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SINGULAR PERTURBATION PROBLEM FOR THE INCOMPRESSIBLE REYNOLDS EQUATION

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Abstract. We study the asymptotic behavior of the solution of a Reynolds equation which describe the behavior of the fluid between two closes surfaces as the distance between the two surfaces locally tends to zero.

1. Introduction

The field of lubricated contact deals with dynamical systems which consist of two (or more) bodies in relative motion. The contact between the bodies is mediated by a lubricant fluid, which in this work is assumed incompressible. The simplest such contact is the wedge (or plane slider), used in thrust bearings. It consists of two planar, rigid surfaces which are not mutually parallel. It is sketched in Fig. 1, in which the bottom surface is assumed to be moving horizontally towards the right. This movement entrains lubricant towards the right into the convergent gap between the surfaces. In turn, this generates a pressure field and consequently a thrust force, which allows to equilibrate a load applied to the top of the device.

Under the thin-film hypothesis (the gap thickness h much smaller than the in-plane dimensions of the contact, with the variations in h also assumed small) the fluid pressure does not depend on the vertical coordinate, which is taken across the gap. Upon normalization and assuming that the system is in a time-independent state, the pressure satisfies the normalized Reynolds equation [5]

$$\nabla \cdot [h(x)^3 \nabla p] = \frac{\partial h}{\partial x_1} \quad x = (x_1, \dots, x_n) \in \Omega \tag{1.1}$$

$$p = 0 \quad x \in \partial\Omega \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^n (n = 1 \text{ or } 2)$ is the domain in which the two surfaces are in proximity, p is the normalized pressure, $h(x)$ is the normalized gap thickness and the relative motion is assumed along the x_1-direction.

Assume, as in Fig. 1, that a vertical force F is applied to the upper surface of the bearing at a point $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$. To equilibrate this load the upper surface

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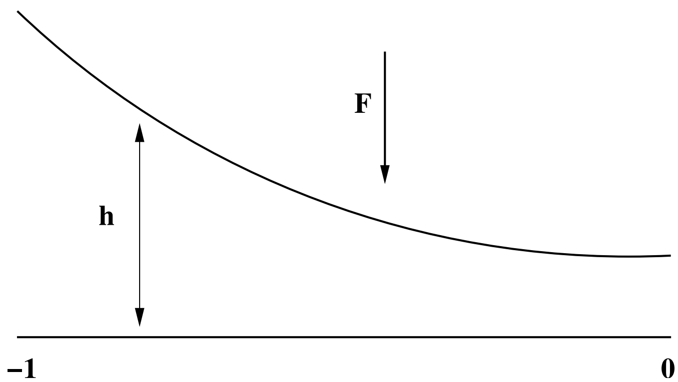

FIGURE 1. Sketch of a slider bearing

changes its position, intuitively getting closer to the lower surface as the applied load increases. Let us define the *reference shape* of the upper surface through a non negative function h_0.
Into this function we incorporate the so-called *attitude* of the slider (pitch and roll angles), so that the gap thickness becomes, simply,

$$h(x) = h_0(x) + \epsilon \tag{1.3}$$

where ϵ represents the minimal distance between the surfaces. With h_0 fixed the pressure becomes a function of ϵ satisfying the problem

$$\begin{gathered} \nabla \cdot \left[\left(h_0(x) + \epsilon \right)^3 \nabla p \right] = \frac{\partial h_0}{\partial x_1} \quad \text{on } \Omega \\ p = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.4}$$

If for any $\epsilon > 0$, we denote

$$g(\epsilon) = \int_\Omega p \, dx \tag{1.5}$$

then for equilibrium to hold in a system in which the only degree of freedom is the vertical position, the upper surface must be placed so as to satisfy

$$g(\epsilon) = F \tag{1.6}$$

It is easy to show both that $\lim_{\epsilon \to +\infty} g(\epsilon) = 0$ and that g is a continuous function, so that an equilibrium position exists for any positive load F smaller than $\max_\epsilon g(\epsilon)$. It is thus extremely important to analyze the behavior of g in the vicinity of zero, in particular the conditions under which $\lim_{\epsilon \to 0} g(\epsilon) = +\infty$. This guarantees the existence of an equilibrium position for any positive F. A finite limit, on the other hand, guarantees the existence of an equilibrium position for any $0 < F \leq g(0)$.

It is also important to study the moments of the force exerted by the pressure for each minimal gap thickness ϵ defined with respect to the point x^0,

$$m_i(\epsilon) = \textstyle\int_\Omega p(x_i - x_i^0) dx \quad i = 1, \dots, n \tag{1.7}$$

because equilibrium also requires that

$$m_i(\epsilon) = 0 \quad i = 1, \dots, n \tag{1.8}$$

and systems with the pitch and roll angles as additional degrees of freedom have their attitudes defined by these conditions.

In this article we will assume $n = 2$, $\Omega =]a_1, b_1[\times]a_2, b_2[$, $h_0 \in C^0(\bar{\Omega})$ with $h_0(x) > 0$ a.e $x \in \Omega$, and

$$\min_{x\in\overline{\Omega}} h_0(x) = 0.$$

The goal is to find the limits of $g(\epsilon)$ and $m_i(\epsilon)$, $i = 1, 2$ as $\epsilon \to 0^+$. Beside its intrinsic importance, this study provides crucial tools for a forthcoming analysis on the existence of equilibria for the dynamical equations of slider bearings.

As we will see later, the results strongly depend on the shape function h_0. We will consider two situations:

(i) when h_0 vanishes on a segment of type $\{x_1 = d_1\}$ only, with $d_1 \in [a_1, b_1]$, which will be called "line contact case". In this case we will assume $h_0 \sim |x_1 - d_1|^\alpha$ in a neighborhood of $\{x_1 = d_1\}$, with $\alpha > 0$.

(ii) when h_0 vanishes only at a single point $d = (d_1, d_2)$ of $\bar{\Omega}$, which will be called "point contact case". We will assume $h_0 \sim |x-d|^\alpha$ in a neighborhood of $\{x = d\}$, with $\alpha > 0$.

In both the "line-contact case" and the "point-contact case" we obtain two types of results: (i) convergence of load and momenta to some finite limits which will be made precise in Section 3 and (ii) divergence to $+\infty$ of load and momenta. Problem (1.4) can be seen as a singular perturbation of the corresponding problem ($\epsilon = 0$) with small parameter ϵ. This kind of problem has been studied in [10, 2, 8, 6].

We can apply here singular perturbation results to obtain the convergence of p to the solution of the limit problem denoted p_0 in a weighted Sobolev space of type $H_0^1(\Omega, h_0^{2\delta_1}, h_0^{2\delta_2})$ (see Section 2 for the definition). This is not sufficient for the convergence of load and momenta; we also need a continuous embedding of $H_0^1(\Omega, h_0^{2\delta_1}, h_0^{2\delta_2})$ into $L^1(\Omega)$ which is obvious if δ_1 is not large.

This singular perturbation approach works only for $\alpha < 1$ in the "line-contact case" and for $\alpha < \frac{3}{2}$ in the "point-contact case". To have a well-posed limit problem we need a Poicaré-like inequality for weighted Sobolev spaces. This subject is well studied in the literature (see [3, 9, 4]). We prefer to give here a new elementary result (Lemma 2.2) well-adapted to our problem.

In cases $\alpha \geq 1$ for "line-contact" and $\alpha \geq \frac{3}{2}$ for "point-contact" (divergent cases) the singular perturbations results cited above are no longer applicable.

This part is more difficult and we use extensively the maximum principle in order to find an appropriate lower bound for p whose integral tends to infinity. This proves the divergence to infinity of the load and we also prove that this divergence, in the "line-contact case", is of order greater than $\epsilon^{2/\alpha-2}$ for $\alpha > 1$ and greater than $\log(1/\epsilon)$ for $\alpha = 1$. In the "point-contact case" the same result holds with α replaced by $\frac{2}{3}\alpha$.

In order to prove the divergence of momenta we also need to prove that p is bounded far from the annulation points of h_0. This result is again proved using the maximum principle, assuming h_0 to be a tensor product. We remark that in the "point-contact case" for $\alpha \in [\frac{3}{2}, 2[$ we have divergence of load and momenta while the limit problem of (1.4) exists in a weighted Sobolev space. This is because the continuous embedding of this space in $L^1(\Omega)$ does not hold.

In some cases the solution of (1.4) is negative, which does not correspond to the actual fluid behavior since cavitation takes place for $p < 0$. To account for cavitation, problem (P1) is replaced by the variational inequality [1]

$$\begin{gathered}\text{Find } p\in\mathcal{K}=\{v\in H_0^1(\Omega): v\geq 0\}\\ \int_\Omega (h_0(x)+\epsilon)^3\nabla p\nabla(\varphi-p)dx \geq \int_\Omega h_0\frac{\partial}{\partial x_1}(\varphi-p)\quad \forall\varphi\in\mathcal{K}\end{gathered}\tag{1.9}$$

The goal is also to find limits of g and $m_i, i=1,2$ (defined as in (1.5) and (1.7) with p now the solution of (1.9)) when ϵ goes to 0. We obtain the same kind of results as in the equation case.

