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POSITIVE SOLUTIONS OF FOUR-POINT BOUNDARY-VALUE PROBLEMS FOR FOUR-ORDER p-LAPLACIAN DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. We study the existence of positive solutions for the nonlinear four-point singular boundary-value problem with p-Laplacian operator on time scales. By using the fixed-point index theory, the existence of positive solution and many positive solutions for nonlinear four-point singular boundary-value problem with p-Laplacian operator are obtained.

1. Introduction

In this paper, we study the existence of positive solutions for the following nonlinear four-point singular boundary-value problem with p-Laplacian operator on time scales

$$
\begin{gathered}
(\phi_p(u^\Delta))^\nabla + g(t)f(u(t)) = 0, \quad t \in (0,T), \\
\alpha\phi_p(u(0)) - \beta\phi_p(u^\Delta(\xi)) = 0, \quad \gamma\phi_p(u(T)) + \delta\phi_p(u^\Delta(\eta)) = 0,
\end{gathered}
\tag{1.1}
$$

where $\phi_p(s)$ is p-Laplacian operator, i.e. $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = \phi_p^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$, $\xi, \eta \in (0,T)$ is prescribed and $\xi < \eta$, $g : (0,1) \to [0,\infty)$, $\alpha > 0$, $\beta \geq 0$, $\gamma > 0$, $\delta \geq 0$.

In recent years, many authors have begun to pay attention to the study of boundary-value problems on time scales. Here, two-point boundary-value problems have been extensively studied; see [1, 2, 3, 4, 8] and the references therein. However, there are not many concerning the p-Laplacian problems on time scales.

A time scale $\mathbf{T}$ is a nonempty closed subset of R. We make the blanket assumption that $0, T$ are point in $\mathbf{T}$. By an internal $(0,T)$, we always mean the intersection of the real internal $(0,T)$ with the given time scale, that is $(0,T) \cap \mathbf{T}$.

Sun and Li [9] considered the existence of positive solution of the following dynamic equations on time scales:

$$
\begin{gathered}
u^{\Delta\nabla}(t) + a(t)f(t,u(t)) = 0, \quad t \in (0,T), \\
\beta u(0) - \gamma u^\Delta(0) = 0, \quad \alpha u(\eta) = u(T),
\end{gathered}
$$

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they obtained the existence of single and multiple positive solutions of the problem by using fixed point theorem and Leggett-Williams fixed point theorem, respectively.

In the rest of the paper, we make the following assumptions:

(H1) $f \in C([0,+\infty),[0,+\infty))$;

(H2) $a(t) \in C_{ld}((0,T),[0,+\infty))$ and there exists $t_0 \in (0,T)$, such that
$$a(t_0) > 0, \quad 0 < \int_0^T a(s)\nabla s < +\infty.$$

In this paper, by constructing one integral equation which is equivalent to the problem (1.1), we research the existence of positive solutions for nonlinear singular boundary-value problem (1.1) when g and f satisfy some suitable conditions. Our main tool of this paper is the following fixed point index theory.

Theorem 1.1 ([5, 6])**.** *Suppose E is a real Banach space, $K \subset E$ is a cone, let $\Omega_r = \{u \in K : \|u\| \le r\}$. Let operator $T : K : \Omega_r \to K$ be completely continuous and satisfy $Tx \ne x$, for all $x \in \partial\Omega_r$. Then*

(i) *If $\|Tx\| \le \|x\|$, $\forall x \in \partial\Omega_r$, then $i(T,\Omega_r,K) = 1$;*

(ii) *If $\|Tx\| \ge \|x\|$, $\forall x \in \partial\Omega_r$, then $i(T,\Omega_r,K) = 0$.*

This paper is organized as follows. In section 2, we present some preliminaries and lemmas that will be used to prove our main results. In section 3, we discuss the existence of single solution of the systems (1.1). In section 4, we study the existence of at least two solutions of the systems (1.1). In section 5, we give two examples as an application.

2. Preliminaries and Lemmas

For convenience, we list here the following definitions which are needed later.

A time scale $\mathbf{T}$ is an arbitrary nonempty closed subset of real numbers R. The operators σ and ρ from $\mathbf{T}$ to $\mathbf{T}$ which is defined in [7],
$$\sigma(t) = \inf\{\tau \in \mathbf{T} : \tau > t\} \in \mathbf{T}, \quad \rho(t) = \sup\{\tau \in \mathbf{T} : \tau < t\} \in \mathbf{T}.$$
are called the forward jump operator and the backward jump operator, respectively.

The point $t \in \mathbf{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

Let $f : \mathbf{T} \to R$ and $t \in \mathbf{T}^k$ (assume t is not left-scattered if $t = \sup \mathbf{T}$), then the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that
$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le |\sigma(t) - s|, \quad \text{for all } s \in U.$$
Similarly, for $t \in \mathbf{T}$ (assume t is not right-scattered if $t = \inf \mathbf{T}$), the nabla derivative of f at the point t is defined in [1] to be the number $f^{\nabla}(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that
$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \le |\rho(t) - s|, \quad \text{for all } s \in U.$$
If $\mathbf{T} = \mathbb{R}$, then $x^{\Delta}(t) = x^{\nabla}(t) = x'(t)$. If $\mathbf{T} = Z$, then $x^{\Delta}(t) = x(t+1) - x(t)$ is the forward difference operator while $x^{\nabla}(t) = x(t) - x(t-1)$ is the backward difference operator.

