MULTIPLICITY OF SOLUTIONS FOR A CLASS OF ELLIPTIC SYSTEMS IN $\mathbb{R}^N$

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Abstract. This article concerns the multiplicity of solutions for the system of equations

$$
-\Delta u + V(\epsilon x)u = \alpha |u|^{\alpha-2}u|v|^\beta,
-\Delta v + V(\epsilon x)v = \beta |u|^{\alpha}|v|^{\beta-2}v
$$

in $\mathbb{R}^N$, where $V$ is a positive potential. We relate the number of solutions with the topology of the set where $V$ attains its minimum. The results are proved by using minimax theorems and Ljusternik-Schnirelmann theory.

1. Introduction

The purpose of this article is to investigate the multiplicity of solutions for the system

$$
-\Delta u + V(\epsilon x)u = \alpha |u|^{\alpha-2}u|v|^\beta \quad \text{in} \quad \mathbb{R}^N,
-\Delta v + V(\epsilon x)v = \beta |u|^{\alpha}|v|^{\beta-2}v \quad \text{in} \quad \mathbb{R}^N,
$$

(1.1)

$$
u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \quad \text{for all} \quad x \in \mathbb{R}^N,
$$

where $\epsilon > 0$, $\alpha, \beta > 1$ such that $\alpha + \beta = p$, $2 < p < 2N/(N-2)$, $N \geq 3$ and the potential $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and satisfies

$$
0 < V_0 := \inf_{x \in \mathbb{R}^N} V(x) < V_\infty := \liminf_{|x| \to \infty} V(x). \tag{1.2}
$$

In this work, we will consider the cases $V_\infty < \infty$ or $V_\infty = \infty$. This kind of hypothesis was introduced by Rabinowitz [16] in the study of a nonlinear Schrödinger equation.

We say that $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ is a weak solution of the system in (1.1) if

$$
\int_{\mathbb{R}^N} \left[ \nabla u \nabla \phi + \nabla v \nabla \psi + V(\epsilon x)(u\phi + v\psi) \right] = \int_{\mathbb{R}^N} \left[ \alpha |u|^{\alpha-2}u|v|^\beta \phi + \beta |u|^{\alpha}|v|^{\beta-2}v \psi \right]
$$

for all $(\phi, \psi) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. 

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In this paper we also relate the number of solutions of (1.1) with the topology of the set of minima of the potential \( V \). In order to present our result we introduce the set of global minima of \( V \), given by

\[
M = \{ x \in \mathbb{R}^N : V(x) = V_0 \}.
\]

Note that, by (1.2), \( M \) is compact. For any \( \delta > 0 \), let \( M_\delta = \{ x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta \} \) be the closed \( \delta \)-neighborhood of \( M \). Our main result is as follows.

**Theorem 1.1.** Suppose that \( V \) satisfies \( (1.2) \). Then, for any \( \delta > 0 \) given, there exists \( \epsilon_\delta > 0 \) such that, for any \( \epsilon \in (0, \epsilon_\delta) \), the system (1.1) has at least \( \text{cat}_{M_\delta}(M) \) solutions.

We recall that, if \( Y \) is a closed set of a topological space \( X \), \( \text{cat}_X(Y) \) is the Ljusternik-Schnirelmann category of \( Y \) in \( X \), namely the least number of closed and contractible set in \( X \) which cover \( Y \).

Existence and concentration of positive solutions for the problem

\[
-\epsilon^2 \Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N
\]

have been extensively studied in recent years, see for example, Ambrosetti, Badiale and Cingolani [4], Del Pino & Felmer [8], Floer [9], Lazzo [11], Oh [13, 14, 15], Rabinowitz [16], Wang [17] and their references.

Cingolani and Lazzo in [6] studied positive solutions for the Schrödinger equation (1.3) with \( f(u) = |u|^q - 2u \), \( \epsilon > 0 \), \( 2 < q < 2^* \), \( V \) satisfying (1.2) and proved a multiplicity result similar to Theorem 1.1. Alves and Monari in [3], proved only the existence and concentration of a nontrivial solutions \((u, v)\) to problem (1.1).

In this work, motivated by [6], [3], and using some recent ideas from [2] and [10], we prove the multiplicity of solutions to (1.1). Our main result completes the study made in [6] in the following sense: We are working with a system of equations and here, in the proof of some lemmas and propositions, we use different arguments than those in [6], for example the proposition 3.6, Lemma 4.4 for appearing along the text and we prove a compactness result on Nehari manifolds. Moreover, we do not know if the problem below has a unique positive solution,

\[
\begin{align*}
-\Delta u + \mu u &= \alpha |u|^{\alpha-2}u + \beta |v|^{\beta-2}v \quad \text{in } \mathbb{R}^N, \\
-\Delta v + \mu v &= \alpha |u|^{\alpha} |v|^{\beta-2}v \quad \text{in } \mathbb{R}^N,
\end{align*}
\]

\( u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \quad \text{for all } x \in \mathbb{R}^N, \mu > 0. \)

This fact is used in a lot of papers in the scalar case.

The paper is organized as follows: In Section 2 we present the abstract framework of the problem as well as some remarks on the autonomous problem. In Section 3 we obtain some compactness properties of the functional associated to the system (1.1). Theorem 1.1 is proved in Section 4.

