r)} |u|^{2n/(n-2)} dy + \frac{1}{2} \frac{d}{dr} \int_{\partial B(x,r)} u^2 ds - \frac{n}{2} \int_{B(x,r)} |u|^{2n/(n-2)} dy
+ \frac{1}{4} \frac{d}{dr} \int_{\partial B(x,r)} u^2 ds - \frac{1}{4} \int_{\partial B(x,r)} u^2 ds
= \frac{1}{2} \int_{B(x,r)} |\nabla u|^2 dy - \frac{1}{2} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds - \frac{n-2}{2n} \int_{B(x,r)} |u|^{2n/(n-2)} dy
+ \frac{1}{4} \frac{d}{dr} \int_{\partial B(x,r)} u^2 ds - \frac{1}{4} \int_{\partial B(x,r)} u^2 ds
= \frac{1}{2} \int_{B(x,r)} (|\nabla u|^2 - \frac{n-2}{2n} |u|^{2n/(n-2)}) dy + \frac{1}{4} \frac{d}{dr} \int_{\partial B(x,r)} u^2 ds
- \frac{1}{4} \int_{\partial B(x,r)} u^2 ds - \frac{1}{2} \int_{\partial B(x,r)} u \frac{\partial u}{\partial r} ds.
\]
We obtain an equivalent formulation of $E_u(x, r)$

$$E_u(x, r) = \frac{1}{2} \int_{B(x, r)} (|\nabla u|^2 - \frac{n-2}{2n} |u|^{2n/(n-2)} + \frac{n-2}{4} \int_{\partial B(x, r)} u^2 \, ds) \, dy \quad (2.5)$$

Moreover, using the fact that

$$\frac{d}{dr} \int_{\partial B(x, r)} u^2 \, ds = 2 \int_{\partial B(x, r)} u \frac{\partial u}{\partial r} \, ds + \frac{n-1}{r} \int_{\partial B(x, r)} u^2 \, ds$$

we obtain

$$\frac{1}{r} \int_{\partial B(x, r)} u^2 \, ds = \frac{1}{n-1} \frac{d}{dr} \int_{\partial B(x, r)} u^2 \, ds - \frac{2}{n-1} \int_{\partial B(x, r)} u \frac{\partial u}{\partial r} \, ds$$

$$= \frac{1}{n-1} \frac{d}{dr} \int_{\partial B(x, r)} u^2 \, ds$$

$$+ \frac{2}{n-1} \left[ \int_{B(x, r)} |u|^{2n/(n-2)} \, dy - \int_{B(x, r)} |\nabla u|^2 \, dy \right]$$

Then $E_u(x, r)$ can also be written

$$E_u(x, r) = \frac{1}{2(n-1)} \int_{B(x, r)} (|\nabla u|^2 + \frac{n-2}{n} |u|^{2n/(n-2)} + \frac{n-2}{4(n-1)} \frac{d}{dr} \int_{\partial B(x, r)} u^2 \, ds) \, dy$$

Proof of Lemma 1.1. To prove that $(x, r) \mapsto E_u(x, r)$ is continuous it suffices to prove that

$$(x, r) \mapsto \int_{\partial B(x, r)} u^2 \, ds$$

is continuous with respect to $x$ and $r$. We have

$$\int_{\partial B(x, r)} u \frac{\partial u}{\partial r} \, ds = \int_{B(x, r)} |\nabla u|^2 - \int_{B(x, r)} |u|^{2n/(n-2)} \, dy$$

Thus $(x, r) \mapsto \int_{\partial B(x, r)} u \frac{\partial u}{\partial r} \, ds$ is continuous, and this allows to get the conclusion.

Now, to prove that $E_u$ is positive, we proceed by contradiction. If the result is not true, then there would exists $x \in \Omega$ and $R > 0$ such that $E_u(x, R) < 0$. For almost every $y$ in some neighborhood of $x$, we have

$$\lim_{r \to 0} \int_{\partial B(x, r)} u \frac{\partial u}{\partial r} \, ds = 0$$

Integrating $E_u(x, r)$ over the interval $[0, R]$ and using the fact that $r \mapsto E_u(x, r)$ is increasing, we obtain

$$\int_0^R E_u(y, r) \, dr = \frac{1}{2(n-1)} \int_0^R \frac{d}{dr} \int_{B(y, r)} (|\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)}) \, dx$$

$$+ \frac{n-2}{4(n-1)} \int_{\partial B(y, R)} u^2 \, ds$$

$$\leq RE_u(y, R) < 0$$

which is not possible. This proves Lemma 1.1. □
Lemma 2.1. There exist $r_0 > 0$ and some constant $c > 0$, depending only on $n$, such that
\[
\int_{B(x,r)} \left( |\nabla u|^2 + |u|^{2n/(n-2)} \right) dy < cE_u(x,r)
\]
for any $r < r_0/2$.