The paper is organized as follows. In Section 2 we present some preliminary results concerning weighted Sobolev spaces, in particular the Poincaré-like inequality. In Section 3 we study the limits of load and momenta in the equation case (problem (1.4)) for the different cases cited above. Finally, Section 4 presents the same study in the inequality case (problem (1.9)).

2. Preliminaries

Let us consider $f_0, f_1\in C^0(\overline{\Omega})$ with $f_k>0$ a.e. $x\in\Omega, k=0,1$. We introduce the weighted Sobolev space

$$H^1(\Omega, f_0, f_1)$$

as the set of all measurable functions $\varphi=\varphi(x)$ defined on Ω with (generalized) derivatives $D^\alpha\varphi$ for $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2$ with $\alpha_1+\alpha_2\leq 1$ such that

$$\int_\Omega f_0(x)\varphi^2(x)dx+\int_\Omega f_1(x)|\nabla\varphi|^2(x)dx<\infty \tag{2.1}$$

$H^1(\Omega, f_0, f_1)$ is a pre-Hilbert space equipped with the scalar product

$$\big(\varphi_1,\varphi_2\big)_{H^1(\Omega,f_0,f_1)}=\int_\Omega f_0(x)\varphi_1(x)\varphi_2(x)dx+\int_\Omega f_1(x)\nabla\varphi_1(x)\cdot\nabla\varphi_2(x)dx\,.$$

Let $H_0^1(\Omega,f_0,f_1)$ be the closure of $\mathcal{D}(\Omega)$ with respect to the norm of $H^1(\Omega,f_0,f_1)$, which is a Hilbert space endowed with the same scalar product as $H^1(\Omega,f_0,f_1)$.

Remark 2.1. If $1/f_k\in L^1_{\rm loc}(\Omega), k=0,1$ then $H^1(\Omega,f_0,f_1)$ is a Hilbert space [7].

We have the following general Poincaré-like inequality.

Lemma 2.2. *Let $f\in C^0(\overline{\Omega})$ with $f>0$ a.e. $x\in\Omega$. Assume that a real $d_1\in[a_1,b_1]$ exists such that f is non-increasing in x_1 on $[a_1,d_1]\times[a_2,b_2]$ and non-decreasing in x_1 on $[d_1,b_1]\times[a_2,b_2]$, with the obvious convention that for $d_1=a_1$ (resp. $d_1=b_1$) the function f is only non-decreasing (resp. non-increasing). Then for any $\delta_1,\delta_2\in\mathbb{R}^+$ such that*

$$\begin{aligned}K=&\sup_{x_2\in[a_2,b_2]}\int_{a_1}^{d_1}\int_{a_1}^{x_1} f^{2(\delta_1-\delta_2)}(s,x_2)dsdx_1\\ &+\sup_{x_2\in[a_2,b_2]}\int_{d_1}^{b_1}\int_{x_1}^{b_1} f^{2(\delta_1-\delta_2)}(s,x_2)dsdx_1<\infty\end{aligned}$$

we have

$$\int_\Omega f^{2\delta_1}u^2\leq K\int_\Omega f^{2\delta_2}\big|\nabla u\big|^2,\quad \forall u\in H_0^1(\Omega,f^{2\delta_1},f^{2\delta_2})$$

Proof. For any $u \in \mathcal{D}(\Omega)$ we have for $x_1 < d_1$:

$$\begin{aligned} f^{\delta_1}(x_1,x_2)|u(x_1,x_2)| &\le f^{\delta_1}(x_1,x_2)\int_{a_1}^{x_1} \big|\frac{\partial u}{\partial x_1}(s,x_2)\big|ds \\ &\le \int_{a_1}^{x_1} f^{\delta_1}(s,x_2)\big|\frac{\partial u}{\partial x_1}(s,x_2)\big|ds, \end{aligned}$$

since f is x_1-non-increasing. We then have

$$\begin{aligned} &f^{2\delta_1}(x_1,x_2)u^2(x_1,x_2) \\ &\le \Big(\int_{a_1}^{x_1} f^{2(\delta_1-\delta_2)}(s,x_2)ds\Big)\Big(\int_{a_1}^{b_1} f^{2\delta_2}(x_1,x_2)\Big(\frac{\partial u}{\partial x_1}(x_1,x_2)\Big)^2 dx_1\Big) \quad \forall x_1 \le d_1 \end{aligned}$$

By integrating in x_1 on $[a_1,d_1]$ first and then in x_2 we obtain

$$\begin{aligned} &\int_{a_1}^{d_1}\int_{a_2}^{b_2} f^{2\delta_1}u^2dx \\ &\le \Big(\sup_{x_2\in[a_2,b_2]}\int_{a_1}^{d_1}\int_{a_1}^{x_1} f^{2(\delta_1-\delta_2)}(s,x_2)\,ds\,dx_1\Big)\Big(\int_\Omega f^{2\delta_2}\big(\frac{\partial u}{\partial x_1}\big)^2dx\Big) \end{aligned} \tag{2.2}$$

In the same manner, using the fact that f is x_1-non-decreasing on $[d_1,b_1]\times[a_2,b_2]$, we obtain

$$\begin{aligned} &\int_{d_1}^{b_1}\int_{a_2}^{b_2} f^{2\delta_1}u^2dx \\ &\le \Big(\sup_{x_2\in[a_2,b_2]}\int_{d_1}^{b_1}\int_{x_1}^{b_1} f^{2(\delta_1-\delta_2)}(s,x_2)\,ds\,dx_1\Big)\Big(\int_\Omega f^{2\delta_2}\big(\frac{\partial u}{\partial x_1}\big)^2dx\Big) \end{aligned} \tag{2.3}$$

Adding (2.2) and (2.3) we obtain the desired inequality and by a density argument we obtain the result. $\square$

Corollary 2.3. *For f,δ_1,δ_2 satisfying assumptions of Lemma 2.2 the semi-norm $\|f^{\delta_2}\nabla\cdot\|_{L^2(\Omega)}$ is a norm on $H_0^1(\Omega,f^{2\delta_1},f^{2\delta_2})$ and is equivalent to the norm of $H^1(\Omega,f^{2\delta_1},f^{2\delta_2})$.*

3. Asymptotic behavior of the equation case (problem (1.4))

In this section we set, for the sake of simplicity, $\Omega=]-1,0[^2$ and we suppose that

$$h_0\ge 0,\quad h_0\in C^0(\bar\Omega) \text{ and } h_0 \text{ is non increasing in } x_1. \tag{3.1}$$

We study the asymptotic behavior, as ϵ tends to 0, of the solution p of (1.4) which is non negative by the maximum principle. We introduce the following limit problem for any $\delta_1\in\mathbb{R}^+$:

$$\begin{gathered} \text{Find } p_0\in H_0^1(\Omega,h_0^{2\delta_1},h_0^3) \text{ such that} \\ \int_\Omega h_0^3\nabla p_0\nabla\varphi = \int_\Omega h_0\frac{\partial\varphi}{\partial x_1}\quad \forall\varphi\in H_0^1(\Omega,h_0^{2\delta_1},h_0^3) \end{gathered} \tag{3.2}$$

Proposition 3.1. *Suppose that $\int_\Omega \frac{dx}{h_0}<+\infty$ and $\delta_1\in\mathbb{R}^+$ is such that*

$$\sup_{x_2\in[-1,0]}\int_{-1}^0\int_{-1}^{x_1} h_0^{2\delta_1-3}(s,x_2)\,ds\,dx_1<+\infty$$

Then problem (3.2) *admits a unique solution* $p_0 \in H_0^1(\Omega; h_0^{2\delta_1}, h_0^3)$ *which is independent on* δ_1.

Proof. From Lemma 2.2 with $d_1 = 0$ and $\delta_2 = \frac{3}{2}$ we deduce that the seminorm $\|h_0^3 \nabla \cdot\|_{L^2(\Omega)}$ is a norm in $H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$ equivalent to the norm in $H^1(\Omega, h_0^{2\delta_1}, h_0^3)$. On the other hand the application

$$\varphi \in H_0^1(\Omega, h_0^{2\delta_1}, h_0^3) \to \int_\Omega h_0 \frac{\partial \varphi}{\partial x_1} dx \in \mathbb{R}$$

is in the dual of $H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$ since we have

$$\Big| \int_\Omega h_0 \frac{\partial \varphi}{\partial x_1} \Big| \leq \Big(\int_\Omega \frac{dx}{h_0} \Big)^{1/2} \Big[\int_\Omega h_0^3 (\frac{\partial \varphi}{\partial x_1})^2 \Big]^{1/2}.$$

By Lax-Milgram theorem we have classically the existence and the uniqueness for any fixed $\delta_1 \in \mathbb{R}^+$. To prove the independence of p_0 with respect to δ_1 we consider p_0^1 and p_0^2 two solutions corresponding respectively to δ_1^1 and δ_1^2 with $\delta_1^2 < \delta_1^1$. We remark that $H_0^1(\Omega, h_0^{2\delta_1^2}, h_0^3) \subset H_0^1(\Omega, h_0^{2\delta_1^1}, h_0^3)$ with dense and continuous embedding. Then p_0^2 satisfies by density the same problem as p_0^1 which by uniqueness gives the result. □

Problem (1.4) can be seen as a singular perturbation of problem (3.2) with the small parameter ϵ. This kind of problem has been studied in [10, 2, 8, 6]. Let us recall a simplified version of a result given in [6] for a more general problem, which allows us to obtain the convergence of p to p_0.