2. The variational framework

Throughout this paper we suppose that the function \( V \) satisfies the conditions (1.2). We write only \( \int u \) instead of \( \int_{\mathbb{R}^N} u(x)dx \). For any \( \epsilon > 0 \), we denote by \( X_\epsilon \) the Sobolev space

\[
X_\epsilon = \{(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int V(\epsilon x)(|u|^2 + |v|^2) < \infty \}
\]
endowed with the norm

\[\|(u,v)\|_\epsilon^2 = \int (|\nabla u|^2 + |\nabla v|^2) + \int V(\epsilon x)(|u|^2 + |v|^2),\]

We will look for solutions of (1.1) by finding critical points of the \(C^2\)-functional \(I_\epsilon : X_\epsilon \to \mathbb{R}\) given by

\[I_\epsilon((u,v)) = \frac{1}{2} \int [||\nabla u||^2 + ||\nabla v||^2 + V(\epsilon x)(|u|^2 + |v|^2)] - \int Q(u,v),\]

where \(Q(u,v) = (u^+)^\alpha (v^+)^\beta\) and \(w^\pm = \max\{\pm w,0\}\) the positive (negative) part of \(w\). By definition of \(Q\), we see that, if \((u,v)\) is a nontrivial critical point of \(I_\epsilon\), then \(u,v\) are positive in \(\mathbb{R}^N\). Indeed, since that

\[\langle I'_\epsilon((u,v)),(\phi,\psi)\rangle = \int [\nabla u \nabla \phi + \nabla v \nabla \psi + V(\epsilon x)(u\phi + v\psi)] - \alpha \int |u|^\alpha - 2 |u|^\beta \phi - \beta \int |v|^\beta - 2 |v|\psi,\]

we have

\[0 = \langle I'_\epsilon((u,v)),(u^-,v^-)\rangle = \|(u^-,v^-)\|_\epsilon^2\]

and therefore \(u,v \geq 0\) in \(\mathbb{R}^N\). By the Maximum Principle in \(\mathbb{R}^N\), \(u,v > 0\) in \(\mathbb{R}^N\).

We introduce the Nehari manifold of \(I_\epsilon\) by setting

\[\mathcal{N}_\epsilon = \{(u,v) \in X_\epsilon \setminus \{(0,0)\} : \langle I'_\epsilon((u,v)),(u,v)\rangle = 0\} .\]

Note that, if \((u,v) \in \mathcal{N}_\epsilon\), we have

\[I_\epsilon((u,v)) = \frac{1}{2} \|(u,v)\|_\epsilon^2 - \int Q(u,v) = \left(\frac{1}{2} - \frac{1}{p}\right) \|(u,v)\|_\epsilon^2 \geq 0,\]

and therefore the following minimization problem is well defined

\[c_\epsilon = \inf_{(u,v) \in \mathcal{N}_\epsilon} I_\epsilon((u,v)).\]

Moreover, we can easily conclude that there exists \(r > 0\), independent of \(\epsilon\), such that

\[\|(u,v)\|_\epsilon \geq r > 0 \text{ for any } \epsilon > 0, (u,v) \in \mathcal{N}_\epsilon. \tag{2.1}\]

We now present some important properties of \(c_\epsilon\) and \(\mathcal{N}_\epsilon\). The proofs can be adapted from [16, Chapter 4] (see also [12, Lemmas 3.1 and 3.2]). First we observe that, for any \((u,v) \in X_\epsilon \setminus \{(0,0)\}\) there exists a unique \(t_{u,v} > 0\) such that \(t_{u,v}(u,v) \in \mathcal{N}_\epsilon\). The maximum of the function \(t \mapsto I_\epsilon(t(u,v))\) for \(t \geq 0\) is achieved at \(t = t_{u,v}\) and the function \((u,v) \mapsto t_{u,v}\) is continuous from \(X_\epsilon \setminus \{(0,0)\}\) to \((0,\infty)\). Note that by conditions on \(\alpha\) and \(\beta\), we have

\[Q(u,v) \leq \frac{\alpha}{p} |u|^p + \frac{\beta}{p} |v|^p. \tag{2.2}\]

Standard calculations imply that \(I_\epsilon\) satisfies the geometry of the Mountain Pass theorem. Arguing as in [16, Theorem 4.2] we can prove that \(c_\epsilon\) is positive, it coincides with the mountain pass level of \(I_\epsilon\) and satisfies

\[c_\epsilon = \inf_{\gamma \in \Gamma_\epsilon} \max_{t \in [0,1]} I_\epsilon(\gamma(t)) = \inf_{(u,v) \in X_\epsilon \setminus \{(0,0)\}} \max_{t \geq 0} I_\epsilon(t(u,v)) > 0, \tag{2.3}\]

where \(\Gamma_\epsilon = \{\gamma \in C([0,1], X_\epsilon) : \gamma(0) = (0,0), I_\epsilon(\gamma(1)) < 0\}\).
We will denote by $\|I'(u,v)\|_*$ the norm of the derivative of $I_\varepsilon$ restricted to $\mathcal{N}_\varepsilon$ at the point $(u,v)$. This norm is given by (see [18, Proposition 5.12])

$$\|I'_\varepsilon((u,v))\|_* = \min_{\lambda \in \mathbb{R}} \|I'_\varepsilon((u,v)) - \lambda J'_\varepsilon((u,v))\|_{X_\varepsilon^*},$$

where $X_\varepsilon^*$ denotes the dual space of $X_\varepsilon$ and $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ is defined as

$$J_\varepsilon((u,v)) = \|(u,v)\|^2_2 - \varepsilon \int Q(u,v). \quad (2.4)$$

As we will see, it is important to compare $c_\varepsilon$ with the minimax level of the autonomous problem \eqref{1.4}. The solutions of \eqref{1.4} are precisely the positive critical points of the functional $E_\mu : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$E_\mu((u,v)) = \frac{1}{2} \int (|\nabla u|^2 + |\nabla v|^2) + \frac{1}{2} \int \mu |u|^2 + |v|^2 - \varepsilon \int Q(u,v).$$