Proof. Using the fact that $(x, r) \mapsto E_u(x, r)$ is nondecreasing, we have
\[
rE_u(x, r) \geq \int_0^r E_u(x, s) ds
\]
\[
= \frac{1}{2n-2} \int_0^r ds \int_{B(x,s)} (|\nabla u|^2 + \frac{n-2}{n} |u|^{2n/(n-2)}) dy
\]
\[
+ \frac{n-2}{4(n-1)} \int_0^r ds \int_{\partial B(x,s)} u^2 d\sigma
\]
\[
\geq \frac{1}{2(n-1)} \frac{n-2}{n} \int_0^r ds \int_{B(x,s)} (|\nabla u|^2 + |u|^{2n/(n-2)}) dy
\]
\[
\geq C(n) \frac{r}{2} \int_{B(x, \frac{r}{2})} (|\nabla u|^2 + |u|^{2n/(n-2)}) dy
\]
where $C(n)$ is a positive constant depending only on $n$. This gives the desired result. \qed

As a consequence of Lemma 2.1 we have the following result.

Lemma 2.2. Assume that there exist $x_0$ and $r_0 > 0$ such that $E_u(x_0, r_0) \leq \varepsilon$ then
\[
\int_{B(x,r)} \left( |\nabla u|^2 + \frac{n-2}{n} |u|^{2n/(n-2)} \right) dy \leq C\varepsilon \quad \forall \quad 0 < r < 2r_0
\]
where $C$ is a positive constant depending only on $n$.

Proof. Let $x_0$ and $r_0$ be such that $E_u(x_0, r_0) \leq \varepsilon$ and let $0 < r < r_0$, then for all $x \in B(x_0, \frac{r}{2})$ we have
\[
B(x, \frac{r}{2}) \subset B(x_0, r) \subset B(x_0, r_0)
\]
Thus
\[
E_u(x_0, r_0) \geq \frac{n-2}{2n(n-1)} \int_{B(x, \frac{r}{2})} |u|^{2n/(n-2)} dy
\]
\[
+ \frac{1}{2(n-1)} \int_{B(x, \frac{r}{2})} |\nabla u|^2 dy + \frac{n-2}{4(n-1)} \frac{d}{dr} \int_{\partial B(x_0, r)} u^2 ds
\]
\[
\geq \frac{1}{2(n-1)} \int_{B(x, \frac{r}{2})} \left( |u|^{2n/(n-2)} + |\nabla u|^2 \right) dy + \frac{n-2}{4(n-1)} \frac{d}{dr} \int_{\partial B(x_0, r)} u^2 ds
\]
Integrating between 0 and \( r \), we obtain
\[
    rE_\mu(x_0, r_0) \geq \frac{1}{2(n-1)} \int_0^r \int_{B(x, \frac{1}{2})} (|u|^2 - |u|^2) dy + \frac{n-2}{4(n-1)} \int_{\partial B(x_0, r)} u^2 ds
\]
\[
    \geq \frac{1}{2(n-1)} \int_0^r \int_{B(x, \frac{1}{2})} (\nabla u)^2 + |u|^{2n/(n-2)} dy
\]
\[
    \geq \frac{1}{2(n-1)} \int_{B(x, \frac{1}{2})} (\nabla u)^2 + |u|^{2n/(n-2)} dy
\]
\[
    \geq \frac{1}{2(n-1)} \frac{1}{2} \int_{B(x, \frac{1}{2})} (\nabla u)^2 + |u|^{2n/(n-2)} dy.
\]

Then
\[
    E_\mu(x_0, r_0) \geq \frac{1}{4(n-1)} \int_{B(x, \frac{1}{2})} (\nabla u)^2 + |u|^{2n/(n-2)} dy
\]
thus
\[
    \int_{B(x, r)} (\nabla u)^2 + |u|^{2n/(n-2)} dy \leq C\varepsilon \quad \forall r < 2r_0.
\]

This proves the desired result. \( \square \)

Proof of Theorem 1.2. Without loss of generality, we can assume that \( x_0 = 0 \) and we denote by \( B_{r_0} \) the ball of radius \( r_0 \) centered at \( x_0 = 0 \).