A function f is left-dense continuous (i.e., ld-continuous), if f is continuous at each left-dense point in $\mathbf{T}$ and its right-sided limit exists at each right-dense point in $\mathbf{T}$. It is well-known that if f is ld-continuous.

If $F^{\nabla}(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

If $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

In the rest of this article, $\mathbf{T}$ is closed subset of $\mathbb{R}$ with $0 \in \mathbf{T}_k$, $T \in \mathbf{T}^k$. And let $E = C_{ld}([0,T],\ R)$ which is a Banach space with the maximum norm $\|u\| = \max_{t\in[0,T]} |u(t)|$. And define the cone $K \subset E$ by

$$K = \big\{u \in E : u(t) \geq 0, \quad u(t) \text{ is concave function, } t \in [0,1]\big\}.$$

We now state and prove several lemmas before stating our main results.

Lemma 2.1. *Suppose condition (H2) holds, then there exists a constant $\theta \in (0, \frac{1}{2})$ satisfies*

$$0 < \int_\theta^{T-\theta} a(t)\nabla t < \infty.$$

Furthermore, the function

$$A(t) = \int_\theta^t \phi_q\Big(\int_s^t a(t)\nabla t\Big)\Delta s + \int_t^{T-\theta} \phi_q\Big(\int_t^s a(t)\nabla t\Big)\Delta s, \quad t \in [\theta, T-\theta] \tag{2.1}$$

is positive continuous functions on $[\theta, T-\theta]$, therefore $A(t)$ has minimum on $[\theta, 1-\theta]$, hence we suppose that there exists $L > 0$ such that $A(t) \geq L$, $t \in [\theta, T-\theta]$.

Proof. At first, it is easily seen that $A(t)$ is continuous on $[\theta, T-\theta]$. Nest, let

$$A_1(t) = \int_\theta^t \phi_q\Big(\int_s^t a(s_1)\nabla s_1\Big)\Delta s, \quad A_2(t) = \int_t^{T-\theta} \phi_q\Big(\int_t^s a(s_1)\nabla s_1\Big)\Delta s.$$

Then, from condition (H2), we have the function $A_1(t)$ is strictly monotone nondecreasing on $[\theta, 1-\theta]$ and $A_1(\theta) = 0$, the function $A_2(t)$ is strictly monotone nonincreasing on $[\theta, T-\theta]$ and $A_2(T-\theta) = 0$, which implies $L = \min_{t\in[\theta,T-\theta]} A(t) > 0$. The proof is complete. $\square$

Lemma 2.2. *Let $u \in K$ and θ of Lemma 2.1, then*

$$u(t) \geq \theta\|u\|, \quad t \in [\theta, T-\theta]. \tag{2.2}$$

Proof. Suppose $\tau = \inf\big\{\xi \in [0,T] : \sup_{t\in[0,T]} u(t) = u(\xi)\big\}$. we shall discuss it from three perspectives.

(i) $\tau \in [0,\theta]$. It follows from the concavity of $u(t)$ that each point on chord between $(\tau, u(\tau))$ and $(T, u(T))$ is below the graph of $u(t)$, thus

$$u(t) \geq u(\tau) + \frac{u(T) - u(\tau)}{T - \tau}(t - \tau), \quad t \in [\theta, T-\theta],$$

then

$$\begin{aligned} u(t) &\geq \min_{t\in[\theta,T-\theta]} \big[u(\tau) + \frac{u(T)-u(\tau)}{T-\tau}(t-\tau)\big] \\ &= u(\tau) + \frac{u(T)-u(\tau)}{T-\tau}(T-\theta-\tau) \\ &= \frac{T-\theta-\tau}{T-\tau}u(T) + \frac{\theta}{T-\tau}u(\tau) \geq \theta u(\tau), \end{aligned}$$

this implies $u(t) \geq \theta\|u\|$, $t \in [\theta, T-\theta]$.

(ii) $\tau \in [\theta, T-\theta]$. If $t \in [\theta, \tau]$, similarly, we have

$$u(t) \geq u(\tau) + \frac{u(\tau)-u(0)}{\tau}(t-\tau), \quad t \in [\theta, \tau],$$

then

$$\begin{aligned} u(t) &\geq \min_{t\in[\theta,1-\theta]} \big[u(\tau) + \frac{u(\tau)-u(0)}{\tau}(t-\tau)\big] \\ &= \frac{\theta}{\tau}u(\tau) + \frac{\tau-\theta}{\tau}u(0) \geq \theta u(\tau), \end{aligned}$$

If $t \in [\tau, T-\theta]$, similarly, we have

$$u(t) \geq u(\tau) + \frac{u(T)-u(\tau)}{T-\tau}(t-\tau), \quad t \in [\tau, T-\theta],$$

then

$$\begin{aligned} u(t) &\geq \min_{t\in[\theta,T-\theta]} \big[u(\tau) + \frac{u(T)-u(\tau)}{T-\tau}(t-\tau)\big] \\ &= \frac{\theta}{T-\tau}u(\tau) + \frac{T-\theta-\tau}{T-\tau}u(T) \geq \theta u(\tau), \end{aligned}$$

this implies $u(t) \geq \theta\|u\|$, $t \in [\theta, 1-\theta]$.