We also define the autonomous minimization problem

$$m(\mu) = \inf_{(u,v) \in \mathcal{M}_\mu} E_\mu((u,v)),$$

where $\mathcal{M}_\mu$ is the Nehari manifold of $E_\mu$, that is

$$\mathcal{M}_\mu = \{(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \setminus \{(0,0)\} : \langle I'_\varepsilon((u,v)), (u,v) \rangle = 0\}.$$ 

The number $m(\mu)$ and the manifold $\mathcal{M}_\mu$ have properties similar to those of $c_\varepsilon$ and $\mathcal{N}_\varepsilon$. Moreover, Alves and Monari in [3, Theorem 4.11] showed that $m(\mu)$ is attained by a solution $(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ of the problem \eqref{1.4}.

### 3. A compactness condition

In this section we obtain some compactness properties of the functional $I_\varepsilon$. We start by recalling the definition of the Palais-Smale condition. So, let $E$ be a Banach space, $\mathcal{V}$ be a $C^1$-manifold of $E$ and $I : E \to \mathbb{R}$ a $C^1$-functional. We say that $I|_{\mathcal{V}}$ satisfies the Palais-Smale condition at level $c$ ((PS)$_c$) if any sequence $(u_n) \subset \mathcal{V}$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_* \to 0$ contains a convergent subsequence.

The next lemma shows a property involving (PS)$_c$ sequences for $I_\varepsilon$. Its proof uses well-know arguments and will be omitted.

**Lemma 3.1.** Let $((u_n,v_n)) \subset X_\varepsilon$ be a (PS)$_c$ sequence for $I_\varepsilon$. Then

(i) $((u_n,v_n))$ is bounded in $X_\varepsilon$,

(ii) there exists $(u,v) \in X_\varepsilon$ such that, up to a subsequence, $(u_n,v_n) \rightharpoonup (u,v)$ weakly in $X_\varepsilon$ and $I'_\varepsilon((u,v)) = 0$,

(iii) $((u_n,v_n^+))$ is also a (PS)$_c$ sequence for $I_\varepsilon$.

Moreover, the same holds if we replace $I_\varepsilon$ and $X_\varepsilon$ which $E_\mu$ and $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, respectively.

**Remark 3.2.** Let $((u_n,v_n))$ be a Palais-Smale sequence for $I_\varepsilon$ (or $E_\mu$). Since we are always interested in the existence of convergent subsequences, we may use the above lemma to suppose that $u_n \geq 0$ and $v_n \geq 0$ for all $n \in \mathbb{N}$. This will be made from now on.

**Lemma 3.3.** Let $((u_n,v_n)) \subset X_\varepsilon$ be a (PS)$_d$ sequence for $I_\varepsilon$. Then we have either

(i) $\|(u_n,v_n)\|_{C^1} \to 0$, or
(ii) there exist a sequence \((y_n) \subset \mathbb{R}^N\) and constants \(R, \gamma > 0\) such that
\[
\liminf_{n \to \infty} \int_{B_R(y_n)} (u_n^2 + v_n^2) \geq \gamma > 0.
\]

The above lemma follows by adapting the arguments of [3, page 171] (see also [2, Theorem 2.1]).

**Remark 3.4.** For future reference we note that, if \(\epsilon_n \to 0\) and \(\((u_n, v_n)\) \subset \mathcal{N}_{\epsilon_n}\) is a bounded sequence in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\) such that \(L_{\epsilon_n}((u_n, v_n)) \to d\), then we can argue along the same lines of the above proof to conclude that either \(\|(u_n, v_n)\|_{\epsilon_n} \to 0\) or (ii) holds. We also have a similar result if \(\((u_n, v_n)\) \subset H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\) is a \((PS)_d\) sequence for the autonomous functional \(E_n\).

**Lemma 3.5.** Consider \(V_\infty < \infty\) and let \(\((u_n, v_n)\) \subset X\) be a \((PS)_d\) sequence for \(I\) such that \((u_n, v_n) \rightharpoonup (0, 0)\) weakly in \(X\). If \((u_n, v_n) \not\to (0, 0)\) in \(X\), then \(d \geq m(V_\infty)\).

**Proof.** Let \((t_n) \subset (0, +\infty)\) be such that \((t_n(u_n, v_n)) \subset \mathcal{M}_{V_\infty}\). We start by proving that \(\limsup_{n \to \infty} t_n \leq 1\). Arguing by contradiction, we suppose that there exist \(\lambda > 0\) and a subsequence, which we also denote by \((t_n)\), such that
\[
t_n \geq 1 + \lambda \quad \text{for all } n \in \mathbb{N}.
\]

Since \(\((u_n, v_n)\)\) is bounded in \(X\), \(\{(I'((u_n, v_n)), (u_n, v_n))\} \to 0\), that is,
\[
\int |\nabla u_n|^2 + |\nabla v_n|^2 + V(\epsilon x)(|u_n|^2 + |v_n|^2) = p \int Q(u_n, v_n) + o_n(1).
\]

Moreover, recalling that \((t_n(u_n, v_n)) \subset \mathcal{M}_{V_\infty}\), we get
\[
\int |\nabla u_n|^2 + |\nabla v_n|^2 + V_\infty(|u_n|^2 + |v_n|^2) = p(t_n^{p-2} - 1) \int Q(u_n, v_n).
\]