We use the idea of Schoen [12]. For \( r < r_0 \), we define
\[
    F(y) = \left( \frac{r}{2} - |y| \right)^{(n-2)/2} u(y)
\]
Clearly \( F \) is continuous over \( B_{\frac{1}{2}} \), then there exist \( y_0 \in B_{\frac{1}{2}} \) such that
\[
    F(y_0) = \max_{y \in B_{\frac{1}{2}}} \left( \frac{r}{2} - |y| \right)^{(n-2)/2} u(y) = \left( \frac{r}{2} - |y_0| \right)^{(n-2)/2} u(y_0)
\]
Let \( 0 < \sigma < \frac{1}{2} \), for all \( y \in B_{\sigma} \), we have
\[
    u(y) \leq \left( \frac{r}{2} - |y_0| \right)^{(n-2)/2} u(y_0)
\]
Then
\[
    \sup_{y \in B_{\sigma}} u(y) \leq \left( \frac{r}{2} - |y_0| \right)^{(n-2)/2} \sup_{y \in B_{\sigma_0}} u(y)
\]
where \( \sigma_0 = |y_0| \). Let \( y_1 \in B_{\sigma_0} \) be such that
\[
    u(y_1) = \sup_{y \in B_{\sigma_0}} u(y)
\]
We claim that
\[
    u(y_1) \leq \frac{2^{(n-2)/2}}{2^{(n-2)/2}}
\]
Indeed, on the contrary case, we get
\[
    (u(y_1))^{-2/(n-2)} \leq \frac{1}{2} \left( \frac{r}{2} - |y_0| \right)
\]
Let \( \mu = (u(y_1))^{-2/(n-2)} \). We have
\[
    B_{\mu}(y_1) \subset B_{\frac{r_0}{2}}
\]
\[ |z - y_1| < \mu \text{ take } |z| < \frac{\mu + |y_0|}{2}. \] Hence
\[ \sup_{y \in B_{\mu}(y_1)} u(y) \leq \frac{\left(\frac{\mu}{2} - |y_0|\right)(n-2)/2}{\left(\frac{\mu + |y_0|}{2}\right)(n-2)/2} u(y_1) = 2^{(n-2)/2} u(y_1) \]

Let \( v(x) = \mu^{(n-2)/2} u(\mu x + y_1) \). Easy computations shows that \( v \) satisfies
\[ \Delta v^{2n/(n-2)} = \frac{2n}{n-2} \left[ \frac{n+2}{n-2} v^{4/(n-2)} \left| \nabla v \right|^2 + v^{n+2} \Delta v \right] \]
\[ \geq \frac{2n}{n-2} \left( \frac{n+2}{n-2} \right)^2 \Delta v = -\frac{2n}{n-2} v^{n+2} \Delta v \]

On the other hand
\[ v^{2n/(n-2)}(0) = \mu^{\frac{n-2}{2} \frac{2n}{n-2}} u^{\frac{2n}{n-2}}(y_1) = 1. \]

Moreover, we have
\[ \sup_{B_1} v(x) = \mu^{(n-2)/2} \sup_{B_1} u(\mu x + y_1) \]
\[ = \mu^{(n-2)/2} \sup_{B_{\mu}(y_1)} u(x) \]
\[ \leq \mu^{(n-2)/2} 2^{(n-2)/2} u(y_1) = 2^{(n-2)/2}. \]

Then \( \sup_{B_1} v^{2n/(n-2)} \leq 2^n \). Therefore,
\[ -\Delta v^{2n/(n-2)} \leq C(n) v^{2n/(n-2)}. \]

We conclude that
\[ 1 = v^{2n/(n-2)}(0) \leq C \int_{B_1} v^{2n/(n-2)}(x) \, dx = C \mu^n \int_{B_{\mu}} u^{2n/(n-2)}(x) \, dx \leq C \varepsilon. \]

For \( \varepsilon \) sufficiently small, we derive a contradiction. It follows that
\[ \sup_{B_{\frac{r}{2}}} u(y) \leq \frac{\left(\frac{\mu}{2} - |y_0|\right)(n-2)/2}{\left(\frac{\mu}{2} - |y_0|\right)(n-2)/2} \cdot \frac{2^{(n-2)/2}}{\left(\frac{\mu}{2} - |y_0|\right)(n-2)/2} = \frac{2^{(n-2)/2}}{\left(\frac{\mu}{2} - |y_0|\right)(n-2)/2}. \]

For \( |y| < r/4 \), we have
\[ \sup_{B_{\frac{r}{2}}} u(y) \leq C(n) / r^{(n-2)/2} \]

This in turns proves the Theorem 1.3 \( \square \)

Proof of Lemma 1.4. We keep the above notations. To show (i), suppose \( x_0 \in B_1 \setminus \Sigma \), then there exists \( r_1 > 0 \) such that
\[ \liminf_{i \to \infty} E_{u_i}(x_0, r_1) < \varepsilon_0. \]

Then, we may find a sequence \( n_j \to \infty \) as \( j \to \infty \) such that
\[ \sup_{n_j} E_{u_{n_j}}(x_0, r_1) < \varepsilon_0. \]