Lemma 3.2. *Let* V, W, H *be three Hilbert spaces, with continuous embeddings* $V \subset W \subseteq H$ *and* V *dense in* W *and in* H. *Let* $b(\epsilon; u, v), 0 < \epsilon \leq \epsilon_0$, *be a sequence of continuous bilinear forms on* V, $b(u, v)$ *a continuous bilinear form on* W *and* $f \in H$. *Under the hypotheses*

(1) $\epsilon \to b(\epsilon; u, v)$ *is continuous and* $\lim_{\epsilon \to 0} b(\epsilon; u, v) = b(u, v)$ $\forall (u, v) \in V \times V$
(2) $\exists \alpha(\epsilon) > 0$ *with* $\alpha(\epsilon) \to 0$ *and* $\beta > 0$ *such that* $b(\epsilon; u, u) \geq \alpha(\epsilon)\|u\|_V^2 + \beta\|u\|_W^2$, $\forall u \in V$
(3) $\exists \gamma > 0$ *such that* $b(u, u) \geq \gamma\|u\|_W^2, \forall u \in W$
(4) *For any sequence* $w_\epsilon \in V$ *for which* $|b(\epsilon; w_\epsilon, w_\epsilon)|$ *is bounded,* $b(\epsilon; w_\epsilon, v) - b(w_\epsilon, v) \to 0, \forall v \in V$
(5) $\exists \delta(\epsilon)$ *with* $\delta(\epsilon) \to 0$ *such that* $b(\epsilon; v, v) - b(v, v) + \delta(\epsilon) b(v, v) \geq 0$

the solution u_ϵ *of the problem*

$$b(\epsilon; u_\epsilon, v) = (f, v) \quad \forall v \in V, \epsilon \leq \epsilon_0$$

converges strongly in W *to the solution* u *of the problem*

$$b(u, v) = (f, v) \quad \forall v \in W.$$

Proposition 3.3. *Under the hypotheses of Proposition 3.1, the solution* p *of* (1.4) *converges, as* ϵ *tends to* 0, *to the solution* p_0 *of* (3.2), *strongly in* $H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$.

Proof. It suffices to apply Lemma 3.2 with $V = H_0^1(\Omega)$, $W = H = H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$ endowed with the norm $\|h_0^3 \nabla \cdot\|_{L^2(\Omega)}$, $u_\epsilon = p$, and

$$b(\epsilon; u, v) = \int_\Omega (h_0 + \epsilon)^3 \nabla u \nabla v dx, \ b(u, v) = \int_\Omega h_0^3 \nabla u \nabla v \, dx.$$

We choose f in the following manner: Since the application $v \in W \to \int_\Omega h_0 \frac{\partial v}{\partial x_1}$ is an element of W', from Riesz's theorem there exists f in W such that

$$\int_\Omega h_0 \frac{\partial v}{\partial x_1} = (f, v)_W, \quad \forall v \in W.$$

Assumptions (1)-(3) and (5) are immediately verified with $\alpha(\epsilon) = \epsilon^3$, $\beta = 1$, $\gamma = 1$ and $\delta(\epsilon) = 0$.

Let us now verify assumption (4). Let w_ϵ be a sequence in $H_0^1(\Omega)$ for which $\int_\Omega (h_0 + \epsilon)^3 \nabla(w_\epsilon)^2$ is bounded, so $\|w_\epsilon\|_W$ is bounded. For all $v \in H_0^1(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega (h_0+\epsilon)^3 \nabla w_\epsilon \nabla v - \int_\Omega h_0^3 \nabla w_\epsilon \nabla v \\
&= \int_\Omega \Big[(h_0+\epsilon)^{3/2} - h_0^{3/2}\Big]\big(h_0+\epsilon\big)^{3/2} \nabla w_\epsilon \nabla v + \int_\Omega \Big[(h_0+\epsilon)^{3/2} - h_0^{3/2}\Big] h_0^{3/2} \nabla w_\epsilon \nabla v
\end{aligned}$$

Since $(h_0 + \epsilon)^{3/2} - h_0^{3/2} \to 0$ in $L^\infty(\Omega)$ we easily obtain the result. □

For simplicity we shall distinguish here two different situations: (i) the function h_0 vanishes on the entire segment $x_1 = 0$ namely the line-contact case, (ii) the function h_0 vanishes on a unique interior point, supposed to be (0,0), namely point-contact case.

3.1. **Line-contact case.**

We assume in this section that, in addition to hypothesis (3.1)

$$\begin{aligned}
h_0(x) = 0 &\quad \text{for } x_1 = 0 \\
h_0(x) > 0 &\quad \text{for } x_1 \neq 0
\end{aligned}$$

We shall prove, under some supplementary hypotheses, that:
For $\int_\Omega \frac{dx}{h_0(x)} < +\infty$ the load and momenta have finite limits (paragraph 3.1.1).
For $\int_\Omega \frac{dx}{h_0(x)} = +\infty$ the load and momenta have infinite limits (paragraph 3.1.2).

Remark 3.4. If h_0 behaves as $(-x_1)^\alpha$ with $\alpha > 0$ in a neighbourhood of $x_1 = 0$, then $\int_\Omega \frac{dx}{h_0(x)}$ is finite for $\alpha < 1$ and infinite for $\alpha \geq 1$.

3.1.1. *Finite limit case.*

Theorem 3.5. *If there exists $\alpha \in]0, 1[$ and $m_0 > 0$ such that*

$$h_0(x) \geq m_0(-x_1)^\alpha \quad \forall x \in \Omega$$

then, for any $(x_1^0, x_2^0) \in \Omega$ and $\epsilon \to 0$, we have

$$\int_\Omega p\, dx \to \int_\Omega p_0 dx, \quad \int_\Omega (x_k - x_k^0) p \to \int_\Omega (x_k - x_k^0) p_0, \quad k = 1, 2$$

where p is the solution of (1.4) *and p_0 the solution of problem* (3.2)

Proof. Since $\alpha \in]0, 1[$, the hypotheses in Propositions 3.1 and 3.3 are satisfied with $\delta_1 = \frac{1}{2}$. Then there exists $p_0 \in H_0^1(\Omega, h_0, h_0^3)$ unique solution of (3.2) with $\delta_1 = \frac{1}{2}$ such that $p \to p_0$ strongly in $H_0^1(\Omega, h_0, h_0^3)$. On the other hand, all $u \in H_0^1(\Omega, h_0, h_0^3)$, we have

$$\int_\Omega |u| dx \leq \Big(\int_\Omega \frac{dx}{h_0}\Big)^{1/2} \Big(\int_\Omega h_0 u^2 dx\Big)^{1/2}$$

that is, the continuous embedding of $H_0^1(\Omega, h_0, h_0^3)$ in $L^1(\Omega)$ holds. This implies that p converges to p_0 strongly in $L^1(\Omega)$ which ends the proof. □

3.1.2. *Infinite-limit case.* For any $\delta > 0$, we shall define

$$\Omega_\delta =]-\delta, 0[\times]-1, 0[$$

In this paragraph we need the following supplementary hypothesis: There exist $\delta_0 \in]0, \frac{1}{2}[$, $\alpha \geq 1$ and $0 < m_1 \leq M_1$ such that

$$h_0 \in W^{1,\infty}(\Omega) \quad \text{and} \quad \alpha m_1(-x_1)^{\alpha-1} \leq -\frac{\partial h_0}{\partial x_1} \leq \alpha M_1(-x_1)^{\alpha-1}, \quad \forall x \in \Omega_{2\delta_0} \tag{3.3}$$

Lemma 3.6. *For any $\phi \in C^2([-2\delta_0, 0])$ with $\phi(-2\delta_0) = 0$ and $q_2 \in C^2([-1, 0])$ with $q_2(-1) = q_2(0) = 0$, there is $c > 0$ small enough such that:*

$$p(x) \geq c q_1(x_1)\phi(x_1)q_2(x_2) \quad \forall x = (x_1, x_2) \in \Omega_{2\delta_0}$$

with

$$q_1(x_1) = \frac{(-x_1)^{\alpha+1}}{\big(M_1(-x_1)^\alpha + \epsilon)^3}$$

Proof. We apply the maximum principle. Since the function $q_1(x_1)\phi(x_1)q_2(x_2)$, vanishes on $\partial\Omega_{2\delta_0}$, it suffices to prove

$$-c\nabla \cdot \Big[(a + h_0)^3 \nabla(q_1(x_1)\phi(x_1)q_2(x_2))\Big] \leq -\frac{\partial h_0}{\partial x_1} \quad \forall x \in \Omega_{2\delta_0}$$