These two equalities imply
\[
p(\frac{p}{n} - 1) \int Q(u_n, v_n) = \int [V_\infty - V(\epsilon x)](|u_n|^2 + |v_n|^2) + o_n(1).
\]

Using the condition \([1.2]\), we have that given \(\delta > 0\), there exists \(R > 0\) such that
\[
V(\epsilon x) \geq V_\infty - \delta \quad \text{for any } |x| \geq R.
\]

Let \(C > 0\) be such that \(\|(u_n, v_n)\|_\epsilon \leq C\). Since \(\|(u_n, v_n)\|_\epsilon \to 0\) in \(H^1(B_R(0)) \times H^1(B_R(0))\) we can use \([3.2]\) and \([3.3]\) to obtain
\[
p(\frac{p}{n} - 1) \int Q(u_n, v_n) \leq \delta C + o_n.
\]

for any \(\delta > 0\). Since \((u_n, v_n) \not\to (0, 0)\), we may invoke Lemma \([3.3]\) to obtain \((y_n) \subset \mathbb{R}^N\) and \(R, \gamma > 0\) such that
\[
\int_{B_R(y_n)} (u_n^2 + v_n^2) \geq \gamma > 0.
\]

If we define \((\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x + y_n), v_n(x + y_n))\) we may suppose that, up to a subsequence,
\[
(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (u, v) \quad \text{weakly in } X,
\]
\[
(\tilde{u}_n, \tilde{v}_n) \to (u, v) \quad \text{in } L^p(B_R(0)) \times L^p(B_R(0)),
\]
\[
(\tilde{u}_n(x), \tilde{v}_n(x)) \to (u(x), v(x)) \quad \text{for a.e. } x \in \mathbb{R}^N,
\]
for some nonnegative functions \( u, v \). Moreover, in view of (3.5), there exists a subset \( \Omega \subset \mathbb{R}^N \) with positive measure such that \( u, v \) are strictly positive in \( \Omega \).

We can use (3.1) to rewrite (3.4) as
\[
0 < p((1 + \lambda)^{p - 2} - 1) \int_{\Omega} |\tilde{u}_n|^\alpha |\tilde{v}_n|^{\beta} \leq \delta C, \quad \forall \delta > 0.
\]
for any \( \delta > 0 \). Letting \( n \to \infty \), using Fatou’s lemma, we obtain
\[
0 < p((1 + \lambda)^{p - 2} - 1) \int_{\Omega} |u|^\alpha |v|^{\beta} \leq \delta C.
\]
for any \( \delta > 0 \). We obtain a contradiction by taking \( \delta \to 0 \). Thus, \( \limsup_{n \to \infty} t_n \leq 1 \), as claimed.

Setting \( t_0 = \limsup_{n \to \infty} t_n \), we consider two complementary cases:

**Case 1**: \( t_0 < 1 \). In this case we may suppose, without loss of generality, that \( t_n < 1 \) for all \( n \in \mathbb{N} \). Thus,
\[
m(V_\infty) \leq E_{V_\infty}(t_n(u_n, v_n)) - \frac{1}{2} E_{V_\infty}((t_n(u_n, v_n))(t_n(u_n, v_n)))
\]
\[
= \frac{p}{2} - 1) \int Q(u_n, v_n) \leq (\frac{p}{2} - 1) \int Q(u_n, v_n)
\]
\[
= I_c((u_n, v_n)) - \frac{1}{2} I_c^*(u_n, v_n), (u_n, v_n)
\]
\[
= d + o_n(1).
\]

Taking the limit we conclude that \( d \geq m(V_\infty) \).

**Case 2**: \( t_0 = 1 \). Up to a subsequence, we may suppose that \( t_n \to 1 \). We first note that
\[
d + o_n(1) \geq m(V_\infty) + I_c((u_n, v_n)) - E_{V_\infty}(t_n(u_n, v_n)).
\]

Note that
\[
I_c((u_n, v_n)) - E_{V_\infty}(t_n(u_n, v_n))
\]
\[
= \int \frac{(1 - t_n^2)}{2} (|\nabla u_n|^2 + |\nabla v_n|^2) + \frac{1}{2} \int V(\varepsilon x)(|u_n|^2 + |v_n|^2)
\]
\[
- \frac{t_n^2}{2} \int V_{\infty}(|u_n|^p + |v_n|^p) - (1 - t_n^2) \int Q(u_n, v_n).
\]

Since \( \|u_n, v_n\|_1 \) is bounded, we have
\[
\int \frac{(1 - t_n^2)}{2} (|\nabla u_n|^2 + |\nabla v_n|^2) = o_n(1),
\]
\[
(1 - t_n^2) \int Q(u_n, v_n) = o_n(1).
\]

Using the condition (1.2), we obtain
\[
d + o_n(1) \geq m(V_\infty) - \delta C + o_n(1),
\]
for any \( \delta > 0 \). By taking \( n \to \infty \) and \( \delta \to 0 \), we conclude that \( d \geq m(V_\infty) \).

We present below two compactness results which we will need for the proof of the main theorem.