We deduce from the \( \varepsilon \)-regularity Theorem (Theorem 1.2) that
\[ \sup_{n_j} \sup_{x \in B_{r_1}(x_0)} |u_{n_j}| \leq \frac{C}{r_1^{(n-2)/2}}, \]

for some constant \( C \) depending only on \( n \). Then
\[ u_{n_j} \to u \text{ in } C^1(B_{r_1}(x_0)) \]
a similar argument allows to show that
\[ \nabla u_{n_j} \rightharpoonup \nabla u \quad \text{in} \ C^1(B_{\frac{r_0}{2}}(x_0)) \]
Then
\[ \mu_{n_j} := \left( \frac{1}{2} |\nabla u_{n_j}|^2 + \frac{n-2}{2n} u_{n_j}^{2n/(n-2)} \right) \rightarrow \left( \frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} u^{2n/(n-2)} \right) \number{2.8} \]
as radon measure. Hence \( \nu = 0 \) on \( B_{\frac{r_0}{2}}(x_0) \) i.e \( x_0 \notin \text{supp}(\nu) \) and then we deduce that \( \text{supp}(\nu) \subset \Sigma \).

To show (ii), let us first recall some properties of the function \( E_u(x, r) \) that has been defined above:
• For all \( x \in \Omega \), there exists \( r_0 > 0 \) and a constant \( C > 0 \) such that
\[ \int_{B(x, r)} \left( \frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)} \right) < CE_u(x, r_0) \quad \forall r < \frac{r_0}{2} \]
This is explained in the proof of Lemma 1.1.
• Using the fact that \( E_u(x, \cdot) \) is increasing on \( r \) together with the fact that \( \lim_{r \searrow 0} E_u(x, r) = 0 \) \( \mathcal{H}^0 \)-a.e. \( x \in \Omega \)
we deduce that for \( \mathcal{H}^0 \)-a.e. \( x \in \Sigma \), \( \lim_{r \searrow 0} \int_{B(x, r)} \nu \) exists. and the density \( \Theta(\eta, \cdot) \) defined by
\[ \Theta(\eta, x) := \lim_{r \searrow 0} \int_{B(x, r)} \left( \frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)} \right) dy \]
exists for every \( x \in \Omega \). Moreover, for \( \mathcal{H}^0 \)-a.e. \( x \in \Omega \), \( \Theta_u(x) = 0 \), where
\[ \Theta_u(x) := \lim_{r \searrow 0} \int_{B(x, r)} \left( \frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)} \right) dy \]
Now, for \( r \) sufficiently small and \( i \) sufficiently large
\[ \int_{B(x, r)} \left( \frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} u_i^{2n/(n-2)} \right) \leq CE_{u_i}(x, r) \leq C(\Lambda, \Omega) \number{2.8} \]
where \( \Lambda \) is given above and \( C(\Lambda, \Omega) \) is a constant depending only on \( \Lambda \) and \( \Omega \). Hence
\[ \eta(B(x, r)) \leq C(\Lambda, \Omega) \quad \text{for } x \in B^n_1 \]
In particular, this implies that \( \eta|\Sigma \) is absolutely continuous with respect to \( \mathcal{H}^0|\Sigma \). Applying Radon-Nikodym’s Theorem \[4\], we conclude that
\[ \eta|\Sigma = \Theta(x)|\mathcal{H}^0|\Sigma \quad \text{for } \mathcal{H}^0\text{-a.e. } x \in \Sigma \]
Using \ref{2.8} we conclude that
\[ \nu(x) = \Theta(x)|\mathcal{H}^0|\Sigma \]
for a \( \mathcal{H}^0 \)-a.e. \( x \in \Sigma \) (recall that \( \eta = (\frac{1}{2} |\nabla u|^2 + \frac{n-2}{2n} |u|^{2n/(n-2)} ) \) \( dx + \nu \) and \( \text{supp}(\nu) \subset \Sigma \). The estimate on \( \Theta \) follows from \ref{2.9}.\ \ \ \ \ \square

For any \( y \in B^n_1 \) and any sufficiently small \( \lambda > 0 \), we define the scaled measure \( \eta_{y, \lambda} \) by
\[ \eta_{y, \lambda}(x) := \eta(y + \lambda x) \]
We have the following lemma.
Lemma 2.3. Assume that \((\lambda_j)_j\) satisfies \(\lim_{j \to \infty} \lambda_j = 0\). Then, there exist a subsequence \((\lambda_{j'})_j\) and a Radon measure \(\chi\) defined on \(\Omega\), such that \(\eta_{y,\lambda_{j'}} \rightharpoonup \chi\) in the sense of measures.