(iii) $\tau \in [T-\theta, T]$. similarly, we have

$$u(t) \geq u(\tau) + \frac{u(\tau)-u(0)}{\tau}(t-\tau), \quad t \in [\theta, T-\theta],$$

then

$$u(t) \geq \min_{t\in[\theta,1-\theta]} \big[u(\tau) + \frac{u(\tau)-u(0)}{\tau}(t-\tau)\big] = \frac{\theta}{\tau}u(\tau) + \frac{\tau-\theta}{\tau}u(0) \geq \theta u(\tau),$$

this implies $u(t) \geq \theta\|u\|$, $t \in [\theta, T-\theta]$. From the above, we know $u(t) \geq \theta\|u\|$, $t \in [\theta, T-\theta]$. The proof is complete. $\square$

Lemma 2.3. *Suppose that conditions (H1), (H2) hold. Then $u(t)$ is a solution of boundary-value problems* (1.1) *if and only if $u(t) \in E$ is a solution of the integral equation*

$$u(t) = \begin{cases} \phi_q\big(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(u(r))\nabla r\big) + \int_0^t \phi_q\big(\int_s^\sigma a(r)f(u(r))\nabla r\big)\Delta s, \\ \qquad \text{if } 0 \leq t \leq \sigma, \\ \phi_q\big(\frac{\delta}{\gamma}\int_\theta^\eta a(r)f(u(r))\nabla r\big) + \int_t^T \phi_q\big(\int_\theta^s a(r)f(u(r))\nabla r\big)\Delta s, \\ \qquad \text{if } \sigma \leq t \leq T. \end{cases} \tag{2.3}$$

Proof. Necessity. By the equation of the boundary condition we have $u^{\Delta}(\xi) \geq 0$, $u^{\Delta}(\eta) \leq 0$, then there exist a constant $\sigma \in [\xi, \eta] \subset (0,1)$ such that $u^{\Delta}(\sigma) = 0$.

First, by integrating the equation of the problems (1.1) on $(\theta, 1)$ we have,

$$\phi_p(u^{\Delta}(t)) = \phi_p(u^{\Delta}(\sigma)) - \int_{\sigma}^{t} a(s)f(u)(s)\nabla s, \tag{2.4}$$

then $u^{\Delta}(t) = u^{\Delta}(\sigma) - \phi_q\Big(\int_{\sigma}^{t} a(s)f(u)(s)\nabla s\Big)$, thus

$$u(t) = u(\sigma) + u^{\Delta}(\sigma)(t-\sigma) - \int_{\sigma}^{t} \phi_q\Big(\int_{\sigma}^{s} a(r)f(u)(r)\nabla r\Big)\Delta s, \tag{2.5}$$

by $u^{\Delta}(\sigma) = 0$, let $t = \eta$ on (2.4), we have $\phi_p(u'(\eta)) = -\int_{\sigma}^{\eta} a(s)f(u)(s)\nabla s$. By the equation of the boundary condition (1.1), we have $\phi_p(u(T)) = -\frac{\delta}{\gamma}\phi_p(u'(\eta))$, then

$$u(T) = \phi_q\Big(\frac{\delta}{\gamma}\int_{\sigma}^{\eta} a(s)f(u)(s)\nabla s\Big). \tag{2.6}$$

by (2.5), (2.6) and let let $t = T$ on (2.5), we have

$$u(\sigma) = \phi_q\Big(\frac{\delta}{\gamma}\int_{\sigma}^{\eta} a(s)f(u)(s)\nabla s\Big) + \int_{\sigma}^{T} \phi_q\Big(\int_{\sigma}^{s} a(r)f(u)(r)\nabla r\Big)\Delta s, \tag{2.7}$$

by (2.5) and (2.7), for $t \in (\sigma, T)$ we know

$$u(t) = \phi_q\Big(\frac{\delta}{\gamma}\int_{\sigma}^{\eta} a(s)f(u)(s)\nabla s\Big) + \int_{t}^{T} \phi_q\Big(\int_{\sigma}^{s} a(r)f(u)(r)\nabla r\Big)\Delta s.$$

Similarly, for $t \in (0, \sigma)$, by integrating the equation of problems (1.1) on $(0, \sigma)$, we have

$$u(t) = \phi_q\Big(\frac{\beta}{\alpha}\int_{\xi}^{\sigma} a(r)f((u)(r))\nabla r\Big) + \int_{0}^{t} \phi_q\Big(\int_{s}^{\sigma} a(r)f((u)(r))\nabla r\Big)\Delta s.$$

Then (2.3) holds.

Sufficiency. Suppose that (2.3) holds. Then by (2.3), we have

$$u^{\Delta}(t) = \begin{cases} \phi_q\Big(\int_{t}^{\sigma} a(r)f((u)(r))\nabla r\Big) \geq 0, & 0 \leq t \leq \sigma, \\ -\phi_q\Big(\int_{\sigma}^{t} a(r)f((u)(r))\nabla r\Big) \leq 0, & \sigma \leq t \leq T, \end{cases} \tag{2.8}$$

So, $(\phi_p(u^{\Delta}))^{\nabla} + a(t)f(u(t)) = 0$, $0 < t < 1$. These imply that the first equation of (1.1) holds. Furthermore, by letting $t = 0$ and $t = T$ on (2.3) and (2.8), we can obtain the boundary value equations of (1.1). The proof of Lemma 2.3 is complete. $\square$