**Proposition 3.6.** The functional \( I_c \) satisfies the \((PS)\) condition at any level \( c < m(V_\infty) \) if \( V_\infty < \infty \) and at any level \( c \in \mathbb{R} \) if \( V_\infty = \infty \).
Proof. Let \((u_n, v_n)\) be such that \(I_e((u_n, v_n)) \to c\) and \(I'_e((u_n, v_n)) \to 0\) in \(X_e^*\). By Lemma 3.1 the weak limit \((u, v)\) of \((u_n, v_n)\) is such that \(I'_e((u, v)) = 0\). Thus,
\[
I_e(u, v) = I_e(u, v) - \frac{1}{2} I'_e((u, v))(u, v) = \left(\frac{p}{2} - 1\right) \int Q(u, v) \geq 0.
\]
Let \(\tilde{u}_n = u_n - u\) and \(\tilde{v}_n = v_n - v\). Arguing as in [1] Lemma 3.3 we can show that \(I'_e(\tilde{u}_n, \tilde{v}_n) \to 0\) and
\[
I_e((\tilde{u}_n, \tilde{v}_n)) = c - I_e((u, v)) = d < m(V_\infty),
\]
where we used that \(c < m(V_\infty)\) and \(I_e((u, v)) \geq 0\). Since \((\tilde{u}_n, \tilde{v}_n) \to (0, 0)\) weakly in \(X_e\) and \(d < m(V_\infty)\), it follows from Lemma 3.5 that \((\tilde{u}_n, \tilde{v}_n) \to (0, 0)\) in \(X_e\), i.e., \((u_n, v_n) \to (u, v)\) in \(X_e\).

The case \(V_\infty = \infty\) follows from [7] Proposition 2.4. This concludes the proof of the proposition.

\begin{proposition}
Proposition 3.7. The functional \(I_e\) restricted to \(N_e\) satisfies the \((PS)_c\) condition at any level \(c < m(V_\infty)\) if \(V_\infty < \infty\) and at any level \(c \in \mathbb{R}\) if \(V_\infty = \infty\).
\end{proposition}

Proof. Let \((u_n, v_n) \in N_e\) be such that \(I_e((u_n, v_n)) \to c\) and \(\|I'_e((u_n, v_n))\|_\ast \to 0\). Then there exists \((\lambda_n) \subset \mathbb{R}\) such that
\[
I'_e((u_n, v_n)) = \lambda_n J'_e((u_n, v_n)) + o_n(1),
\]
where \(J_e\) was defined in (2.4). Thus
\[
0 = I'_e((u_n, v_n))(u_n, v_n) = \lambda_n J'_e((u_n, v_n))(u_n, v_n) + o_n(1).
\]
Since
\[
J'_e((u_n, v_n))(u_n, v_n) = (2 - p)\|u_n, v_n\|_e^2 < 0,
\]
and \(\|u_n, v_n\|_e^2 \to 0\) by (2.1), we have \(\lambda_n = o_n(1)\). By using (3.6), we conclude that \(I'_e((u_n, v_n)) \to 0\) in \(X_e\), that is, \((u_n, v_n)\) is a \((PS)_c\) sequence for \(I_e\). The result follows from Proposition 3.6.

\begin{corollary}
Corollary 3.8. The critical points of functional \(I_e\) on \(N_e\) are critical points of \(I_e\) in \(X_e\).
\end{corollary}

The proof of the above corollary follows by using similar arguments explored in the previous proposition.

4. Multiplicity of solutions

For any \(\mu > 0\), we denote by \(\|\cdot\|_{H_\mu}\) the following norm in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\)
\[
\|u, v\|_{H_\mu} = \left\{ \int \left[ |\nabla u|^2 + |\nabla v|^2 + \mu(|u|^2 + |v|^2) \right] \right\}^{1/2}
\]
which is well defined and equivalent to the standard norm of \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\).

Let \((w_1, w_2)\) be a ground state solution of the problem \((AP_{\eta})\) and consider \(\eta : [0, \infty) \to \mathbb{R}\) a cut-off function such that \(0 \leq \eta \leq 1\), \(\eta(s) = 1\) if \(0 \leq s \leq \delta/2\) and \(\eta(s) = 0\) if \(s \geq \delta\). We recall that \(M\) denotes the set of global minima points of \(V\) and define, for each \(y \in M\), \(\Psi_{\eta, \epsilon, y} : \mathbb{R}^N \to \mathbb{R}\) by setting
\[
\Psi_{\eta, \epsilon, y}(x) = \eta(|x - y|)w_i\left(\frac{\epsilon x - y}{\epsilon}\right), \quad i = 1, 2.
\]
Let $t_{\epsilon}$ be the unique positive number satisfying
\[
\max_{t \geq 0} I_{\epsilon}(t(\psi_{1,\epsilon,y}, \psi_{2,\epsilon,y})) = I_{\epsilon}(t_{\epsilon}(\psi_{1,\epsilon,y}, \psi_{2,\epsilon,y})),
\]
and define the map $\Phi_{\epsilon} : M \rightarrow N_{\epsilon}$ in the following way
\[
\Phi_{\epsilon}(y) = \Phi_{\epsilon,y} = (t_{\epsilon}(\psi_{1,\epsilon,y}, \psi_{2,\epsilon,y})).
\] (4.1)

In view of the definition of $t_{\epsilon}$ we have that the above map is well defined. Moreover, the following holds.