Proof. For each \(i \in \mathbb{N}\), we define the scaled function \(u_{i,y,\lambda}\) by

\[
u_{i,y,\lambda}(x) := \lambda^{\frac{n-2}{2}} u_i(\lambda x + y) \quad \text{for } y \in B_1^n.\]  

Then \(u_{i,y,\lambda}\) is a solution of

\[-\Delta u = u|u|^4/(n-2) \quad \text{on } B_1^n.\]

In addition, for any \(r > 0\) sufficiently small, we have

\[
\int_{B_r(0)} \left( \frac{1}{2} |\nabla u_{i,y,\lambda}|^2 + \frac{n-2}{2n} |u_{i,y,\lambda}|^{2(n/(n-2))} \right) dx
= \int_{B_{\lambda r}(y)} \left( \frac{1}{2} |\nabla u_i|^2 + \frac{n-2}{2n} |u_i|^{2n/(n-2)} \right) dx \leq C(\Lambda, \Omega). \tag{2.14}
\]

Finally for fixed \(\lambda\),

\[
\left( \frac{1}{2} |\nabla u_{i,y,\lambda}|^2 + \frac{n-2}{2n} |u_{i,y,\lambda}|^{2n/(n-2)} \right)(x) dx
= \lambda^n \left( \frac{1}{2} |\nabla u_i|^2 - \frac{n-2}{2n} |u_i|^{2n/(n-2)} \right)(\lambda x + y) dx
\to \eta(\lambda x + y) = \eta_{y,\lambda}(x)
\]

in the sense of measures as \(i \to \infty\). On the other hand letting \(i\) tends to infinity in (2.14), we conclude that for any \(r > 0\)

\[
\eta_{y,\lambda}(B_r(0)) \leq C(\Omega, \Lambda). \tag{2.15}
\]

Hence, we may find a subsequence \(\{\lambda'_j\}\) of \(\{\lambda_j\}\) and a Radon measure \(\chi\) such that \(\eta_{y,\lambda'_j}\) converge weakly to \(\chi\) as Radon measure on \(\Omega\). Then

\[
\lim_{j \to \infty} \lim_{i \to \infty} \left( \frac{1}{2} |\nabla u_{i,y,\lambda'_j}|^2 + \frac{n-2}{2n} |u_{i,y,\lambda'_j}|^{2n/(n-2)} \right) dx = \lim_{j \to \infty} \eta_{y,\lambda'_j}(x) = \chi
\]

Using a diagonal subsequence argument, we may find a subsequence \(i_j \to \infty\), such that

\[
\lim_{j \to \infty} \left( \frac{1}{2} |\nabla u_{i_j,y,\lambda'_j}|^2 + \frac{n-2}{2n} |u_{i_j,y,\lambda'_j}|^{2n/(n-2)} \right) dx = \chi
\]

This proves the Lemma. \qed

Remark 2.4. Observe that

\[
\chi(B_r(0)) = \lim_{j \to \infty} \eta_{y,\lambda'_j}(B_r(0)) = \lim_{j \to \infty} \eta(B_{\lambda'_j r}(y)) = \Theta(\eta, y)
\]

In particular, we deduce that \(\chi(B_r(0))\) is independent of \(r\).
3. Proof of Theorem 1.5

The idea of the proof comes from Rivière [11] in the context of Yang-Mills Fields. To simplify notation and since the result is local, we assume that $\Omega$ is the unit ball $B^n$ of $\mathbb{R}^n$. Let $(u_k)$ be a sequence of smooth solutions of (1.1) such that

$$\left(\|u_k\|_{H^1(\Omega)} + \|u_k\|_{L^{2n/(n-2)}(\Omega)}\right)$$

is bounded and let $\nu$ be the defect measure defined above. We claim that for $\delta > 0$, we have

$$\lim_{k \to \infty} \sup_{y \in B(\delta, x_0)} \int_{B_\delta(y_0)} \left(|u_k|^{2n/(n-2)} + |\nabla u_k|^2\right) \geq \varepsilon(n)$$

where $\varepsilon(n)$ is given by Theorem 1.5. Indeed if (3.1) would not hold, we have for $\delta > 0$ and $k \in \mathbb{N}$ large enough

$$\sup_{y \in B(\delta, x_0)} \int_{B_\delta(y_0)} \left(|u_k|^{2n/(n-2)} + |\nabla u_k|^2\right) \leq \varepsilon(n)$$

and by Theorem 1.2 we have

$$\|\nabla u_k\|_{L^\infty(B_{\delta/2}(y))} \leq C(\varepsilon)/r^{n/2}$$

This contradict the concentration phenomenon and the claim is proved. We then conclude that there exists sequences $\delta_k \to 0$ as $k \to \infty$ and $(y_k) \subset B_1(x_0)$ such that

$$\int_{B_{\delta_k}(y_0)} \left(|u_k|^{2n/(n-2)} + |\nabla u_k|^2\right) dx = \sup_{y \in B(\delta, x_0)} \int_{B_{\delta_k}(y_0)} \left(|u_k|^{2n/(n-2)} + |\nabla u_k|^2\right) dx$$

$$= \frac{\varepsilon(n)}{2}. \tag{3.2}$$

In other words, $y_k$ is located at a bubble of characteristic size $\delta_k$. More precisely, if one introduces the function

$$\tilde{u}_k(x) = \delta_k^{(n-2)/2} u_k(\delta_k x + y_k);$$

we have, up to a subsequence, that

$$\tilde{u}_k \to u_\infty \quad \text{in } C^\infty_\text{loc}(\mathbb{R}^n) \quad \text{as } k \to \infty,$$

$$\nabla \tilde{u}_k \to \nabla u_\infty \quad \text{in } C^\infty_\text{loc}(\mathbb{R}^n) \quad \text{as } k \to \infty.$$