Now, we define a mapping $T : K \to E$ given by

$$(T(u))(t) = \begin{cases} \phi_q\big(\frac{\beta}{\alpha}\int_{\xi}^{\sigma} a(r)f(u(r))\nabla r\big) + \int_{0}^{t} \phi_q\big(\int_{s}^{\sigma} a(r)f(u(r))\nabla r\big)\Delta s, \\ \qquad \text{if } 0 \leq t \leq \sigma, \\ \phi_q\big(\frac{\delta}{\gamma}\int_{\sigma}^{\eta} a(r)f(u(r))\nabla r\big) + \int_{t}^{T} \phi_q\big(\int_{\sigma}^{s} a(r)f(u(r))\nabla r\big)\Delta s, \\ \qquad \text{if } \sigma \leq t \leq T. \end{cases}$$

Because of

$$(T(u))'(t) = \begin{cases} \phi_q\big(\int_{t}^{\sigma} a(r)f((u)(r))\nabla r\big) \geq 0, & 0 \leq t \leq \sigma, \\ -\phi_q\big(\int_{\sigma}^{t} a(r)f((u)(r))\nabla r\big) \leq 0, & \sigma \leq t \leq T, \end{cases}$$

the operator T is monotone decreasing continuous and $(T(u)^{\Delta})(\sigma)=0$, and for any $u\in K$, we have
$$(\phi_q(T(u))^{\Delta})^{\nabla}(t)=-a(t)f((u)(t),\quad \text{a.e. } t\in(0,1),$$
and $(T(u))(\sigma)=\|T(u)\|$. Therefore, $T(K)\subset K$.

Lemma 2.4. *$T:K\to K$ is completely continuous.*

Proof. Suppose $D\subset K$ is a bounded set, Let $M>0$ such that $\|u\|\le M$, $u\in D$. For any $u\in D$, we have
$$\begin{aligned}
\|Tu\|&=(Tu)(\sigma)\\
&=\phi_q\Big(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(u(r))\nabla r\Big)+\int_0^\sigma\phi_q\Big(\int_s^\sigma a(r)f(u(r))\nabla r\Big)\Delta s,\\
&\le\Big[\phi_q\Big(\frac{\beta}{\alpha}\int_0^\sigma a(r)\nabla r\Big)+\int_0^\sigma\phi_q\Big(\int_s^\sigma a(r)\nabla r\Big)\Delta s\Big]\phi_q\Big(\sup_{u\in D}f(u)\Big).
\end{aligned}$$
Then $T(D)$ is bounded. Furthermore it is easy see by Arzela-ascoli Theorem and Lebesgue dominated convergence Theorem that $T:K\to K$ is completely continuous. The proof is complete. $\square$

For convenience, we set $\theta^*=2/L$ and
$$\theta_*=\frac{1}{\Big(T+\phi_q\big(\frac{\beta}{\alpha}\big)\Big)\phi_q\Big(\int_0^T a(r)\nabla r\Big)}.$$

3. The Existence of Positive Solutions

The main results of this part are the following three Theorems.

Theorem 3.1. *Suppose that conditions (H1), (H2) hold. Assume that f also satisfies*

- (A1) *$f(u)\ge(mr)^{p-1}$, $\theta r\le u\le r$;*
- (A2) *$f(u)\le(MR)^{p-1}$, $0\le u\le R$, where $m\in(\theta^*,\infty)$, $M\in(0,\theta_*)$. Then, the boundary-value problem* (1.1) *has a positive solution u such that $\|u\|$ is between r and R.*

Theorem 3.2. *Suppose that conditions (H1), (H2) hold. Assume that f also satisfy*

- (A3) *$f_\infty=\lambda\in\big((\frac{2\theta^*}{\theta})^{p-1},\infty\big)$;*
- (A4) *$f_0=\varphi\in[0,(\frac{\theta_*}{4})^{p-1})$. Then, the boundary-value problem* (1.1) *has a positive solution u such that $\|u\|$ is between r and R.*

Theorem 3.3. *Suppose that conditions (H1), (H2) hold. Assume that f also satisfy*

- (A5) *$f_0=\varphi\in\big((\frac{2\theta^*}{\theta})^{p-1},\infty\big)$;*
- (A6) *$f_\infty=\lambda\in\big[0,(\frac{\theta_*}{4})^{p-1}\big)$.*

Then, the boundary-value problem (1.1) *has a positive solution u such that $\|u\|$ is between r and R.*

Proof of Theorem 3.1. . Without loss of generality, we suppose that $r<R$. For any $u\in K$, by Lemma 2.2, we have
$$u(t)\ge\theta\|u\|,\quad t\in[\theta,T-\theta]. \tag{3.1}$$

we define two open subset Ω_1 and Ω_2 of E,

$$\Omega_1 = \{u \in K : \|u\| < r\}, \quad \Omega_2 = \{u \in K : \|u\| < R\}$$

For $u \in \partial\Omega_1$, by (3.1), we have

$$r = \|u\| \geq u(t) \geq \theta\|u\| = \theta r, \quad t \in [\theta, T-\theta].$$

For $t \in [\theta, T-\theta]$, if (A1) hold, we shall discuss it from three perspectives.