Dividing by $(-x_1)^{\alpha-1}$ and using (3.3) it suffices to prove the existence of a positive constant K_1 independent of ϵ such that

$$-\frac{\nabla \cdot \big[(a + h_0)^3 \nabla(q_1(x_1)\phi(x_1)q_2(x_2))\big]}{(-x_1)^{\alpha-1}} \leq K_1$$

which is equivalent to

$$\begin{aligned} &3(\epsilon + h_0)^2 q_2 \phi q_1^{'} \big[\frac{-\frac{\partial h_0}{\partial x_1}}{(-x_1)^{\alpha-1}}\big] - (\epsilon + h_0)^3 q_2 \phi \frac{q_1''}{(-x_1)^{\alpha-1}} - 2(\epsilon + h_0)^3 q_2 \phi^{'} \frac{q_1^{'}}{(-x_1)^{\alpha-1}} \\ &+ 3(\epsilon + h_0)^2 q_1 q_2 \phi^{'} \big[\frac{-\frac{\partial h_0}{\partial x_1}}{(-x_1)^{\alpha-1}}\big] - (\epsilon + h_0)^3 q_2 \phi'' \frac{q_1}{(-x_1)^{\alpha-1}} \\ &- (\epsilon + h_0)^3 \phi q_2'' \frac{q_1}{(-x_1)^{\alpha-1}} - 3(\epsilon + h_0)^2 q_1 \phi q_2^{'} \big[\frac{\frac{\partial h_0}{\partial x_2}}{(-x_1)^{\alpha-1}}\big] \leq K_1 \end{aligned} \tag{3.4}$$

On the other hand, it is easy to see that a constant K_2 independent of ϵ exists such that:

$$\begin{aligned} |q_1^{'}(x_1)| &\leq K_2 \frac{(-x_1)^\alpha}{\big(M_1(-x_1)^\alpha + \epsilon\big)^3} \quad \forall x_1 \in [-2\delta_0, 0], \\ |q_1''(x_1)| &\leq K_2 \frac{(-x_1)^{\alpha-1}}{\big(M_1(-x_1)^\alpha + \epsilon\big)^3} \quad \forall x_1 \in [-2\delta_0, 0]. \end{aligned}$$

Integrating (3.3) on $[x_1, 0]$ we obtain

$$m_1(-x_1)^\alpha \leq h_0(x) \leq M_1(-x_1)^\alpha \quad \forall x \in \Omega_{2\delta_0} \tag{3.5}$$

Using the above inequalities we obtain (3.4) which concludes the proof. $\square$

Now we are able to give a first result in the case $\alpha \geq 1$.

Theorem 3.7. *For $\alpha \geq 1$ we have $\int_\Omega p\,dx \to +\infty$ as $\epsilon \to 0$. Moreover there exists $K > 0$ such that for ϵ small enough we have*

$$\int_\Omega p\,dx \geq K\epsilon^{\frac{2}{\alpha}-2} \quad \text{for } \alpha > 1,$$
$$\int_\Omega p\,dx \geq K\log(\frac{1}{\epsilon}) \quad \text{for } \alpha = 1.$$

Proof. We apply Lemma 3.6 with $\phi = 1$ on $[-\delta_0, 0]$ and $q_2 \geq 0$ with

$$q_2 \in C^2([-1,0]) \cap H_0^1(]-1,0[)$$

and $\int_{-1}^0 q_2(x_2) > 0$. We deduce that there exists a constant $c > 0$ independent of ϵ such that

$$p(x_1,x_2) \geq cq_2(x_2)\phi(x_1)q_1(x_1), \quad \forall x \in \Omega_{2\delta_0}$$

with q_1 given in Lemma 3.6. Taking into account the fact that p is non negative on all of Ω we obtain

$$\int_\Omega p\,dx \geq c\int_{\Omega_{\delta_0}} q_2(x_2)q_1(x_1)dx = c\int_{-1}^0 q_2(x_2)dx_2 \int_{-\delta_0}^0 q_1(x_1)dx_1\,. \tag{3.6}$$

On the other hand, for $\alpha > 1$ and ϵ small enough we have

$$\int_{-\delta_0}^0 q_1(x_1)dx_1 \geq \int_{-\epsilon^{1/\alpha}}^0 q_1(x_1)dx_1 \geq \frac{\epsilon^{1+\frac{2}{\alpha}}}{(M_1+1)^3\epsilon^3(\alpha+2)}$$

which, with (3.6), gives the result for $\alpha > 1$.

For $\alpha = 1$ an elementary calculation gives

$$\begin{aligned}\int_{-\delta_0}^0 q_1(x_1)dx_1 =& \frac{-1}{3M_1^3}\int_{-\delta_0}^0 \frac{\frac{d}{dx_1}\big((-M_1x_1+\epsilon)^3\big)}{(-M_1x_1+\epsilon)^3}dx_1 \\ &- \frac{2\epsilon}{M_1^2}\int_{-\delta_0}^0 \frac{-M_1x_1}{(-M_1x_1+\epsilon)^3}dx_1 - \frac{\epsilon^2}{M_1^2}\int_{-\delta_0}^0 \frac{dx_1}{(-M_1x_1+\epsilon)^3}\end{aligned}$$

We easily prove that the last two terms of the right-hand side are bounded by a constant independent of ϵ. With the help of (3.6) we obtain the result. $\square$

In the particular case when h_0 is symmetric in the x_2 direction we have the following asymptotic behavior of the x_2-momentum.

Theorem 3.8. *Suppose that h_0 is symmetric in x_2 with respect to $x_2 = -1/2$ and $\alpha \geq 1$. Then for $\epsilon \to 0$, we have*

$$\int_\Omega (x_2 - x_2^0)p\,dx \to +\infty \quad \text{if } x_2^0 \in]-1, -\frac{1}{2}[,$$
$$\int_\Omega (x_2 - x_2^0)p\,dx \to -\infty \quad \text{if } x_2^0 \in]-\frac{1}{2}, 0[,$$
$$\int_\Omega (x_2 - x_2^0)p\,dx = 0 \quad \text{if } x_2^0 = -\frac{1}{2}\,.$$

Proof. By symmetry of h_0 the function $\bar{p}(x_1,x_2) = p(x_1,-1-x_2)$ is also a solution of problem (1.4), so that by uniqueness $p = \bar{p}$. We then have

$$\int_\Omega (x_2 - x_2^0)p\,dx = \int_\Omega (x_2 + \frac{1}{2})p\,dx - (x_2^0 + \frac{1}{2})\int_\Omega p\,dx\,.$$

The first integral of the right-hand side is equal to 0 by symmetry. Then we have the result by Theorem 3.7. □

Now we shall give the behavior of the x_1-moment in the particular case when h_0 is a tensor product. This is often the case in practice for the contact line. We begin by the following lemma which means that p is bounded uniformly in ϵ far from the line contact.

Lemma 3.9. *Suppose that* $h_0(x_1, x_2) = a_1(x_1)a_2(x_2)$ *with* $a_2 \in C^0([-1, 0])$, $a_2 > 0$, $a_1 \in H^1(]-1, 0[)$, $a_1 \geq 0$, $\frac{\partial a_1}{\partial x_1} \leq 0$. *Then there exists* $C > 0$ *such that*

$$p(x) \leq C \int_{-1}^{x_1} \frac{ds}{(\bar{h}_0(s) + \epsilon)^2}, \quad \forall x \in \Omega$$

where $\bar{h}_0(x_1) = a_{2m}a_1(x_1)$ *with* $a_{2m} = \min_{x_2 \in [-1,0]} a_2(x_2)$

Proof. We apply again the maximum principle. Let $q(x_1) = C \int_{-1}^{x_1} 1/(\bar{h}_0(s) + \epsilon)^2 \, ds$. Since $q \geq 0$ and $p = 0$ on $\partial\Omega$ it suffices to show that for $C > 0$ large enough we have

$$-\frac{\partial}{\partial x_1}\big[(h_0 + \epsilon)^3 \frac{\partial q}{\partial x_1}\big] \geq -\frac{\partial h_0}{\partial x_1} \quad \forall x \in \Omega,$$

that is

$$-C\frac{\partial h_0}{\partial x_1}\frac{(h_0 + \epsilon)^3}{(\bar{h}_0 + \epsilon)^3}E(x) \geq -\frac{\partial h_0}{\partial x_1} \tag{3.7}$$

with

$$E(x) = 3\frac{\bar{h}_0 + \epsilon}{h_0 + \epsilon} - 2\frac{a_{2m}}{a_2(x_1)}$$

