EXPONENTIAL DECAY FOR THE SEMILINEAR WAVE EQUATION WITH SOURCE TERMS

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Abstract. In this paper, we prove that for a semilinear wave equation with source terms, the energy decays exponentially as time approaches infinity. For this end we use the multiplier method.

1. Introduction

Main results. Let \( \Omega \) be a bounded subset of \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). We are concerned with the mixed problems

\[
\begin{align*}
  u_{tt} - \Delta u + \delta u_t &= |u|^{p-1}u, & x \in \Omega, & t \geq 0, \\
  u(0, x) &= u_0(x) \in H^1_0(\Omega), & u_t(0, x) = u_1(x) \in L^2(\Omega), & x \in \Omega, \\
  u(t, x)|_{\partial \Omega} &= 0, & \text{for } t \geq 0.
\end{align*}
\]

Here \( \delta > 0 \) and \( 1 < p \leq \frac{n}{n-2} \) (\( n \geq 3 \)), \( 1 < p \) (\( n = 1, 2 \)). Set

\[
\begin{align*}
  I(u) &= \int_\Omega (|\nabla u|^2 - |u|^{p+1})dx, \\
  J(u) &= \int_\Omega \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right)dx, \\
  E(t) &= \frac{1}{2} \int_\Omega |u_t|^2 dx + J(u).
\end{align*}
\]

Also let the Nehari manifold

\[
N := \{ u \in H^1_0(\Omega) : I(u) = 0, \ u \neq 0 \};
\]

and the potential depth

\[
d := \inf_{u \in N} J(u).
\]

For problem (1.1)-(1.3), Ikehata and Suzuki have shown the following results:

\[
d > 0;
\]

\[
E(t) + \int_0^t \int_\Omega \delta |u_t|^2 dx dt = E(0);
\]
If \( E(0) < d \) and \( I(u(0, x)) > 0 \) then we have
\[
E(t) < d \quad \text{and} \quad I(u(t, x)) > 0, \quad \forall t \in [0, \infty); \quad (1.11)
\]
\[
\theta \int_\Omega |\nabla u|^2 dx \geq \int_\Omega |u|^{p+1} dx, \quad \theta \in (0, 1), \quad \forall t \in [0, \infty); \quad (1.12)
\]
\[
\lim_{t \to +\infty} \int_\Omega (|u_t|^2 + |\nabla u|^2) dx = 0; \quad (1.13)
\]
\[
\int_0^t \int_\Omega |\nabla u|^2 dx \, dt \leq C. \quad (1.14)
\]

In this paper we will use the multiplier technique to prove the following result.

**Theorem 1.1.** If \( E(0) < d \) and \( I(u(0, x)) > 0 \), then there exists positive constant \( \gamma \) and \( C > 1 \) such that
\[
E(t) \leq Ce^{-\gamma t}, \quad \forall t \in [0, \infty). \quad (1.15)
\]

**Our results and their relationship to the literature.** The Problem
\[
\begin{align*}
&u_{tt} - \Delta u + a(x)|u_t|^{m-1}u_t + |u|^{p-1}u = 0, \quad \text{in } \Omega, \\
&u|_{\partial \Omega} = 0, \quad (u, u_t)|_{t=0} = (u_0, u_1)
\end{align*}
\]
has been studied, among others, by Nakao [2, 3] and Zuazua [4]. In [2, 3, 4], the authors assumed that \( a(x) \geq 0 \) in \( \Omega \), \( \inf a(x) > 0 \) in \( \Omega_0 \subset \subset \Omega \) and \( m = 1 \). The case \( m > 1 \) is still open [4].

The following problem, with \( m > 1 \) and \( a(x) \geq a_0 > 0 \) in \( \Omega \),
\[
\begin{align*}
&u_{tt} - \Delta u + a(x)|u_t|^{m-1}u_t = |u|^{p-1}u, \quad \text{in } \Omega, \\
&u|_{\partial \Omega} = 0, \quad (u, u_t)|_{t=0} = (u_0, u_1)
\end{align*}
\]
has been studied by many authors, Ball [5], Ikehata [6], Ikehata and Tanizawa [7], Levine [8, 9], Georgiev and Todorova [10], Georgiev and Milani [11], Todorova [12], Barbu et al [13], Todorova and Vitillaro [14], Messaoudi [15], Serrin [16], Kawashima et al [17]. Ball [5] proved the existence of a global attractor when \( m = 1 \). In [6, 7, 14, 17], the authors obtained a time-decay result when \( \Omega = \mathbb{R}^N \). In [8, 9, 10, 11, 12, 13, 15, 16], the authors mainly concerned the existence or nonexistence of global weak (or strong) solutions.

By the multiplier method in [18], Benaissa and Mimouni [19] studied very recently the decay properties of the solutions to the wave equation of \( p \)-Laplacian type with a weak nonlinear dissipative.