(i) If $\theta \in [\theta, T-\theta]$, thus for $u \in \partial\Omega_1$, by (A1) and Lemma 2.1, we have

$$\begin{aligned}
2\|T(u)\| &= 2(T(u))(\sigma)\\
&\geq \int_0^\theta \phi_q\Big(\int_s^\theta a(r)f((u)(r))\nabla r\Big)\Delta s + \int_\theta^T \phi_q\Big(\int_\theta^s a(r)f((u)(r))\nabla r\Big)\Delta s\\
&\geq (mr)\Big(\int_\theta^\theta \phi_q\Big(\int_s^\theta a(r)\nabla r\Big)\Delta s\Big) + (mr)\Big(\int_\theta^{T-\theta} \phi_q\Big(\int_\theta^s a(r)\nabla r\Big)\Delta s\Big)\\
&\geq mrA(\theta) \geq mrL\\
&> 2r = 2\|u\|.
\end{aligned}$$

(ii) If $\theta \in (T-\theta, T]$, thus for $u \in \partial\Omega_1$, by (A1) and Lemma 2.1, we have

$$\begin{aligned}
\|T(u)\| &= (T(u))(\theta)\\
&\geq \int_0^\theta \phi_q\Big(\int_s^\theta a(r)f((u)(r))\nabla r\Big)\Delta s\\
&\geq \int_\theta^{T-\theta} \phi_q\Big(\int_s^{T-\theta} a(r)f((u)(r))\nabla r\Big)\Delta s\\
&\geq mr\int_\theta^{T-\theta} \phi_q\Big(\int_s^{T-\theta} a(r)\nabla r\Big)\Delta s\\
&= mrA(T-\theta) \geq mrL\\
&> 2r > r = \|u\|.
\end{aligned}$$

(iii) If $\theta \in [0, \theta)$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 2.1, we have

$$\begin{aligned}
\|T(u)\| &= (T(u))(\theta)\\
&\geq \int_\theta^T \phi_q\Big(\int_\theta^s a(r)f((u)(r))\nabla r\Big)\Delta s\\
&\geq \int_\theta^{T-\theta} \phi_q\Big(\int_\theta^s a(r)f((u)(r))\nabla r\Big)\Delta s\\
&\geq mr\int_\theta^{T-\theta} \phi_q\Big(\int_\theta^s a(r)\nabla r\Big)\Delta s\\
&= mrA(\theta) \geq mrL\\
&> 2r > r = \|u\|.
\end{aligned}$$

Therefore, $\|Tu\| \geq \|u\|$, for all $u \in \partial\Omega_1$. Then by Theorem 3.1,

$$i(T, \Omega_1, K) = 0. \tag{3.2}$$

On the other hand, since $u \in \partial\Omega_2$, we have $u(t) \le \|u\| = R$. By (A2), we know

$$
\begin{aligned}
\|T(u)\| &= (T(u))(\theta)\\
&= \phi_q\Big(\frac{\beta}{\alpha}\int_\xi^\theta a(r)f((u)(r))\nabla r\Big) + \int_0^t \phi_q\Big(\int_0^t a(r)f((u)(r))\nabla r\Big)\Delta s\\
&\le \phi_q\Big(\frac{\beta}{\alpha}\int_0^T a(r)f((u)(r))\nabla r\Big) + \int_0^T \phi_q\Big(\int_0^T a(r)f((u)(r))\nabla r\Big)\Delta s\\
&= \big[T+\phi_q(\frac{\beta}{\alpha})\big]\phi_q\Big(\int_0^T a(r)f((u)(r))\nabla r\Big)\\
&\le \big[T+\phi_q(\frac{\beta}{\alpha})\big]MR\phi_q\Big(\int_0^T a(r)\nabla r\Big)\\
&= \big[T+\phi_q(\frac{\beta}{\alpha})\big]MR\phi_q\Big(\int_0^T a(r)\nabla r\Big)\\
&\le R = \|u\|.
\end{aligned}
$$

Therefore, $\|T(u)\| \le \|u\|$, for all $u \in \partial\Omega_2$. Then by Theorem 3.1,

$$
i(T,\Omega_2,K) = 1. \tag{3.3}
$$

Therefore, by (3.2), (3.3) and $r < R$, we have

$$
i(T,\Omega_2 \setminus \overline{\Omega}_1, K) = 1.
$$

Then T has a fixed point $u \in (\Omega_2 \setminus \overline{\Omega}_1)$. Obviously, u is positive solution of problem (1.1) and $r < \|u\| < R$. The proof of Theorem 3.1 is complete. $\square$

Proof of Theorem 3.2. Firstly, by $f_0 = \lim_{u\to 0}\frac{f(u)}{u^{p-1}} = \varphi$, for $\epsilon = (\frac{\theta_*}{4})^{p-1} - \varphi$, there exists an adequate small positive number ρ. Since $0 \le u \le \rho$, $u \ne 0$, we have

$$
f(u) \le (\varphi+\epsilon)u^{p-1} \le (\frac{\theta_*}{4})^{p-1}(2\rho)^{p-1} = \big(\frac{\theta_*}{2}\rho\big)^{p-1}. \tag{3.4}
$$

Let $R = \rho$, $M = \frac{\theta_*}{2} \in (0,\theta_*)$, thus by (3.4), we have

$$
f(u) \le (MR)^{p-1}, \quad 0 \le u \le R.
$$

Then condition (A2) holds. Next, by condition (A3),

$$
f_\infty = \lim_{u\to 0}\frac{f(u)}{u^{p-1}} = \lambda \in \big((\frac{2\theta^*}{\theta})^{p-1}.\infty\big)
$$

Then for $\epsilon = \lambda - (\frac{2\theta^*}{\theta})^{p-1}$, there exists an adequate big positive number $r \ne R$. Since $u \ge \theta r$, we have

$$
f(u) \ge (\lambda-\epsilon)u^{p-1} \ge \big(\frac{2\theta^*}{\theta}\big)^{p-1}(\theta r)^{p-1} = (2\theta^* r)^{p-1}. \tag{3.5}
$$

