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NONLINEAR PSEUDODIFFERENTIAL EQUATIONS ON A HALF-LINE WITH LARGE INITIAL DATA

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Abstract. We study the initial-boundary value problem for nonlinear pseudodifferential equations, on a half-line,

$$\begin{gathered} u_t + \lambda |u|^\sigma u + \mathcal{L}u = 0, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^+, \end{gathered}$$

where $\lambda > 0$ and pseudodifferential operator $\mathcal{L}$ is defined by the inverse Laplace transform. The aim of this paper is to prove the global existence of solutions and to find the main term of the asymptotic representation in the case of the large initial data.

1. Introduction

We consider the initial-boundary value problem for a general class of the nonlinear nonlocal equations, on a half-line,

$$\begin{gathered} u_t + \lambda |u|^\sigma u + \mathcal{L}u = 0, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^+, \end{gathered} \tag{1.1}$$

where $\lambda > 0$, $\sigma > 0$. The linear operator $\mathcal{L}$ is a pseudodifferential operator defined by the inverse Laplace transform as follows

$$\mathcal{L}u = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} C_\alpha p^\alpha \big(\hat{u}(p,t) - \frac{u(0,t)}{p}\big)\, dp, \tag{1.2}$$

where $\alpha \in (1,2)$ and

$$\hat{u}(p) = \int_0^{+\infty} e^{-px} u(x) dx$$

denotes the Laplace transform of u. We assume that the symbol $K(p) = C_\alpha p^\alpha$ is dissipative, i.e. $\Re(K(p)) > 0$ for $\Re(p) = 0$. Here and below p^α is the main branch of the complex analytic function in the half-complex plane $\Re p \geq 0$, so that $1^\alpha = 1$ (we make a cut along the negative real axis $(-\infty, 0)$).

The initial-boundary value problem (1.1) is of great interest from the physical point of view , since it describes many physical phenomena , such as the focusing of laser beams , waves on water (some other applications can be found [28]) . A great number of publications have dealt with asymptotic representations of solutions to

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the Cauchy problem for nonlinear evolution equations in the last twenty years. While not attempting to provide a complete review of this publications, we do list some known results [3]-[10],[12] ,[13],[15] -[18], [21], [22], [26]-[35], where there were obtained optimal time decay estimates and asymptotic formulas of solutions to different nonlinear local and nonlocal dissipative equations. The asymptotic theory of the initial-boundary value problems for the nonlinear pseudodifferential equations is relatively new and traditional questions of a general theory are far from their conclusion . A description of the large time asymptotic behavior of solutions for the initial-boundary value problems requires new approaches and the reorientation of the points of view compared to the Cauchy problem.

The initial-boundary value homogeneous problems for nonlinear pseudodifferential equations were studied in the book [20]. In the present paper we continue the study of pseudodifferential equations on a half-line, considering the case of a large initial data .

The aim of this paper is to prove a global existence of solutions to the initial-boundary value problem (1.1), and to find the main term of the asymptotic representation of solutions. We will obtain the a priory optimal time decay estimates of solutions in the usual Lebesgue spaces $\mathbf{L}^r$ for $1 \leq r \leq \infty$. These type of estimates enable us to consider the critical and sub critical cases in future works.

To state our results we give some notations. The weighted Sobolev space is

$$\mathbf{H}_r^{k,s} = \{f \in \mathbf{L}^r : \|f\|_{\mathbf{H}_r^{k,s}} = \|\langle i\partial_x\rangle^k \langle x\rangle^s f\|_{\mathbf{L}^r} < \infty\},$$

where $\langle x\rangle = \sqrt{1+x^2}$.

We introduce the function $\Lambda_0(s) \in \mathbf{L}^\infty$,

$$\Lambda_0(s) = (-1)^{\frac{\{\alpha\}}{\alpha}} \frac{C_\alpha}{2\pi^2 i} \frac{A_1}{\pi - A_1} \sin(\frac{\pi\{\alpha\}}{\alpha}) \int_{-i\infty}^{i\infty} dz e^{sz} z^{\{\alpha\}} \int_0^\infty dq e^{-q} \frac{q^{-\frac{\{\alpha\}}{\alpha}}}{K(z)+q}, \tag{1.3}$$

where

$$A_1 = (-1)^{\{\alpha\}+1}(-C_\alpha)^{-\frac{[\alpha]}{\alpha}} \Gamma(1-\{\alpha\})\Gamma(\{\alpha\}) \sin \pi\{\alpha\}.$$

We prove the following theorem.