Now we have

$$E(x) = \frac{a_{2m}}{a_2(x_2)}\big[3\frac{a_1 + \frac{\epsilon}{a_{2m}}}{a_1 + \frac{\epsilon}{a_2}} - 2\big] = \frac{a_{2m}}{a_2(x_2)}\frac{a_1 + \epsilon(\frac{3}{a_{2m}} - \frac{2}{a_2})}{a_1 + \frac{\epsilon}{a_2}}\,.$$

We easily obtain

$$E(x) \geq \frac{a_{2m}}{a_{2M}} \quad \text{with} \quad a_{2M} = \max_{x_2 \in [-1,0]} a_2(x_2)$$

which proves (3.7) by taking $C \geq a_{2M}/a_{2m}$ since $h_0 + \epsilon \geq \bar{h}_0 + \epsilon$ and $\frac{\partial h_0}{\partial x_1} \leq 0$. □

Theorem 3.10. *Assuming* (3.3) *to hold, and assuming further that the function* h_0 *is of the form* $h_0(x) = g_1(x_1)g_2(x_2)$ *with* $g_1, g_2 \in W^{1,\infty}(]-1, 0[)$, *then*

$$\int_\Omega (x_1 - x_1^0)p \, dx \to +\infty \quad \textit{as } \epsilon \to 0 \quad \textit{for any } x_1^0 \in]-1, 0[\,.$$

Proof. We choose $\delta > 0$ such that $-\delta > \max(x_1^0, -\delta_0)$. Then we have

$$\int_\Omega (x_1 - x_1^0)p \, dx \geq (-\delta - x_1^0)\int_{\Omega_\delta} p \, dx + \int_{-1}^{-\delta}\int_{-1}^{0}(x_1 - x_1^0)p \, dx$$

We prove as in Theorem 3.7 that the first integral of the right-hand side tends to $+\infty$ since $-\delta - x_1^0 > 0$. Applying Lemma 3.9 with $a_1 = (-x_1)^\alpha g_1$ and $a_2 = g_2$ we easily prove that the second integral is bounded by a constant independent of ϵ. We then have the result. □

Remark 3.11. An interesting open question is to obtain an upper bound for p which allows to say that p is bounded uniformly in ϵ far from $x_1 = 0$, without the global hypotheses that h_0 is a tensor product.

3.2. **Point-contact case.** We now assume that h_0 is, in a neighbourhood of $x = 0$, equivalent to $|x|^\alpha$ with $\alpha > 0$, where $|\cdot|$ denotes the Euclidean norm. For simplicity we make here the following (non-essential) hypothesis

$$h_0(x) = |x|^\alpha h_1(x)$$

with $h_1 \in W^{1,\infty}(\Omega)$ and $h_1 > 0$. We denote

$$m = \inf_{x\in\bar\Omega} h_1(x), \quad M = \sup_{x\in\bar\Omega} h_1(x)$$

We shall prove that for $0 < \alpha < \frac{3}{2}$ (paragraph 3.2.1) we have finite limits of load and momenta while for $\alpha \geq \frac{3}{2}$ (paragraph 3.2.2) they tend to $+\infty$. We begin by the following existence, uniqueness and convergence results.

Proposition 3.12.

- *If $0 < \alpha < \frac{4}{3}$ then for any $0 < \delta_1 < \frac{3}{4}$ there is a unique solution $p_0 \in H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$ of* (3.2) *and $p \to p_0$ in $H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$.*
- *If $\frac{4}{3} \leq \alpha < 2$ then for any $\delta_1 > \frac{3}{2} - \frac{1}{\alpha}$ there is a unique solution $p_0 \in H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$ of* (3.2) *and $p \to p_0$ in $H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$.*

Proof. The first hypothesis of Proposition 3.1 is obvious. The second one is evident for $\frac{4}{3} \leq \alpha < 2$ and $\delta_1 \geq \frac{3}{2} - \frac{1}{\alpha}$. For $0 < \alpha < \frac{4}{3}$ and $0 < \delta_1 < \frac{3}{4}$ we use the inequality $\sqrt{s^2 + x_2^2} \geq |s|$ and the result is immediate. □

3.2.1. *Finite-limit case ($\alpha < \frac{3}{2}$).* In the following we prove that the limits of load and momenta are finite for $\alpha < \frac{4}{3}$ without supplementary hypotheses. For $\frac{4}{3} \leq \alpha < \frac{3}{2}$ we prove the same result but adding a restrictive supplementary assumption on h_0.

Theorem 3.13. *For $0 < \alpha < \frac{4}{3}$ we have for any $(x_1^0, x_2^0) \in \Omega$:*

$$\int_\Omega p \to \int_\Omega p_0, \quad \int_\Omega (x_k - x_k^0)p \to \int_\Omega (x_k - x_k^0)p_0; \quad k = 1,2$$

with p_0 solution of the limit problem (3.2)*.*

Proof. From Proposition 3.12 we have $p \to p_0$ in $H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$-strongly for any δ_1 such that

$$0 < \delta_1 < \frac{3}{4} \tag{3.8}$$

On the other hand we remark that if

$$0 < \delta_1 < \frac{1}{\alpha} \tag{3.9}$$

then $\int_\Omega h_0^{-2\delta_1}$ is finite, which by Cauchy-Schwartz inequality gives the continuous embedding of $H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$ in $L^1(\Omega)$. This will prove the three desired convergence. Now the existence of at least a δ_1 satisfying (3.8) and (3.9) is assured if $3/4 < 1/\alpha$ which is equivalent to $\alpha < 4/3$. □

Theorem 3.14. *For $4/3 \leq \alpha < 3/2$, under the supplementary hypothesis $\triangle h_0 \geq 0$ on Ω we have the same convergence as in Theorem 3.13.*

Proof. We need here an estimation of p in a stronger norm than $\|h_0^{3/2}\nabla\cdot\|_{L^2(\Omega)}$ in order to obtain $\|h_0^{\delta_1}p\|_{L^2(\Omega)}$ bounded with a better parameter δ_1 than in Theorem 3.13. We prove in the following that $\|h_0^{(3-\delta)/2}\nabla p\|_{L^2(\Omega)}$ is bounded for an appropriate $\delta>0$. Taking $\varphi=\big(h_0+\epsilon\big)^{-\delta}p$ with $0<\delta<\frac{2}{\alpha}-1\leq\frac{1}{2}$ as a test function in the variational formulation of (1.4) we obtain

$$\int_\Omega\big(h_0+\epsilon\big)^{3-\delta}\big|\nabla p\big|^2=\delta\int_\Omega\big(h_0+\epsilon\big)^{2-\delta}\nabla h_0p\nabla p-\int_\Omega\frac{\partial h_0}{\partial x_1}\big(h_0+\epsilon\big)^{-\delta}p=:I_1+I_2 \tag{3.10}$$

Using Green's formula we deduce

$$\begin{aligned}I_1&=-\frac{\delta}{2}\int_\Omega\nabla\cdot\Big[\big(h_0+\epsilon\big)^{2-\delta}\nabla h_0\Big]p^2\\&=-\frac{\delta}{2}\int_\Omega\Big[(2-\delta)\big(h_0+\epsilon\big)^{1-\delta}\big|\nabla h_0\big|^2+\big(h_0+\epsilon\big)^{2-\delta}\Delta h_0\Big]p^2\end{aligned}$$

which is negative, thanks to the additional hypothesis $\Delta h_0\geq0$. On the other hand

$$\begin{aligned}|I_2|&\leq\frac{1}{1-\delta}\int_\Omega\big|(h_0+\epsilon)^{1-\delta}\frac{\partial p}{\partial x_1}\big|\\&=\frac{1}{1-\delta}\int_\Omega(h_0+\epsilon)^{(3-\delta)/2}\left|\frac{\partial p}{\partial x_1}\right|(h_0+\epsilon)^{-(1+\delta)/2}\\&\leq\frac{1}{1-\delta}\Big(\int_\Omega(h_0+\epsilon)^{3-\delta}\big|\frac{\partial p}{\partial x_1}\big|^2\Big)^{1/2}\Big(\int_\Omega(h_0+\epsilon)^{-1-\delta}\Big)^{1/2}\\&\leq\frac{1}{2}\int_\Omega(h_0+\epsilon)^{3-\delta}\big|\frac{\partial p}{\partial x_1}\big|^2dx+\frac{1}{2}\frac{1}{(1-\delta)^2}\int_\Omega\frac{dx}{(h_0+\epsilon)^{1+\delta}}dx\end{aligned}$$

The last integral of the above inequality is bounded uniformly in ϵ due to the hypotheses on δ. We deduce from (3.10) that

$$\Big(\int_\Omega h_0^{3-\delta}\big|\nabla p\big|^2dx\Big)^{1/2}\leq C$$

Applying Lemma 2.2 with $f=h_0,\delta_2=\frac{3}{2}-\frac{\delta}{2}$ and $\delta_1>\frac{3}{2}-\frac{\delta}{2}-\frac{1}{\alpha}$ we deduce that p is bounded in $H_0^1(\Omega,h_0^{2\delta_1},h_0^{3-\delta})$. We then infer the existence of $\xi\in H_0^1(\Omega,h_0^{2\delta_1},h_0^{3-\delta})$ and of a subsequence of ϵ such that $p\to\xi$ weakly in $H_0^1(\Omega,h_0^{2\delta_1},h_0^{3-\delta})$. From the continuous embedding of $H_0^1(\Omega,h_0^{2\delta_1},h_0^{3-\delta})$ in $H_0^1(\Omega,h_0^{2\delta_1},h_0^{3})$ and by identification and uniqueness of p_0 we deduce that $p\to p_0$ weakly in $H_0^1(\Omega,h_0^{2\delta_1},h_0^{3-\delta})$ for the entire sequence.