**Lemma 4.1.** $\lim_{\epsilon \to 0} I_{\epsilon}(\Phi_{\epsilon,y}) = m(V_0)$, uniformly in $y \in M$.

**Proof.** Suppose, by contradiction, that the lemma is false. Then there exist $\lambda > 0, (y_n) \subset M$ and $\epsilon_n \to 0$ such that
\[
|I_{\epsilon_n}(\Phi_{\epsilon_n,y_n}) - m(V_0)| \geq \lambda > 0.
\] (4.2)

Since $\langle I_{\epsilon_n}(\Phi_{\epsilon_n,y_n}, \Phi_{\epsilon_n,y_n}) \rangle = 0$, we have that
\[
\|\langle \psi_{1,\epsilon_n,y_n}, \psi_{2,\epsilon_n,y_n} \rangle \| = \| (w_1, w_2) \|_{H_{V_0}}^2
\]
Moreover, making the change of variables $z = (\epsilon_n x - y_n)/\epsilon_n$ and using the Lebesgue theorem, we can check that
\[
\lim_{n \to \infty} \int M_{V_0} Q(\psi_{1,\epsilon_n,y_n}, \psi_{2,\epsilon_n,y_n}) = \int Q(w_1, w_2).
\]
Thus, up to a subsequence, we have $t_n \to t_0 > 0$ and
\[
\| (w_1, w_2) \|_{H_{V_0}}^2 = t_0^{p-2} \int Q(w_1, w_2).
\]

Since $(w_1, w_2) \in M_{V_0}$, we obtain $t_0 = 1$. Letting $n \to \infty$, we get
\[
\lim_{n \to \infty} I_{\epsilon_n}(\Phi_{\epsilon_n,y_n}) = E_{V_0}(w_1, w_2) = m(V_0),
\]
which contradicts (4.2) and proves the lemma. \qed

For any $\delta > 0$, let $\rho = \rho_{\delta} > 0$ be such that $M_{\delta} \subset B_\rho(0)$. Let $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be defined as $\chi(x) = x$ for $|x| < \rho$ and $\chi(x) = \rho x/|x|$ for $|x| \geq \rho$. Finally, let us consider the barycenter map $\beta_{\epsilon} : N_{\epsilon} \to \mathbb{R}^N$ given by
\[
\beta_{\epsilon}(u, v) = \frac{\int \chi(\epsilon x) u(x)^2}{\int |u(x)|^2} + \frac{\int \chi(\epsilon x) v(x)^2}{\int |v(x)|^2}.
\]

**Lemma 4.2.** $\lim_{\epsilon \to 0} \beta_{\epsilon}(\Phi_{\epsilon,y}) = y$ uniformly for $y \in M$.

**Proof.** Arguing by contradiction, we suppose that there exist $\lambda > 0, (y_n) \subset M$ and $\epsilon_n \to 0$ such that
\[
|\beta_{\epsilon_n}(\Phi_{\epsilon_n,y_n}) - y_n| \geq \lambda > 0.
\] (4.3)

By using the change of variables $z = (\epsilon_n x - y_n)/\epsilon_n$, we get
\[
\beta_{\epsilon}(\Phi_{\epsilon_n,y_n}) = y_n + \frac{\int [\chi(\epsilon_n z + y_n) - y_n]|\eta(\epsilon_n z)| w_1(z)^2}{\int |\eta(\epsilon_n z)| w_1(z)^2} + \frac{\int [\chi(\epsilon_n z + y_n) - y_n]|\eta(\epsilon_n z)| w_2(z)^2}{\int |\eta(\epsilon_n z)| w_2(z)^2}.
\]
Since \((y_n) \subset M \subset B_\rho(0)\) we have that \(\chi(\epsilon_n z + y_n) - y_n = o_n(1)\). Hence, by the Lebesgue theorem, we conclude that
\[\beta_{\epsilon_n}(\Phi_{\epsilon_n y_n}) - y_n = o_n(1),\]
which contradicts (4.3) and proves the lemma.

**Lemma 4.3** (A Compactness Lemma). Let \((\{u_n, v_n\}) \subset \mathcal{M}_\mu\) be a sequence satisfying \(E_\mu(u_n, v_n) \to m(\mu)\). Then,
\begin{enumerate}[(a)]
  
  \item \([u_n, v_n]\) has a subsequence strongly convergent in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\), or
  
  \item there exists a sequence \((\tilde{y}_n) \subset \mathbb{R}^N\) such that, up to a subsequence,
  \begin{align*}
    (\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x + \tilde{y}_n), v_n(x + \tilde{y}_n))
  \end{align*}
  converges strongly in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\).
\end{enumerate}

In particular, there exists a minimizer for \(m(\mu)\).

**Proof.** Applying Ekeland’s variational principle [18 Theorem 8.5], we may suppose that \((\{u_n, v_n\})\) is a \((PS)_{m(\mu)}\) sequence for \(E_\mu\). Thus, going to a subsequence if necessary, we have that \((u_n, v_n) \rightharpoonup (u, v)\) weakly in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\) with \((u, v)\) being a critical point of \(E_\mu\).

If \((u, v) \neq (0, 0)\), it is easy to check that \((u, v)\) is a ground state solution of the autonomous problem \((1.4)\), that is, \(E_\mu(u, v) = m(\mu)\).

We now consider the complementary case \((u, v) = (0, 0)\). In this case, by Remark 3.4, there exist a sequence \((\tilde{y}_n) \subset \mathbb{R}^N\) and constants \(R, \gamma > 0\) such that
\[\liminf_{n \to \infty} \int_{B_R(\tilde{y}_n)} (|u_n|^2 + |v_n|^2) \geq \gamma > 0.\]

Defining \(\tilde{u}_n(x) = u_n(x + \tilde{y}_n)\) and \(\tilde{v}_n(x) = v_n(x + \tilde{y}_n)\) we have that \((\tilde{u}_n, \tilde{v}_n)\) is also a \((PS)_{m(\mu)}\) sequence of \(E_\mu\) such that \((\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v}) \neq (0, 0)\). It follows from the first part of the proof that, up to a subsequence, \((\tilde{u}_n, \tilde{v}_n)\) converges in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\). The lemma is proved.