Here it should be noted that our main result Theorem 1.1 is also true for the locally damping case i.e., \( \delta = \delta(x) \geq 0 \) in \( \Omega \) and \( \delta(x) \geq \delta_0 > 0 \) in \( \Omega_0 \subset \subset \Omega \). We did not find references for the case with boundary damping term.

2. **Proof of the Main Result**

Take \( x_0 \in \mathbb{R}^n \) and set \( m(x) := x - x_0 \). Let \( \nu \) denote the outward normal vector to \( \partial \Omega \). Set
\[
\Gamma(x_0) := \{ x \in \partial \Omega : (x - x_0) \cdot \nu > 0 \},
\]
\[
\chi := \int_\Omega (u_t (m \cdot \nabla u) + \frac{n}{p+1} u(u_t + \frac{\delta}{2} u)) \bigg|_{x_0}^T dx.
\]
Combining (2.3) and (2.4) we obtain
\[ \int_0^T E(t) dt \leq C \int_0^T \int_{\Gamma(x_0)} (m \cdot \nu) \frac{\partial u}{\partial \nu} \, d\Gamma dt + \int_0^T \int_{\Omega} |u|^2 \, dx \, dt + |\chi|. \] (2.1)

Proof. Multiplying (1.1) by \( q(x) \cdot \nabla u \) and integrating by parts gives, \[ \int_0^T \int_{\Omega} (q(x) \cdot \nabla u) \partial_t u - (q(x) \cdot \nabla u) \partial_t u \, dx \, dt \]
\[ + \int_0^T \int_{\Omega} (m \cdot \nabla u) \frac{\partial u}{\partial \nu} \, d\Gamma dt + \int_0^T \int_{\partial \Omega} (q(x) \cdot \nabla u) \partial_t u \, d\nu dt \]
\[ = \frac{1}{2} \int_0^T \int_{\partial \Omega} (m \cdot \nu) \frac{\partial u}{\partial \nu} \, d\Gamma dt. \] (2.2)

Here \( q(x) \in W^{1,\infty}(\Omega) \). Applying identity (2.2) with \( q(x) = m(x) \), we deduce
\[ \left( \int_{\Omega} u_t (m \cdot \nabla u) dx \right)_0^T + \int_0^T \int_{\Omega} |u_t|^2 - |\nabla u|^2 \, dx \, dt + \int_0^T \int_{\Omega} |\nabla u|^2 \, dx \, dt \]
\[ + \frac{n}{p+1} \int_0^T \int_{\Omega} |u|^{p+1} \, dx \, dt \]
\[ = \frac{1}{2} \int_0^T \int_{\partial \Omega} (m \cdot \nu) \frac{\partial u}{\partial \nu} \, d\Gamma dt \]
\[ \leq \frac{1}{2} \int_0^T \int_{\Gamma(x_0)} (m \cdot \nu) \frac{\partial u}{\partial \nu} \, d\Gamma dt. \] (2.3)

We now multiply (1.1) by \( u \) and integrate by parts, then we have
\[ \left( \int_{\Omega} u_t (u + \frac{\delta}{2} u) dx \right)_0^T = \int_0^T \int_{\Omega} |u_t|^2 - |\nabla u|^2 \, dx \, dt + \int_0^T \int_{\Omega} |u|^{p+1} \, dx \, dt. \] (2.4)

Combining (2.3) and (2.4) we obtain
\[ \chi + \left( \frac{n}{2} - \frac{n}{p+1} \right) \int_0^T \int_{\Omega} |u_t|^2 \, dx \, dt + (1 + \frac{n}{p+1} - \frac{n}{2}) \int_0^T \int_{\Omega} |\nabla u|^2 \, dx \, dt \]
\[ + \int_0^T \int_{\Omega} \delta u_t (m \cdot \nabla u) dx \, dt \]
\[ \leq \frac{1}{2} \int_0^T \int_{\Gamma(x_0)} (m \cdot \nu) \frac{\partial u}{\partial \nu} \, d\Gamma dt. \] (2.5)

On the other hand, for any given \( \varepsilon > 0 \),
\[ \int_0^T \int_{\Omega} \delta u_t (m \cdot \nabla u) dx \, dt \]
\[ \leq \varepsilon \|m\|_{L^\infty(\Omega)} \int_0^T \int_{\Omega} |\nabla u|^2 \, dx \, dt + \frac{\delta^2}{2\varepsilon} \int_0^T \int_{\Omega} |u_t|^2 \, dx \, dt. \] (2.6)

Taking \( \varepsilon \) sufficiently small in (2.6), then substituting (2.6) into (2.5) we obtain (2.1). \( \square \)
Lemma 2.2. With the above notation,  
\[ E(t) \leq C \int_0^T \int_\Omega (|u_t|^2 + |u|^{p+1}) \, dx \, dt. \]  
\text{(2.7)}