Choosing now $\delta=\frac{2}{\alpha}-1-\eta,\delta_1=\frac{3}{2}-\frac{\delta}{2}-\frac{1}{\alpha}+\frac{\eta}{2}=2-\frac{2}{\alpha}+\eta$ with $0<\eta<\frac{3}{\alpha}-2$ we obtain $\int_\Omega h_0^{-2\delta_1}<+\infty$ which implies the continuous embedding of $H_0^1(\Omega,h_0^{2\delta_1},h_0^{3-\delta})$ in $L^1(\Omega)$.

We then obtain $p\to p_0$ weakly in $L^1(\Omega)$ which gives the desired convergence. $\square$

3.2.2. *Infinite-limit case*($\alpha\geq3/2$). In this paragraph we use the polar coordinates r,θ:

$$x_1=r\cos\theta,\quad x_2=r\sin\theta$$

and we denote $\tilde\Omega$, the image of Ω by this change of variables. For simplicity notations we use the same notation as in cartesian coordinates (for example $h_0(r,\theta)$ means

$h_0(r\cos\theta, r\sin\theta))$. We set:

$$\tilde{\Omega}_r =]0, r[\times] -\pi, -\frac{\pi}{2}[\subset \tilde{\Omega}, \quad \forall r \in]0,1[$$

The problem (1.4) becomes

$$\begin{gathered}\frac{\partial}{\partial r}\big[(a+h_0(r,\theta))^3 r\frac{\partial p}{dr}\big] + \frac{\partial}{\partial\theta}\big[\frac{(a+h_0(r,\theta))^3}{r}\frac{\partial p}{\partial\theta}\big]\\ = r\frac{\partial h_0}{\partial r}\cos\theta - \sin\theta\frac{\partial h_0}{\partial\theta} \quad \text{in } \tilde{\Omega}\\ p = 0 \quad \text{in } \partial\tilde{\Omega}\,.\end{gathered}\tag{3.11}$$

Let us remark that from the relation $h_0 = r^\alpha h_1(r,\theta)$ there exists a positive constant $K > 0$ and $r_1 \in]0,1[$ such that

$$\frac{\partial h_0}{\partial r} \geq K r^{\alpha-1}, \quad \forall (r,\theta) \in \tilde{\Omega}_{r_1}\,. \tag{3.12}$$

We recall that h_0 is non-increasing in x_1 which is equivalent in polar coordinates to

$$r\frac{\partial h_0}{\partial r}\cos\theta \leq \sin\theta\frac{\partial h_0}{\partial\theta} \quad \text{ on } \tilde{\Omega} \tag{3.13}$$

We need here the following supplementary local condition: There exists $r_2 > 0$ and $\beta \in]0,1[$ such that

$$\beta r\frac{\partial h_0}{\partial r}\cos\theta \leq \sin\theta\frac{\partial h_0}{\partial\theta} \quad \text{ on } \tilde{\Omega}_{r_2}\,. \tag{3.14}$$

Remark 3.15. Hypothesis (3.14) is a little stronger locally than (3.13) and is true if for example $\frac{\partial h_0}{\partial\theta} \leq 0$ locally (in particular if h_0 is radial) since $\frac{\partial h_0}{\partial r} > 0$ locally and $\cos\theta \leq 0$

We now give an analog of Lemma 3.6 in the line-contact case.

Lemma 3.16. *Suppose that* (3.14) *is fulfilled and set* $r_0 = \min\{r_1, r_2\}$. *Then for any* $\phi \in C^2([0, r_0])$ *with* $\phi(r_0) = 0$, *a constant* $c > 0$ *exists such that*

$$p(r,\theta) \geq c q_1(r)\phi(r) q_2(\theta) \quad \forall (r,\theta) \in \tilde{\Omega}_{r_0}$$

with

$$q_1(r) = \frac{r^{\alpha+1}}{(Mr^\alpha + \epsilon)^3}, \quad \textit{and } q_2(\theta) = \cos^3\theta\sin\theta$$

Proof. It suffices to prove the inequality

$$\frac{\partial}{\partial r}\Big[(\epsilon+h_0)^3 r\frac{\partial}{\partial r}\big(cq_1q_2\phi\big)\Big] + \frac{\partial}{\partial\theta}\Big[\frac{(\epsilon+h_0)^3}{r}\frac{\partial}{\partial\theta}\big(cq_1q_2\phi\big)\Big] \geq r\frac{\partial h_0}{\partial r}\cos\theta - \sin\theta\frac{\partial h_0}{\partial\theta} \tag{3.15}$$

From (3.14), we have

$$r\frac{\partial h_0}{\partial r}\cos\theta - \sin\theta\frac{\partial h_0}{\partial\theta} \leq (1-\beta) r\frac{\partial h_0}{\partial r}\cos\theta\,.$$

Carrying out the differentiations in the left-hand side of (3.15) and dividing by $r\frac{\partial h_0}{\partial r}\cos\theta$, we obtain the following inequality in $\tilde{\Omega}_{r_0}$,

$$\begin{aligned}
&\frac{q_2(\theta)}{\cos\theta}\phi\Big[3(\epsilon+h_0)^2 q_1^{'}(r)+(a+h_0)^2 q_1^{'}(r)\frac{(\epsilon+h_0)}{r}\Big(\frac{\partial h_0}{\partial r}\Big)^{-1}\\
&+(\epsilon+h_0)^3 q_1''(r)\Big(\frac{\partial h_0}{\partial r}\Big)^{-1}\Big]\\
&+\frac{q_2(\theta)}{\cos\theta}\phi^{'}(r)\Big[3\big(\epsilon+h_0\big)^2 q_1+\big(\epsilon+h_0\big)^3\frac{q_1}{r}\Big(\frac{\partial h_0}{\partial r}\Big)^{-1}+2\big(\epsilon+h_0\big)^3 q_1^{'}(r)\Big(\frac{\partial h_0}{\partial r}\Big)^{-1}\Big]\\
&+\frac{q_2(\theta)}{cos\theta}(\epsilon+h_0)^3 q_1\phi''\Big(\frac{\partial h_0}{\partial r}\Big)^{-1}+3\frac{q_2^{'}(\theta)}{\cos\theta}\phi(\epsilon+h_0)^2 q_1\frac{\partial h_0}{\partial\theta}\frac{1}{r^2}\Big(\frac{\partial h_0}{\partial r}\Big)^{-1}\\
&+\frac{q_2''}{\cos\theta}\phi\frac{(\epsilon+h_0)^3 q_1}{r^2}\Big(\frac{\partial h_0}{\partial r}\Big)^{-1}\\
&\le\frac{1-\beta}{c}.
\end{aligned} \tag{3.16}$$

Remark also that there is a constant $K_1>0$ such that

$$|q_1^{'}(r)|\le K_1\frac{r^\alpha}{\big(Mr^\alpha+\epsilon\big)^3},\quad |q_1''(r)|\le K_1\frac{r^{\alpha-1}}{\big(Mr^\alpha+\epsilon\big)^3} \tag{3.17}$$

Using now (3.12), (3.17) and the expression of q_2 we obtain that the absolute value of the left-hand side of (3.16) is bounded by a constant. Taking c small enough we obtain the result. $\square$

Theorem 3.17. *Under hypothesis* (3.14) *we have* $\int_\Omega p\,dx\to+\infty$ *for* $\epsilon\to 0$ *Moreover there exists* $K>0$ *such that for* ϵ *small enough we have*

$$\begin{gathered}
\int_\Omega p\,dx\ge K\epsilon^{\frac{3}{\alpha}-2}\quad \text{for } \alpha>\frac{3}{2},\\
\int_\Omega p\,dx\ge K\log(\frac{1}{\epsilon})\quad \text{for } \alpha=\frac{3}{2}\,.
\end{gathered}$$

Proof. Using polar coordinates and the non-negativity of p we have

$$\int_\Omega pdx\ge\int_{\tilde{\Omega}_\rho} rp(r,\theta)\,dr\,d\theta,\quad \forall\rho\in]0,1]$$