**Lemma 4.4.** Let \(\epsilon_n \to 0\) and \((\{u_n, v_n\}) \subset \mathcal{N}_{\epsilon_n}\) be such that \(I_{\epsilon_n}(\{u_n, v_n\}) \to m(V_0)\). Then there exists a sequence \((\tilde{y}_n) \subset \mathbb{R}^N\) such that \((\tilde{u}_n, \tilde{v}_n)(x) = (u_n(x + \tilde{y}_n), v_n(x + \tilde{y}_n))\) has a convergent subsequence in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\). Moreover, up to a subsequence, \((y_n) = (\epsilon_n \tilde{y}_n)\) is such that \(y_n \to y \in M\).

**Proof.** Arguing as in Remark 3.4 we obtain a sequence \((\tilde{y}_n) \subset \mathbb{R}^N\) such that
\[\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in} \quad H^1(\mathbb{R}^N)\]
and \(\tilde{v}_n \rightharpoonup \tilde{v} \quad \text{in} \quad H^1(\mathbb{R}^N)\),
where \(\tilde{u}_n = u_n(x + \tilde{y}_n)\) and \(\tilde{v}_n = v_n(x + \tilde{y}_n)\) with \(\tilde{u} \neq 0\) and \(\tilde{v} \neq 0\).

Let \((t_n) \subset (0, +\infty)\) be such that \((\tilde{u}_n, \tilde{v}_n) = t_n(\tilde{u}_n, \tilde{v}_n) \in \mathcal{M}_{V_0}\). Defining \(y_n = \epsilon_n \tilde{y}_n\), changing variables and recalling that \((u_n, v_n) \in \mathcal{N}_{\epsilon_n}\), we get
\begin{align*}
E_{V_0}(\tilde{u}_n, \tilde{v}_n)) \\
\leq \frac{1}{2} \int [\nabla \tilde{u}_n]^2 + [\nabla \tilde{v}_n]^2 + V(\epsilon_n x + y_n)(|\tilde{u}_n|^2 + |\tilde{v}_n|^2) - \int Q(\tilde{u}_n, \tilde{v}_n) \\
= I_{\epsilon_n}(\tilde{u}_n, \tilde{v}_n) \\
= I_{\epsilon_n}(t_n(\tilde{u}_n, \tilde{v}_n)) \\
\leq I_{\epsilon_n}(u_n, v_n) = m(V_0) + o_n(1).
\end{align*}
Hence

\[ E_{V_0}((\hat{u}_n, \hat{v}_n)) \to m(V_0). \]

Since \((t_n)\) is bounded, the sequence \((\hat{u}_n, \hat{v}_n)\) is also bounded in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\), thus for some subsequence, \((\hat{u}_n, \hat{v}_n) \to (\hat{u}, \hat{v})\) in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\). Moreover, reasoning as in [10], up to some subsequence, still denote by \((t_n)\), we can assume that \(t_n \to t_0 > 0\), and this limit implies that \((\hat{u}, \hat{v}) \neq (0,0)\). From Lemma 4.3, \((\hat{u}_n, \hat{v}_n) \to (\hat{u}, \hat{v})\) in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\) and so, \((\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}, \tilde{v})\) in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\).

To complete the proof of the lemma, it suffices to check that \((y_n) = (\epsilon_n \tilde{y}_n)\) has a subsequence such that \(y_n \to y \in M\). Indeed, suppose by contradiction that \((y_n)\) is not bounded, then there exists a subsequence, still denoted by \((y_n)\), such that \(|y_n| \to \infty\). Considering firstly the case \(V_\infty = \infty\), the inequality

\[
\int V(\epsilon_n x + y_n)(|u_n|^2 + v_n^2) \\
\leq \int (|\nabla u_n|^2 + \nabla v_n|^2) + \int V(\epsilon_n x + y_n)(|u_n|^2 + v_n^2) \\
= p \int Q(u_n, v_n),
\]

together with Fatou’s Lemma imply

\[
\infty = p \liminf_{n \to \infty} \int Q(u_n, v_n),
\]

which is an absurd, because the sequence \(Q(u_n, v_n)\) is bounded in \(L^1(\mathbb{R}^N)\).

Now, let us consider the case \(V_\infty < \infty\). Since \((\hat{u}_n, \hat{v}_n) \to (\hat{u}, \hat{v})\) in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\) and \(V_0 < V_\infty\), we have

\[
m(V_0) = \frac{1}{2} \int (|\nabla \hat{u}|^2 + \nabla |\hat{v}|^2) + \frac{1}{2} \int V_0(|\hat{u}|^2 + |\hat{v}|^2) - \int Q(\hat{u}, \hat{v}) \\
< \frac{1}{2} \int (|\nabla \tilde{u}|^2 + \nabla |\tilde{v}|^2) + \frac{1}{2} \int V_\infty(|\tilde{u}|^2 + |\tilde{v}|^2) - \int Q(\tilde{u}, \tilde{v}) \\
\leq \liminf_{n \to \infty} \left[ \frac{1}{2} \left( \int (|\nabla \tilde{u}_n|^2 + \nabla |\tilde{v}_n|^2) + V(\epsilon_n x + y_n)(|\tilde{u}_n|^2 + |\tilde{v}_n|^2) \right) \\
- \int Q(\tilde{u}_n, \tilde{v}_n) \right],
\]

or, equivalently,

\[
m(V_0) < \liminf_{n \to \infty} \left[ \frac{1}{2} \left( \int (|\nabla \tilde{u}_n|^2 + \nabla |\tilde{v}_n|^2) + V(\epsilon_n x + y_n)(|\tilde{u}_n|^2 + |\tilde{v}_n|^2) \right) \\
- \int Q(t_n \tilde{u}_n, t_n \tilde{v}_n) \right].
\]