Let $m = 2\theta^* > \theta^*$, thus by (3.5), condition (A1) holds. Therefore, by Theorem 3.1, we know that the results of Theorem 3.2 holds. The proof of Theorem 3.2 is complete. $\square$

Proof of Theorem 3.3. Firstly, by condition $f_0 = \varphi$, then for $\epsilon = \varphi - (\frac{2\theta^*}{\theta})^{p-1}$, there exists an adequate small positive number r. Since $0 \le u \le r, u \ne 0$, we have

$$
f(u) \ge (\varphi-\epsilon)u^{p-1} = \big(\frac{2\theta^*}{\theta}\big)^{p-1}u^{p-1}.
$$

Thus when $\theta r \le u \le r$, we have

$$f(u) \ge \big(\frac{2\theta^*}{\theta}\big)^{p-1}(\theta r)^{p-1} = (2\theta^* r)^{p-1}. \tag{3.6}$$

Let $m = 2\theta^* > \theta^*$, so by (3.6), condition (A1) holds.

Next, by condition (A6): $f_\infty = \lambda$, then for $\epsilon = (\frac{\theta_*}{4})^{p-1} - \lambda$, there exists an adequate big positive number $\rho \ne r$. Since $u \ge \rho$, we have

$$f(u) \le (\lambda+\epsilon)u^{p-1} \le \big(\frac{\theta_*}{4}\big)^{p-1}u^{p-1}. \tag{3.7}$$

If f is non-boundary, by the continuation of f on $[0,\infty)$, then exists constant $R(\ne r) \ge \rho$, and a point $u_0 \in [0,\infty)$ such that $\rho \le u \le R$ and $f(u) \le f(u_0)$, $0 \le u \le R$. Thus, by $\rho \le u_0 \le R$, we know

$$f(u) \le f(u_0) \le \big(\frac{\theta_*}{4}\big)^{p-1}u_0^{p-1} \le \big(\frac{\theta_*}{4}R\big)^{p-1}.$$

Let $M = \theta_*/4 \in (0,\theta_*)$, we have

$$f(u) \le (MR)^{p-1}, \quad 0 \le u \le R.$$

If f is boundary, we suppose $f(u) \le \overline{M}^{p-1}$, $u \in [0,\infty)$. There exists an adequate big positive number $R > \frac{4}{\theta_*}\overline{M}$, then let $M = \frac{\theta_*}{4} \in (0,\theta_*)$, we have

$$f(u) \le \overline{M}^{p-1} \le \big(\frac{\theta_*}{4}R\big)^{p-1} = (MR)^{p-1}, \quad 0 \le u \le R.$$

Therefore, condition (A2) holds. Therefore, by Theorem 3.1, we know that the results of Theorem 3.3 holds. The proof of Theorem 3.3 is complete. ☐

4. The Existence of Many Positive Solutions

Now, we will discuss the existence of many positive solutions.

Theorem 4.1. *Suppose that conditions (H1), (H2) and (A2) in Theorem 3.1 hold. Assume that f also satisfy*

- (A7) $f_0 = +\infty$;
- (A8) $f_\infty = +\infty$.

Then, the boundary-value problem (1.1) *has at least two solutions u_1, u_2 such that*

$$0 < \|u_1\| < R < \|u_2\|.$$

Proof. Firstly, by condition (A7), for any $M > \frac{2}{L}$, there exists a constant $\rho_* \in (0,R)$ such that

$$f(u) \ge (Mu)^{p-1}, \quad 0 < u \le \rho_*, \quad u \ne 0. \tag{4.1}$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for any $u \in \partial\Omega_{\rho_*}$, by (4.1) and Lemma 2.2, similar to the previous proof of Theorem 3.1, we can have from three perspectives

$$\|Tu\| \ge \|u\|, \quad \forall u \in \partial\Omega_{\rho_*}.$$

Then by Theorem 3.1, we have

$$i(T,\Omega_{\rho_*},K) = 0. \tag{4.2}$$

Next, by condition (A_8), for any $\overline{M} > \frac{2}{L}$, there exists a constant $\rho_0 > 0$ such that

$$f(u) \ge (\overline{M}u)^{p-1}, \quad u > \rho_0. \tag{4.3}$$

We choose a constant $\rho^* > \max\{R, \frac{\rho_0}{\theta}\}$, obviously, $\rho_* < R < \rho^*$. Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$. For any $u \in \partial\Omega_{\rho^*}$, by Lemma 2.2, we have

$$u(t) \geq \theta\|u\| = \theta\rho^* > \rho_0, \quad t \in [\theta, 1-\theta].$$

Then by (4.3) and also similar to the previous proof of Theorem 3.1, we can also have from three perspectives

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_{\rho^*}.$$

Then by Theorem 3.1, we have

$$i(T, \Omega_{\rho^*}, K) = 0. \tag{4.4}$$

Finally, set $\Omega_R = \{u \in K : \|u\| < R\}$, For any $u \in \partial\Omega_R$, by (A2), Lemma 2.2 and also similar to the latter proof of Theorem 3.1, we can also have