Therefore,

$$-\Delta u_\infty = u_\infty |u_\infty|^{4/(n-2)} \quad \text{in } \mathbb{R}^n.$$

This is the first bubble we detect. On the other hand, we have clearly that

$$\int_{\mathbb{R}^n} \left(|u_\infty|^{2n/(n-2)} + |\nabla u_\infty|^2\right) dx = \lim_{R \to \infty} \lim_{k \to \infty} \int_{B_{R\delta_k}(y_k)} \left(|u_k|^{2n/(n-2)} + |\nabla u_k|^2\right) dx.$$
Indeed:

\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{B_R \setminus B_{kR_k}(y_k)} \left( |u_k|^{2n/(n-2)} + |\nabla u_k|^2 \right) dx \\
= \lim_{R \to \infty} \lim_{k \to \infty} \int_{B_R(0)} \left( |u_k|^{2n/(n-2)} + |\nabla u_k(\delta_k x + y_k)\delta_k^n | dx \\
= \lim_{R \to \infty} \lim_{k \to \infty} \int_{B_R(0)} \left( |\tilde{u}_k(x)|^{2n/(n-2)} + |\nabla \tilde{u}_k(x)|^2 \right) dx \\
= \lim_{R \to \infty} \int_{B_R(0)} \left( |u_\infty(x)|^{2n/(n-2)} + |\nabla u_\infty(x)|^2 \right) dx \\
= \int_{\mathbb{R}^n} \left( |u_\infty(x)|^{2n/(n-2)} + |\nabla u_\infty(x)|^2 \right) dx.
\]

Assume first that we have only one bubble of characteristic \(\delta_k\). We have shown that

\[
\Theta = \lim_{k \to \infty} \int_{B^*_R(0) \setminus B_{R\delta_k}(y_k)} \left( u_k(x) \right)^{2n/(n-2)} + |\nabla u_k(x)|^2 dx = \int_{\mathbb{R}^n} \left( |\nabla u_\infty|^2 + |u_\infty|^{2n/(n-2)} \right) dx,
\]

where \(\Theta\) is defined above. It suffices to prove that

\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{B^*_R(0) \setminus B_{kR_k(y_k)}} \left( |u_k(x)|^{2n/(n-2)} + |\nabla u_k(x)|^2 \right) dx = 0.
\]

In other words there is no “neck” of energy which is quantized.

To simplify notation, we assume that \(y_k = 0\). We claim that for any \(\varepsilon > 0\) small enough, there exists \(R > 0\) and \(k_0 \in \mathbb{N}\) such that for any \(k \geq k_0\) and \(R\delta_k \leq r \leq \frac{1}{2}\), we have

\[
\int_{B_{2r}(0) \setminus B_r(0)} \left( |u_k(x)|^{2n/(n-2)} + |\nabla u_k(x)|^2 \right) dx \leq \varepsilon
\]

Indeed, if is not the case, we may find \(\varepsilon_0 > 0\), a subsequence \(k' \to \infty\) (Still denoted \(k\)) and a sequence \(r_k\) such that

\[
\int_{B_{2r_k}(0) \setminus B_r(0)} \left( |u_k(x)|^{2n/(n-2)} + |\nabla u_k(x)|^2 \right) dx \geq \varepsilon_0, \\
\frac{r_k}{\delta_k} \to \infty \quad \text{as} \quad k \to \infty
\]

Let \(\alpha_k \to 0\) such that \(r_k/\alpha_k = o(1)\) and \(\alpha_k r_k/\delta_k \to \infty\) and let

\[v_k(x) = r_k^{n-2/2} u_k(r_k x)\]

clearly \(v_k\) satisfies

\[-\Delta v_k = v_k |v_k|^{4/(n-2)} \quad \text{in} \quad B_{2\alpha_k} \setminus B_{\alpha_k} \]

Therefore,

\[
\int_{B_{2\alpha_k}(0) \setminus B_{\alpha_k}(0)} \left( |v_k(x)|^{2n/(n-2)} + |\nabla v_k(x)|^2 \right) dx \geq \varepsilon(n)
\]

and then we have a second bubble. This contradicts our assumption.
We deduce from (3.7) and Theorem 1.2 that for any $\varepsilon < \varepsilon(n)$, there exist $R > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $|x| \geq R\delta_k$

$$|\nabla u_k|(x) \leq C(\varepsilon)/|x|^n/2$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then

$$|\nabla u_k|^2(x) \leq C(\varepsilon)/|x|^n. \quad (3.8)$$

We define $E^k_\lambda$ by

$$E^k_\lambda = \text{meas}\{x \in \mathbb{R}^n : |\nabla u_k|(x) \geq \lambda\}$$