Applying Lemma 3.16 with $\phi=1$ on $[0,\frac{r_0}{2}]$ we show that there exists a $c>0$ such that

$$\int_\Omega pdx\ge c\int_{-\pi}^{-\pi/2} q_2(\theta)d\theta\cdot\int_0^{r_0/2} rq_1(r)dr\,.$$

As in the proof of Theorem 3.7 with some elementary computations we obtain the result. $\square$

Theorem 3.18. *Under hypothesis* (3.14) *if moreover* $h_1(r,\theta)=g_1(r)g_2(\theta)$ *with* $g_1\in C^1[0,\sqrt{2}]$, $g_2\in C^1[-\pi,-\frac{\pi}{2}]$, $g_1(r)>0, g_2(\theta)>0$ *and* $\frac{d}{dr}\big(r^\alpha g_1(r)\big)\ge 0$ *we have*

$$\int_\Omega (x_k-x_k^0)p\,dx\to+\infty,\quad \text{as } \epsilon\to 0,\quad k=1,2\,.$$

Proof. First we prove that there exists $K > 0$ large enough such that

$$p(r,\theta) \le K \int_r^{\sqrt{2}} \frac{ds}{(\overline{h}_0(s)+\epsilon)^2}, \quad \forall (r,\theta) \in \tilde{\Omega} \tag{3.18}$$

with

$$\overline{h}_0(r) = g_{2m} r^\alpha g_1(r), \quad \text{and} \quad g_{2m} = \min_{\theta \in [-\pi, -\frac{\pi}{2}]} g_2(\theta)\,.$$

We use the maximum principle as in the proof of Lemma 3.9. It suffices to prove the following inequality

$$K \frac{(\epsilon+h_0)^3}{(\epsilon+\overline{h}_0)^2} + Kr\frac{\partial h_0}{\partial r}\frac{(\epsilon+h_0)^3}{(\epsilon+\overline{h}_0)^3} E(r,\theta) \ge -r\frac{\partial h_0}{\partial r}\cos\theta + \sin\theta \frac{\partial h_0}{\partial \theta} \tag{3.19}$$

with

$$E(r,\theta) = 3\frac{\epsilon+\overline{h}_0}{\epsilon+h_0} - 2\frac{g_{2m}}{g_2(\theta)}.$$

We consider two situations

Case 1: $r \le r_1$ with r_1 given in (3.12). As in the proof of Lemma 3.9 we have $E(r,\theta) \ge \frac{g_{2m}}{g_{2M}}$ with $g_{2M} = \max_{\theta\in[-\pi,-\frac{\pi}{2}]} g_2(\theta)$. Then it suffices to prove for $r \le r_1$

$$Kr\frac{\partial h_0}{\partial r}\frac{g_{2m}}{g_{2M}} \ge -r\frac{\partial h_0}{\partial r}\cos\theta + \sin\theta\frac{\partial h_0}{\partial\theta} \tag{3.20}$$

with $K > 0$ large enough. From (3.12) the function $\left|\frac{\partial h_0}{\partial\theta}/\left(r\frac{\partial h_0}{\partial r}\right)\right|$ is bounded for $r \le r_1$. Now dividing by $r\frac{\partial h_0}{\partial r}$ the inequality (3.20) is obvious, which proves (3.19) for $r \le r_1$.

Case 2: $r > r_1$. We shall prove

$$K(\epsilon+\bar{h}_0) \ge -r\frac{\partial h_0}{\partial r}\cos\theta + \sin\theta\frac{\partial h_0}{\partial\theta} \quad \text{for } r > r_1 \tag{3.21}$$

for $K > 0$ large enough, which implies (3.19) for $r > r_1$. We have, from hypothesis on h_0, $\overline{h}_0(r) \ge \overline{h}_0(r_1)$ so $K(\epsilon+\overline{h}_0(r)) \ge K\overline{h}_0(r_1)$ for $r \ge r_1$. Since the right-hand side of (3.21) is bounded, the result is obvious.

From the tow cases above, the proof of (3.18) is complete. $\square$

Now we have

$$\int_\Omega (x_k - x_k^0)p\,dx = \int_{\tilde{\Omega}_\delta} r(r\cos\theta - x_1^0)p\,dr\,d\theta + \int_{\tilde{\Omega}-\tilde{\Omega}_\delta} r(r\cos\theta - x_1^0)p\,dr\,d\theta\,. \tag{3.22}$$

We choose $0 < \delta < \min(r_0, \frac{|x_1^0|}{2})$ with r_0 given in Lemma 3.16. We have $r\cos\theta - x_1^0 \ge -r + |x_1^0| \ge \frac{|x_1^0|}{2}$ for $r < \delta$, so

$$\int_{\tilde{\Omega}_\delta} r(r\cos\theta - x_1^0)p\,dr\,d\theta \ge \frac{|x_1^0|}{2}\int_{\tilde{\Omega}_\delta} rp\,dr\,d\theta$$

and this last integral goes to $+\infty$ as in the proof of Theorem 3.17. Now using (3.18) we easily prove that the second integral of the right-hand side of (3.22) is bounded which ends the proof for $k = 1$. The case $k = 2$ is similar.

4. Asymptotic behavior in the inequality case (Problem (1.9))

In this section we suppose for simplicity that $\Omega =]-1,1[^2$ and that h_0 is non-increasing in x_1 on Ω_1 and non-decreasing in x_1 on Ω_2 where we denote

$$\Omega_1 =]-1,0[\times]-1,1[\quad \text{and} \quad \Omega_2 =]0,1[\times]-1,1[\,.$$

We study the asymptotic behaviour of the solution p of (1.9) when $\epsilon \to 0$. In order to introduce the limit problem we define $\mathcal{K}_{\delta_1}$ as the closure of $\mathcal{K} = \{\varphi \in H_0^1(\Omega) : \varphi \geq 0\}$ with respect to the norm of $H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$. We remark that $\mathcal{K}_{\delta_1}$ is a closed convex set in $H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$.

We now define the limit problem

$$\begin{gathered}\text{Find } p_0 \in \mathcal{K}_{\delta_1} \text{ such that}\\ \int_\Omega h_0^3 \nabla p_0 \nabla(\varphi - p_0) dx \geq \int_\Omega h_0 \frac{\partial}{\partial x_1}(\varphi - p) \quad \forall \varphi \in \mathcal{K}_{\delta_1}\end{gathered} \tag{4.1}$$

We now give the following existence, uniqueness and convergence results.

Proposition 4.1. *Suppose that $\int_\Omega \frac{dx}{h_0} < +\infty$ and $\delta_1 \in \mathbb{R}^+$ is such that*

$$\begin{aligned} K = &\sup_{x_2 \in [-1,1]} \int_{-1}^0 \int_{-1}^{x_1} h_0^{2\delta_1 - 3}(s, x_2)\, ds\, dx_1 \\ &+ \sup_{x_2 \in [-1,1]} \int_0^1 \int_0^{x_1} h_0^{2\delta_1 - 3}(s, x_2)\, ds\, dx_1 < \infty \end{aligned}$$

Then Problem (4.1) admits an unique solution $p_0 \in \mathcal{K}_{\delta_1}$ which is independent of δ_1. Also the solution p of problem (1.9) converges, when $\epsilon \to 0$, to p_0 strongly in $H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$.

Proof. We apply Lemma 2.2 with $d_1 = 0$ and $\delta_2 = \frac{3}{2}$ and we obtain classically the first result. Taking $\varphi = 0$ in (1.9) we obtain

$$\int_\Omega (h_0 + \epsilon)^3 |\nabla p|^2 dx \leq \int_\Omega h_0 \frac{\partial p}{\partial x_1} = \int_\Omega (h_0 + \epsilon) \frac{\partial p}{\partial x_1}$$

which leads to

$$\left\| (h_0 + \epsilon)^{3/2} \nabla p \right\|_{L^2(\Omega)} \leq \Big(\int_\Omega \frac{dx}{h_0(x)} \Big)^{1/2} \tag{4.2}$$

which implies

$$\left\| h_0^{3/2} \nabla p \right\|_{L^2(\Omega)} \leq \Big(\int_\Omega \frac{dx}{h_0(x)} \Big)^{1/2}. \tag{4.3}$$

From Lemma 2.2 with $d_1 = 0$ and $\delta_2 = 3/2$ we deduce that p is bounded in $H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$. Then an element $\xi \in \mathcal{K}_{\delta_1}$ exists such that, up to a subsequence, $p \to \xi$ weakly in $H_0^1(\Omega, h_0^{2\delta_1}, h_0^3)$. We now pass to the limit in all terms in the inequality

$$\int_\Omega (h_0 + \epsilon)^3 \nabla p \cdot \nabla \varphi \geq \int_\Omega (h_0 + \epsilon)^3 |\nabla p|^2 + \int_\Omega h_0 \frac{\partial}{\partial x_1}(\varphi - p) \quad \forall \varphi \in \mathcal{K}. \tag{4.4}$$