The last inequality implies

\[
m(V_0) < \liminf_{n \to \infty} I_{\epsilon_n}((t_n u_n, t_n v_n)) \leq \liminf_{n \to \infty} I_{\epsilon_n}((u_n, v_n)) = m(V_0),
\]

which is impossible. Hence, \((y_n)\) is bounded and, up to a subsequence, \(y_n \to y \in \mathbb{R}^N\). If \(y \not\in M\), then \(V(y) > V_0\) and we obtain a contradiction arguing as above. Thus, \(y \in M\) and the lemma is proved. \(\square\)
Following [6], we introduce a subset of $\mathcal{N}_\epsilon$ which will be useful in the future. We take a function $h : [0, \infty) \to [0, \infty)$ such that $h(\epsilon) \to 0$ as $\epsilon \to 0$ and set

$$
\Sigma_\epsilon = \{(u, v) \in \mathcal{N}_\epsilon : I_\epsilon((u, v)) \leq m(V_0) + h(\epsilon)\}.
$$

Given $y \in M$, we can use Lemma 4.1 to conclude that $h(\epsilon) = |I_\epsilon(\Phi_{\epsilon,y}) - m(V_0)|$ is such that $h(\epsilon) \to 0$ as $\epsilon \to 0$. Thus, $\Phi_{\epsilon,y} \in \Sigma_\epsilon$ and we have that $\Sigma_\epsilon \neq \emptyset$ for any $\epsilon > 0$.

**Lemma 4.5.** For any $\delta > 0$ we have that

$$
\lim_{\epsilon \to 0} \sup_{(u,v) \in \Sigma_\epsilon} \text{dist}(\beta_\epsilon(u, v), M_\delta) = 0.
$$

**Proof.** Let $(\epsilon_n) \subset \mathbb{R}$ be such that $\epsilon_n \to 0$. By definition, there exists $((u_n, v_n)) \subset \Sigma_{\epsilon_n}$ such that

$$
\text{dist}(\beta_{\epsilon_n}(u_n, v_n), M_\delta) = \sup_{(u,v) \in \Sigma_{\epsilon_n}} \text{dist}(\beta_{\epsilon_n}(u, v), M_\delta) + o_n(1).
$$

Thus, it suffices to find a sequence $(y_n) \subset M_\delta$ such that

$$
|\beta_{\epsilon_n}(u_n, v_n) - y_n| = o_n(1).
$$

(4.4)

To obtain such sequence, we note that $((u_n, v_n)) \subset \Sigma_{\epsilon_n} \subset \mathcal{N}_{\epsilon_n}$. Thus, recalling that $m(V_0) \leq \epsilon_n$, we get

$$
m(V_0) \leq \epsilon_n \leq I_{\epsilon_n}((u_n, v_n)) \leq m(V_0) + h(\epsilon_n),
$$

from which follows that $I_{\epsilon_n}((u_n, v_n)) \to m(V_0)$. We may now invoke the Lemma 4.4 to obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $(y_n) = (\epsilon_n \tilde{y}_n) \subset M_\delta$ for $n$ sufficiently large. Hence,

$$
\beta_\epsilon(u_n, v_n) = y_n + \frac{\int \chi(\epsilon_n z + y_n) - y_n |\tilde{u}_n(z)|^2}{\int |\tilde{u}_n(z)|^2} + \frac{\int \chi(\epsilon_n z + y_n) - y_n |\tilde{v}_n(z)|^2}{\int |\tilde{v}_n(z)|^2},
$$

Since $\epsilon_n z + y_n \to y \in M$, we have that $\beta_{\epsilon_n}(u_n, v_n) = y_n + o_n(1)$ and therefore the sequence $(y_n)$ verifies (4.4). The lemma is proved. \(\square\)

We are now ready to present the proof of the multiplicity result and the technique used here is due to Benci and Cerami [5].

**Proof of Theorem 1.1.** Given $\delta > 0$ we can use Lemmas 4.1, 4.2, 4.3 and argue as in [6] Section 6 to obtain $\epsilon_\delta > 0$ such that, for any $\epsilon \in (0, \epsilon_\delta)$, the diagram

$$
M \xrightarrow{\Phi_\epsilon} \Sigma_\epsilon \xrightarrow{\beta_{\epsilon_\delta}} M_\delta
$$

is well defined and $\beta_\epsilon \circ \Phi_\epsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Moreover, using the definition of $\Sigma_\epsilon$ and taking $\epsilon_\delta$ small if necessary, we may suppose that $I_\epsilon$ satisfies the Palais-Smale condition in $\Sigma_\epsilon$. Standard Ljusternik-Schnirelmann theory and Corollary 3.8 provide at least $\text{cat}_{\Sigma_\epsilon}(\Sigma_\epsilon)$ solutions of the problem (1.1). The inequality

$$
\text{cat}_{\Sigma_\epsilon}(\Sigma_\epsilon) \geq \text{cat}_{\beta_{\epsilon_\delta}}(M)
$$

follows from arguments used in [3] Lemma 4.3. \(\square\)

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References


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