We have $E^k_\lambda \leq C(\varepsilon)/\lambda^2$; indeed

$$\{x \in \mathbb{R}^n : |\nabla u_k|(x) \geq \lambda\} \subset \{x \in \mathbb{R}^n : |x|^n \leq \frac{C(\varepsilon)}{\lambda^2}\}$$

and

$$\text{meas}\left\{x \in \mathbb{R}^n : |x|^n \leq \frac{C(\varepsilon)}{\lambda^2}\right\} \leq \frac{C(\varepsilon)}{\lambda^2}$$

We deduce from (3.8) that

$$\|\nabla u_k\|_{L^2,\infty(CBR_d)} \leq C(\varepsilon) \quad (3.9)$$

where $L^{2,\infty}$ is the Lorentz space defined in [14], the weak $L^2$ space, and $\|\cdot\|_{L^2,\infty}$ is the weak norm defined by

$$\|f\|_{L^2,\infty} = \sup_{0 < t < \infty} t^{1/2} f^*(t)$$

where $f^*$ is the nonincreasing rearrangement of $|f|$. Indeed

$$\|\nabla u_k\|_{L^2,\infty(CBR_d)} = \sup_{0 < t < \infty} t^{1/2} (\nabla u_k)^*(t)$$

by definition,

$$(\nabla u_k)^*(t) = \inf\{\lambda > 0 : E^k_\lambda \leq t\}$$

For all $t > 0$ such that $\frac{C(\varepsilon)}{\lambda^2} \leq t$, we have $E^k_\lambda \leq t$. Then

$$\inf\{\lambda > 0 : E^k_\lambda \leq t\} \leq \inf\left\{\lambda > 0 : \frac{C(\varepsilon)}{\lambda^2} \leq t\right\}$$

$$\leq \inf\left\{\lambda > 0 : \lambda \geq \frac{(C(\varepsilon))^{1/2}}{t^{1/2}}\right\}$$

$$= \frac{(C(\varepsilon))^{1/2}}{t^{1/2}}$$

Hence $t^{1/2}(\nabla u_k)^*(t) \leq C(\varepsilon)$ and so

$$\|\nabla u_k\|_{L^2,\infty(CBR_d)} \leq C(\varepsilon) \quad (3.10)$$

We claim that the sequence $(\nabla u_k)$ is uniformly bounded in the Lorentz space $L^{2,1}(B^n_1)$ (see [14] for the definition). We prove this claim using an iteration proceeding; Indeed, the sequence $(u_k)$ is bounded in $L^{\frac{2n}{n-2}}(B^n_1)$. Then

$$\Delta u_k = -u_k |u_k|^{4/(n-2)}$$
is bounded in $L^{\frac{2n}{n+2}}(B^n_1)$ which implies by the elliptic regularity Theorem that the sequence $(u_k)$ is bounded in $W^{2, \frac{2n}{n+2}}(B^n_1)$. Using the imbedding Theorem for Sobolev spaces

$$W^{m,p}(B^n_1) \subset W^{r,s}(B^n_1) \quad \text{if } m \geq r, \ p \geq s \text{ and } m - \frac{n}{p} = r - \frac{n}{s}.$$ 

In particular, $W^{2, \frac{2n}{n+2}}(B^n_1)$ is continuously imbedded in $W^{1,2}(B^n_1)$. On the other hand by Proposition 4 in [14], we have

$$W^{1,2}(B^n_1) \hookrightarrow L^{2,2}(B^n_1) = L^{\frac{2n}{n+2},2}(B^n_1)$$

continuously. We then deduce that

$$\Delta u_k = -u_k|u_k|^{4/(n-2)}$$

is bounded in $L^{\frac{2n}{(n-2)+2}}(B^n_1)$. Here, we have used the following lemma.

**Lemma 3.1.** If $f \in L^{p,q}(B^n_1)$ and $\alpha \in \mathbb{Q}^+$, then $f^\alpha \in L^{\frac{pq}{p+q}}(B^n_1)$.

**Proof.** In the case where $\alpha \in \mathbb{N}$, the result follows from the fact that

$$f \in L^{a,b}(B^n_1) \text{ and } g \in L^{c,d}(B^n_1) \Rightarrow f.g \in L^{a,c}(B^n_1),$$

where $\frac{1}{q} = \frac{1}{a} + \frac{1}{b}$ and $\frac{1}{r} = \frac{1}{c} + \frac{1}{d}$ (see [2]). The general case is a consequence of the fact that the increasing rearrangement of the function $|f|^{\beta}$ is equal to the puissance $\beta$ of the increasing rearrangement of $|f|$ since $(f^{\beta})^\ast$ is the only one function verifying

$$\text{meas}\{x \in \mathbb{R}^n : f^\beta(x) \geq \lambda\} = \text{meas}\{t > 0 : (f^{\beta})^\ast(x) \geq \lambda\}$$