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_R.$$

Then by Theorem 3.1,

$$i(T, \Omega_R, K) = 1. \tag{4.5}$$

Therefore, by (4.2), (4.4), (4.5) and $\rho_* < R < \rho^*$, we have

$$i(T, \Omega_R \setminus \overline{\Omega}_{\rho_*}, k) = 1, \quad i(T, \Omega_{\rho^*} \setminus \overline{\Omega}_R, k) = -1.$$

Then T has fixed points $u_1 \in \Omega_R \setminus \overline{\Omega}_{\rho_*}$, and fixed point $u_2 \in \Omega_{\rho^*} \setminus \overline{\Omega}_R$. Obviously, $u_1,\ u_2$ are all positive solutions of problem (1.1),(2.1) and $0 < \|u_1\| < R < \|u_2\|$. The proof of Theorem 4.1 is complete. □

Theorem 4.2. *Suppose that conditions (H1), (H2) and (A1) in Theorem 3.1 hold. Assume that f also satisfy*

- (A9) $f_0 = 0$;
- (A10) $f_\infty = 0$.

Then the boundary-value problem (1.1) *has at least two solutions* $u_1,\ u_2$ *such that*

$$0 < \|u_1\| < r < \|u_2\|.$$

Proof. Firstly, by $f_0 = 0$, for $\epsilon_1 \in (0, \theta_*)$, there exists a constant $\rho_* \in (0, r)$ such that

$$f(u) \leq (\epsilon_1 u)^{p-1}, \quad 0 < u \leq \rho_*. \tag{4.6}$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for any $u \in \partial\Omega_{\rho_*}$, by (4.6), we have

$$\begin{aligned}
\|Tu\| &= (Tu)(\delta) \\
&= \phi_q\Big(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(u(r))\nabla r\Big) + \int_0^\delta \phi_q\Big(\int_s^\sigma a(r)f(u(r))\nabla r\Big)\Delta s \\
&\leq \phi_q\Big(\frac{\beta}{\alpha}\int_0^T a(r)f(u(r))\nabla r\Big) + T\phi_q\Big(\int_0^T a(r)f(u(r))\nabla r\Big) \\
&\leq \big(\phi_q(\frac{\beta}{\alpha}) + T\big)\epsilon_1\rho_*\phi_q\Big(\int_0^T a(r)\nabla r\Big) \\
&\leq \rho_* = \|u\|.
\end{aligned}$$

i.e., $\|Tu\| \leq \|u\|$, for all $u \in \partial\Omega_{\rho_*}$. Then by Theorem 3.1, we have

$$i(T, \Omega_{\rho_*}, K) = 1. \tag{4.7}$$

Next, let $f^*(x) = \max_{0\le u\le x} f(u)$, note that $f^*(x)$ is monotone increasing with respect to $x \ge 0$. Then from $f_\infty = 0$, it is easy to see that

$$\lim_{x\to\infty} \frac{f^*(x)}{x^{p-1}} = 0.$$

Therefore, for any $\epsilon_2 \in (0, \theta_*)$, there exists a constant $\rho^* > r$ such that

$$f^*(x) \le (\epsilon_2 x)^{p-1}, \quad x \ge \rho^*. \tag{4.8}$$

Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$. For any $u \in \partial\Omega_{\rho^*}$, by (4.8), we have

$$\begin{aligned}
\|Tu\| &= (Tu)(\delta)\\
&= \phi_q\Big(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(u(r))\nabla r\Big) + \int_0^\delta \phi_q\Big(\int_s^\sigma a(r)f(u(r))\nabla r\Big)\Delta s\\
&\le \phi_q\Big(\frac{\beta}{\alpha}\int_0^T a(r)f(u(r))\nabla r\Big) + T\phi_q\Big(\int_0^T a(r)f(u(r))\nabla r\Big)\\
&\le \phi_q\Big(\frac{\beta}{\alpha}\int_0^T a(r)f^*(\rho^*))\nabla r\Big) + T\phi_q\Big(\int_0^T a(r)f^*(\rho^*))\nabla r\Big)\\
&\le \big(\phi_q(\frac{\beta}{\alpha}) + T\big)\epsilon_\rho^*\phi_q\Big(\int_0^T a(r)\nabla r\Big) \le \rho^* = \|u\|.
\end{aligned}$$

i.e., $\|Tu\| \le \|u\|$, for all $u \in \partial\Omega_{\rho^*}$. Then by Theorem 3.1, we have

$$i(T, \Omega_{\rho^*}, K) = 1. \tag{4.9}$$

Finally, set $\Omega_r = \{u \in K : \|u\| < r\}$. For any $u \in \partial\Omega_r$, by (A1), Lemma 2.2 and also similar to the previous proof of Theorem 3.1, we have

$$\|Tu\| \ge \|u\|, \quad \forall u \in \partial\Omega_r.$$

Then by Theorem 3.1, we have

$$i(T, \Omega_r, K) = 0. \tag{4.10}$$

Therefore, by (4.7), (4.9), (4.10), $\rho_* < r < \rho^*$ we have

$$i(T, \Omega_r \setminus \overline{\Omega}_{\rho_*}, k) = -1, \quad i(T, \Omega_{\rho^*} \setminus \overline{\Omega}_r, k) = 1.$$

Then T has fixed points $u_1 \in \Omega_r \setminus \overline{\Omega}_{\rho_*}$, and $u_2 \in \Omega_{\rho_*} \setminus \overline{\Omega}_r$. Obviously, u_1, u_2 are all positive solutions of problem (1.1), (2.1) and $0 < \|u_1\| < r < \|u_2\|$. The proof of Theorem 4.2 is complete. □

Similar to Theorem 3.1, we obtain the following Theorems.