Proof. First, we construct a function \( h(x) \in W^{1,\infty}(\Omega) \) such that \( h(x) = \nu \) on \( \Gamma(x_0) \); \( h(x) \cdot \nu > 0 \) a.e in \( \partial \Omega \); see [4]. Applying (2.2) with \( q(x) = h(x) \), we have  
\[ \int_0^T \int_{\Gamma(x_0)} |\partial u / \partial \nu|^2 \, d\Gamma \, dt \leq C \int_0^T \int_\Omega (|u_t|^2 + |u|^{p+1}) \, dx \, dt + 2 \left( \int_\Omega (u_t(\cdot \nabla u) \, dx \right)_0^T. \]  
\text{(2.8)}

From (2.4), we see that  
\[ \int_0^T \int_{\Gamma(x_0)} |\partial u / \partial \nu|^2 \, d\Gamma \, dt \leq C \int_0^T \int_\Omega (|u_t|^2 + |u|^{p+1}) \, dx \, dt + \chi \]  
\text{(2.9)}

where  
\[ \chi = \left( \int_\Omega (u_t + \delta/2 \, u) / \, dx \right)_0^T + 2 \left( \int_\Omega (u_t(\cdot \nabla u) \, dx \right)_0^T. \]  
\text{(2.10)}

Combining (2.1), (2.9) and (1.10) we obtain  
\[ T E(T) \leq \int_0^T E(t) \, dt \]  
\[ \leq C \int_0^T \int_\Omega (|u_t|^2 + |u|^{p+1}) \, dx \, dt + |\chi| + |\chi| \]  
\[ \leq C \int_0^T \int_\Omega (|u_t|^2 + |u|^{p+1}) \, dx \, dt + C(E(0) + E(T)) \]  
\[ \leq C \int_0^T \int_\Omega (|u_t|^2 + |u|^{p+1}) \, dx \, dt + C(2E(T) + \delta \int_0^T \int_\Omega |u_t|^2 \, dx \, dt). \]  
\text{(2.11)}

Taking \( T \) sufficiently large we get (2.7). \( \Box \)

Lemma 2.3.  
\[ \int_0^T \int_\Omega |u|^{p+1} \, dx \, dt \leq C \int_0^T \int_\Omega |u_t|^2 \, dx \, dt. \]  
\text{(2.11)}

Proof. We argue by contradiction. If (2.11) is not satisfied for some \( C > 0 \), then there exists a sequence of solutions \( \{u_n\} \) of (1.1)-(1.3) with  
\[ \lim_{n \to \infty} \int_0^T \int_\Omega |u_n|^{p+1} \, dx \, dt = \infty. \]  
\text{(2.12)}

From (1.12) and (1.14) we have  
\[ \int_0^T \int_\Omega |u_n|^{p+1} \, dx \, dt \leq \theta \int_0^T \int_\Omega |\nabla u_n|^2 \, dx \, dt \leq C. \]  
\text{(2.13)}

Thus we get  
\[ \lim_{n \to \infty} \int_0^T \int_\Omega |u_n|^2 \, dx \, dt = 0. \]  
\text{(2.14)}
We extract a subsequence (still denote by \{u_n\}) such that
\[
\begin{align*}
u_n &\rightharpoonup u \text{ weakly in } H^1(\Omega \times (0, T)), \\u_n &\to u \text{ strongly in } L^2(\Omega \times (0, T)), \\u_n &\to u \text{ a.e. in } \Omega \times (0, T), \\
|u_n|^{p-1}u_n &\to |u|^{p-1}u \text{ strongly in } L^\infty(0, T; L^r(\Omega)) \quad (2.18)
\end{align*}
\]

where \(r \in [1, \frac{2n}{n-2})\) if \(n \geq 3\) and \(r \in [1, \infty)\) if \(n = 1, 2\). From (2.14) we know that
\[
u_t = 0, \quad \text{a.e. in } \Omega \times (0, T) \quad (2.19)
\]

and so we have
\[
\begin{align*}
-\Delta u &= |u|^{p-1}u, \quad \text{in } \Omega \times (0, T) \quad (2.20) \\
u &= 0, \quad \text{on } \partial \Omega \times (0, T). \quad (2.21)
\end{align*}
\]

From (2.13) we get
\[
\int_0^T \int_\Omega |u|^{p+1} dx \, dt \leq \theta \int_0^T \int_\Omega |\nabla u|^2 dx \, dt < \int_0^T \int_\Omega |\nabla u|^2 dx \, dt \quad (2.22)
\]

which contradicts (2.20) and (2.21). This proves (2.11). \(\square\)

By Lemmas 2.2 and 2.3 we obtain
\[
E(T) \leq C \int_0^T \int_\Omega |u_t|^2 dx \, dt. \quad (2.23)
\]

This inequality, (1.10), and semigroup properties complete the proof of Theorem 1.1. For properties of semigroups, we refer the reader to [21].

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