This in turns proves Lemma 3.1. \qed

Now, using in [14] Theorem 8, we deduce from (3.7) that $(\nabla u_k)$ is uniformly bounded in the space $L^{\left(\frac{2n}{n+2}\right)^2, \frac{2(n-2)}{n+2}}(B^n_1) = L^{\frac{2n}{n+2}, \frac{2n}{n+2}}(B^n_1)$. Hence $(u_k)$ is bounded in $L^{2, \frac{2n}{(n+2)^2}}(B^n_1)$. Then

$$\Delta u_k = -u_k|u_k|^{4/(n-2)}$$

is bounded in $L^{\frac{2n}{(n-2)+2}, \frac{2(n-2)^2}{(n+2)^2}}(B^n_1)$. Hence, again by [14] Theorem 8], the sequence $(\nabla u_k)$ is bounded in $L^\alpha(\frac{2n}{(n+2)^2}, \frac{2n}{(n+2)^2})(B^n_1)$ and by elliptic regularity Theorem

$$\Delta u_k = -u_k|u_k|^{4/(n-2)}$$

is bounded in $L^{\frac{2n}{(n-2)+2}, \frac{2(n-2)^3}{(n+2)^3}}(B^n_1)$. We obtain after $p$ iterations that

$$\Delta u_k = -u_k|u_k|^{4/(n-2)}$$

is bounded in $L^{\frac{2n}{(n-2)+2}, \frac{2(n-2)^p}{(n+2)^p}}(B^n_1)$. We choose $p > 0$ such that $6p > n$, we have in particular $\frac{2(n-2)^p}{(n+2)^p} < 1$ which gives

$$\Delta u_k = -u_k|u_k|^{4/(n-2)}$$

is bounded in $L^{\frac{2n}{(n-2)+2}, \frac{1}{(n+2)^3}}(B^n_1)$. Here we have used the fact that

$$L^{p,q_1}(B^n_1) \subset L^{p,q_2}(B^n_1) \quad \text{if } q_1 < q_2$$
We deduce from (3.10), (3.11) together with the $L^1 - L^\infty$ duality that
\[ \|\nabla u_k\|_{L^2,1(B^n_1)} \leq C \]
(3.11)
We deduce from (3.10), (3.11) together with the $L^2,1 - L^2,\infty$ duality that
\[ \|\nabla u_k\|_{L^2(B^n_1 \setminus B_{R\delta_k})} \leq \|\nabla u_k\|_{L^2,1(B^n_1 \setminus B_{R\delta_k})} \|\nabla u_k\|_{L^2,\infty(B^n_1 \setminus B_{R\delta_k})} \leq C(\varepsilon) \]
for a constant $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now, we use the embedding $H^1 \hookrightarrow L^{2n/(n-2)}$ continuously, we obtain
\[ \|u_k\|_{L^{2n/(n-2)}(B^n_1 \setminus B_{R\delta_k})} \leq C(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0. \]
We deduce that
\[ \lim_{R \to \infty} \lim_{k \to \infty} \int_{B^n_1 \setminus B_{R\delta_k}(y_k)} (|u_k|^{2n/(n-2)} + |\nabla u_k|^2) \, dx = 0 \]
This proves Theorem 1.5 in the case of one bubble.

The case of more than one bubble can be handled in a very similar way and we just give few details for $m = 2$. The proof starts the same until (3.4) which cannot hold any more otherwise we would have had one bubble only as it is (3.4) holds. It remains to show that: for any $\varepsilon \geq 0$, there are sufficiently large $R > 0$ and a sequence $r_i \to 0$ such that for any $R\delta_i \leq r_i \leq 1/2$,
\[ \lim_{R \to \infty} \lim_{i \to \infty} \int_{(0) \times B^n_1 \setminus B_{R\delta_i}(0)} \left( \frac{1}{2} |\nabla v_i|^2 + \frac{n-2}{2n} |v_i|^{2n/(n-2)} \right) \, dx = 0, \]
\[ \lim_{i \to \infty} \int_{(0) \times B^n_{1/2} \setminus B^n_1(0)} \left( \frac{1}{2} |\nabla v_i|^2 + \frac{n-2}{2n} |v_i|^{2n/(n-2)} \right) \, dx = 0 \]
(3.12)
where $v_i$ is defined by $v_i(y) = r_i^{(n-2)/2} u_i(r_i y), \ y \in \mathbb{R}^n$.

The proof of (3.12) can be done exactly as the proof of (3.4), the case of 2 bubbles is then proved. To prove the general case, for any number $m \geq 2$, one can follow exactly the same strategy.

 References


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