Theorem 4.3. *Suppose that conditions (H1), (H2) and (A2) in Theorem 3.1, (A4) in Theorem 3.2 and (A6) in Theorem 3.3 hold. Then the boundary-value problem (1.1) has at last two solutions u_1, u_2 such that $0 < \|u_1\| < R < \|u_2\|$.*

Theorem 4.4. *Suppose that conditions (H1), (H2) and (A1) in Theorem 3.1, (A3) in Theorem 3.2 and (A5) in Theorem 3.3 hold. Then the boundary-value problem (1.1) has at last two solutions u_1, u_2 such that $0 < \|u_1\| < r < \|u_2\|$.*

5. Applications

Example 5.1. Consider the following singular boundary-value problem (SBVP) with p-Laplacian:

$$\begin{gathered}
(\phi_p(u^\Delta))^\nabla + \frac{1}{4}t^{-\frac{1}{2}}u^{1/2}\big[\frac{1}{3} + \frac{64e^{2u}}{120+7e^u+e^{2u}}\big] = 0, \quad 0 < t < \frac{3}{2}, \\
4\phi_p(u(0)) - \phi_p(u^\Delta(\frac{1}{4})) = 0, \quad \phi_p(u(\frac{3}{2})) + \delta\phi_p(u^\Delta(\frac{1}{2})) = 0,
\end{gathered} \tag{5.1}$$

where $\beta = \gamma = 1$, $\alpha = 4$, $p = \frac{3}{2}$, $\delta \geq 0$, $\xi = \frac{1}{4}$, $\eta = \frac{1}{2}$, $T = \frac{3}{2}$,

$$a(t) = \frac{1}{4}t^{-1/2}, \quad f(u) = u^{1/2}\big[\frac{1}{3} + \frac{64e^{2u}}{120+7e^u+e^{2u}}\big].$$

Then obviously,

$$\begin{gathered}
q = 3, \quad f_0 = \varepsilon = \lim_{u\to 0^+}\frac{f(u)}{u^{p-1}} = \frac{5}{6}, \\
f_\infty = \lim_{u\to\infty}\frac{f(u)}{u^{p-1}} = 64 + \frac{1}{3}, \quad \int_0^T a(t)\nabla t = \frac{\sqrt{6}}{4},
\end{gathered}$$

so conditions (H1), (H2) hold. Next,

$$\theta_* = \frac{1}{\big(T + \phi_q(\frac{\beta}{\alpha})\big)\phi_q\big(\int_0^T a(r)\nabla r\big)} = \frac{32\sqrt{6}}{75},$$

then $\varepsilon \in [0, (\frac{\theta_*}{4})^{p-1}) = [0, 1.97)$, so conditions (A4) holds. We choose $\theta = 1/4$, then it is easy see by calculating that

$$L = \min_{t\in[\theta,1-\theta]} A(t) = \frac{1}{16}\big(\frac{7}{36} + \frac{\sqrt{3}}{3}\big).$$

Because of

$$\big(\frac{2\theta^*}{\theta}\big)^{p-1} = 96 \times \big(\frac{1}{7+12\sqrt{3}}\big)^{1/2} < 64 + \frac{1}{3},$$

then $f_\infty = \lambda \in \big((\frac{2\theta^*}{\theta})^{p-1}, \infty\big)$, so conditions (A3) holds. Then by Theorem 3.2, SBVP (5.1) has at least a positive solution.

Example 5.2. Consider the following singular boundary-value problem (SBVP) with p-Laplacian

$$\begin{gathered}
(\phi_p(u'))' + \frac{3}{256\pi^3}t^{-\frac{1}{2}}(1-t)[u^2+u^4] = 0, \quad 0 < t < 1, \\
2\phi_p(u(0)) - \phi_p(u'(\frac{1}{4})) = 0, \ \phi_p(u(1)) + \delta\phi_p(u'(\frac{1}{2})) = 0,
\end{gathered} \tag{5.2}$$

where $\beta = \gamma = 1$, $\alpha = 2$, $p = 4$, $\delta \geq 0$, $\xi = \frac{1}{4}$, $\eta = \frac{1}{2}$, $T = 1$,

$$a(t) = \frac{3}{256\pi^3}t^{-1/2}(1-t),$$

and $f(u) = u^2 + u^4$. Then obviously,

$$q = \frac{4}{3}, \quad \int_0^T a(t)\nabla t = \frac{1}{64\pi^3}, \quad f_\infty = +\infty, \quad f_0 = +\infty,$$

so conditions (H1), (H2), (A7), (A8) hold. Next,

$$\phi_q\Big(\int_0^T a(t)\nabla t\Big) = \frac{1}{4\pi}, \quad \theta_* = \frac{4\pi}{1+\sqrt[3]{2}},$$

we choose $R = 3$, $M = 2$. Because $f(u)$ is monotone increasing on $[0,\infty)$, we have $f(u) \le f(3) = 90$, for $0 \le u \le 3$. Therefore, because $M \in (0, \theta_*)$, $(MR)^{p-1} = (6)^3 = 216$. Also we know that

$$f(u) \le (MR)^{p-1}, \quad 0 \le u \le 3,$$

so conditions (A2) holds. Then by Theorem 4.1, SBVP (5.2) has at least two positive solutions u_1, u_2 and $0 < \|u_1\| < 3 < \|u_2\|$.

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