Writing for any $\varphi \in \mathcal{K}$,

$$\begin{aligned} \int_\Omega (h_0 + \epsilon)^3 \nabla p \cdot \nabla \varphi = &\int_\Omega \big((h_0 + \epsilon)^{3/2} - h_0^{3/2}\big)(h_0 + \epsilon)^{3/2} \nabla p \cdot \nabla \varphi \\ &+ \int_\Omega \big((h_0 + \epsilon)^{3/2} - h_0^{3/2}\big) h_0^{3/2} \nabla p \cdot \nabla \varphi + \int_\Omega h_0^3 \nabla p \cdot \nabla \varphi \end{aligned}$$

and using (4.2) and (4.3) we deduce

$$\int_\Omega (h_0+\epsilon)^3 \nabla p \cdot \nabla \varphi \to \int_\Omega h_0^3 \nabla \xi \cdot \nabla \varphi \quad \forall \varphi \in \mathcal{K}\,. \tag{4.5}$$

Writing also

$$\int_\Omega h_0 \frac{\partial}{\partial x_1}(\varphi - p) = \int_\Omega h_0^{-1/2} h_0^{3/2} \frac{\partial}{\partial x_1}(\varphi - p),$$

we obtain

$$\int_\Omega h_0 \frac{\partial}{\partial x_1}(\varphi - p) \to \int_\Omega h_0 \frac{\partial}{\partial x_1}(\varphi - \xi) \quad \forall \varphi \in \mathcal{K}. \tag{4.6}$$

Finally we have

$$\int_\Omega (h_0+\epsilon)^3 |\nabla p|^2 \ge \int_\Omega h_0^3 |\nabla p|^2$$

which gives

$$\liminf \int_\Omega (h_0+\epsilon)^3 |\nabla p|^2 \ge \int_\Omega h_0^3 |\nabla \xi|^2\,. \tag{4.7}$$

From (4.4)-(4.7) we deduce

$$\int_\Omega h_0^3 \nabla \xi \cdot \nabla \varphi \ge \int_\Omega h_0^3 |\nabla \xi|^2 + \int_\Omega h_0 \frac{\partial}{\partial x_1}(\varphi - \xi) \quad \forall \varphi \in \mathcal{K}\,.$$

By denseness and uniqueness we deduce that $\xi = p_0$ and that the entire sequence p converges to p_0. It remains to prove the strong convergence. We have

$$\int_\Omega h_0^3 |\nabla (p-p_0)|^2 \le \int_\Omega (h_0+\epsilon)^3 |\nabla p|^2 + \int_\Omega h_0^3 |\nabla p_0|^2 - 2\int_\Omega h_0^3 \nabla p \cdot \nabla p_0\,.$$

Taking $\varphi = 0$ in (1.9) and (4.1) and passing to the limit we deduce

$$\lim_{\epsilon\to 0} \int_\Omega h_0^3 |\nabla (p-p_0)|^2 \le 2\int_\Omega h_0 \frac{\partial p_0}{\partial x_1} - 2\int_\Omega h_0^3 |\nabla p_0|^2.$$

The right hand-side of the above inequality is 0 (take $\varphi = 0$ and $\varphi = 2p_0$ in (4.1)) which proves the result. □

In the following we shall use the classical notation

$$\begin{aligned} \Omega_\epsilon^0 &= \{x \in \Omega : p(x) = 0\} \quad \text{(cavitation zone)} \\ \Omega_\epsilon^+ &= \{x \in \Omega : p(x) > 0\} \quad \text{(active zone)} \end{aligned}$$

It is well known that if $x \in \Omega_\epsilon^0$ then $\frac{\partial h_0}{\partial x_1} \le 0$ which implies the inclusion $\Omega_1 \subset \Omega_\epsilon^+$ so that in Ω_1, p satisfies

$$\nabla \cdot \big[(h_0+\epsilon)^3 \nabla p\big] = \frac{\partial h_0}{\partial x_1}. \tag{4.8}$$

The next lemma will be useful for the proofs in the infinite-limit cases.

Lemma 4.2. *Let Ω^* be an open subset of Ω_1 with Lipschitz boundary and p^* the solution of* (4.8) *with $p^* = 0$ on $\partial\Omega^*$. Then $p \ge p^*$ on Ω^*.*

Proof. Since p and p^* satisfy (4.8) on Ω^*, we have the result by the maximum principle since $p \ge 0$ on $\partial\Omega^*$ and $p^* = 0$ on $\partial\Omega^*$. □

4.1. **Line-contact case.** We suppose for simplicity $h_0(x) = (-x_1)^\alpha h_1(x), \forall x \in \Omega$ with $\alpha > 0$, $h_1 \in W^{1,\infty}(\Omega)$ and $h_1 > 0$. We have the following result.

Theorem 4.3. *For any $\alpha \in]0,1[$ and $(x_1^0, x_2^0) \in \Omega$ we have*

$$\int_\Omega p\, dx \to \int_\Omega p_0 dx$$

$$\int_\Omega (x_k - x_k^0) p\, dx \to \int_\Omega (x_k - x_k^0) p_0 dx, \quad k = 1, 2$$

Proof. The hypotheses of Proposition 4.1 are satisfied with $\delta_1 = 1/2$. Then the proof is exactly as the proof of Theorem 3.5. $\square$

Now for the infinite-limit case we use Lemma 4.2 with $\Omega^* = \Omega_1$. Performing for p^* the same kind of estimates as for p in paragraph 3.1.2 we easily obtain the following result.

Theorem 4.4. *For $\alpha \geq 1$ we have*

(1) $\int_\Omega p\, dx \to +\infty$
(2) *If h_0 is symmetric in x_2 with respect to $x_2 = 0$ then*
- $\int_\Omega (x_2 - x_2^0) p\, dx \to +\infty$ *for* $x_2^0 < 0$
- $\int_\Omega (x_2 - x_2^0) p\, dx \to -\infty$ *for* $x_2^0 > 0$
- $\int_\Omega (x_2 - x_2^0) p\, dx = 0$ *for* $x_2^0 = 0$

(3) *We assume that h_1 is of the form $h_1(x) = g_1(x_1) g_2(x_2)$, $g_2 \in C^0([-1,1])$, $g_1 \in H^1(]-1,1[)$, $g_2 > 0$, $g_1 > 0$ and $\frac{d}{dx_1}((-x_1)^\alpha g_1) \leq 0$. Then for any $x_1^0 \in]-1, 0[$ we have*

$$\int_\Omega (x_1 - x_1^0) p\, dx \to +\infty \quad \text{as } \epsilon \to 0$$

Remark 4.5. In the above theorem we obtained the behaviour of the x_1-moment for $x^0 \in \Omega_1$ only. The problem is open when x^0 is such that $x_1^0 \geq 0$.

4.2. **Point-contact case.** We suppose $h_0(x) = |x|^\alpha h_1(x)$ with h_1 as in Section 3.2. The analogous of Theorem 3.13 for the inequality problem is the following.

Theorem 4.6. *For $0 < \alpha < 4/3$ we have for any $(x_1^0, x_2^0) \in \Omega$*

$$\int_\Omega p\, dx \to \int_\Omega p_0\, dx\,,$$

$$\int_\Omega (x_k - x_k^0) p\, dx \to \int_\Omega (x_k - x_k^0) p_0 dx, \quad k = 1, 2\,.$$

The proof the above theorem uses Proposition 4.1 and is exactly as the proof of Theorem 3.13.

For the infinite-limit case we pass again to polar coordinates. We have the following result which is immediate applying Lemma 4.2 with $\Omega^* =]-1, 0[^2$ which reduces the problem to the equation case.

Theorem 4.7. *Suppose that $r_2 > 0$ and $\beta \in]0,1[$ exist such that*

$$\beta r \frac{\partial h_0}{\partial r} \cos\theta \leq \sin\theta \frac{\partial h_0}{\partial \theta}, \quad \forall (r, \theta) \in [0, r_2] \times]-\pi, -\frac{\pi}{2}[\,.$$

Then for $\alpha \geq 3/2$, we have

$$\int_\Omega p\, dx \to +\infty\,.$$

Also fir h_1 *of the form* $h_1(r,\theta) = g_1(r)g_2(\theta)$ *with* $g_1 \in C^1[0,\sqrt{2}]$, $g_2 \in C^1[-\pi, -\frac{\pi}{2}]$, $g_1 > 0$, $g_2 > 0$ *and* $\frac{d}{dr}(r^\alpha g_1(r)) \geq 0$, *we have*

$$\int_\Omega (x_k - x_k^0) p \, dx \to +\infty \quad \text{as } \epsilon \to 0$$

for all $x_k^0 \in]-1, 0[$, $k = 1, 2$ *.*

We remark that for $\alpha \in [4/3, 3/2[$, we are not able to obtain a result as in Theorem 3.14, since we can not take a test function φ in (1.9) such that $\varphi - p = -c(h_0 + \epsilon)^{-\delta} p$ with c independent of ϵ.

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