Theorem 1.1. *Let $\alpha \in (1,2)$, $\sigma > \alpha$,$\lambda > 0$ and real valued function $u_0 \in \mathbf{H}_2^{1,0} \cap \mathbf{H}_\infty^{0,\mu} \cap \mathbf{H}_1^{0,0}$, where $\mu \in [0,1]$. Then there exists a unique real valued solution of* (1.1) *such that*

$$u(x,t) \in C([0,\infty); \mathbf{H}_2^{1,0} \cap \mathbf{H}_\infty^{0,\mu} \cap \mathbf{H}_1^{0,0}).$$

Moreover there exists a constant A such that

$$u(x,t) = t^{-1/\alpha} A \Lambda_0(xt^{-1/\alpha}) + O(t^{-\frac{1+\mu}{\alpha}}) \tag{1.4}$$

as $t \to \infty$ uniformly with respect to $x > 0$, where

$$A = \int_0^{+\infty} u_0 \, dy + \lambda \int_0^{+\infty} dt \int_0^\infty |u|^\sigma u \, dy.$$

Remark 1.2. We can guarantee that the coefficient $A \neq 0$ in the asymptotic representation (1.4) if $\int_0^{+\infty} u_0 \, dy \neq 0$ and $|u|^\sigma u$ is small. It can occur that $A = 0$, for instance, for convective equations $\int_0^\infty |u|^\sigma u \, dy \equiv 0$ if the initial data have zero mean value $\int_0^{+\infty} u_0 \, dy = 0$. In the last case formula (1.4) gives us only some time decay estimate for the solutions.

2. Preliminaries

We introduce the function $\kappa(\xi) = K^{-1}(-\xi)$, such that $\operatorname{Re}\kappa(\xi) > 0$ for all $\operatorname{Re}\xi \geq 0$. We denote

$$\mathcal{G}(t_1)\phi(t_2) = \int_0^{+\infty} G(x,y,t_1)\phi(y,t_2)\,dy,$$

where Green function $G(x,y,t)$ is defined by

$$\begin{aligned} G(x,y,t) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-K(p)t+p(x-y)}\,dp \\ &\quad + \frac{1}{4\pi^2}\int_{-i\infty}^{i\infty} e^{\xi t}\kappa(\xi)\xi^{-1}e^{-\kappa(\xi)y}\int_{-i\infty}^{i\infty}\frac{e^{px}K(p)}{p(K(p)+\xi)}d\xi\,dp. \end{aligned} \tag{2.1}$$

The solution u of the problem (1.1) can be represented as follows (see [20], pp. 23-24)

$$u(x,t) = \mathcal{G}(t)u_0 + \int_0^t d\tau\mathcal{G}(t-\tau)\mathcal{N}(u)(\tau), \tag{2.2}$$

where $\mathcal{N}(u) = \lambda|u|^\sigma u$. We introduce complete metric space

$$\begin{gathered} \mathbf{X} = \{\phi(x,t) \in \mathbf{C}([0,\infty);\mathbf{H}_r^{0,\mu}) \cap \mathbf{C}((0,\infty);\mathbf{H}_r^{1,0}) | \|\phi\|_{\mathbf{X}} < +\infty,\}, \\ \|\phi\|_{\mathbf{X}} = \sup_{t>0}\langle t\rangle^{\frac{1}{\alpha}}(\langle t\rangle^{-\frac{1}{\alpha r}-\frac{\mu}{\alpha}}\|x^\mu\phi\|_{\mathbf{L}^r} + t^{\frac{1}{\alpha}-\frac{1}{\alpha r}}\|\phi_x\|_{\mathbf{L}^r}), \end{gathered}$$

where $\mu \in [0,1]$, $1 \leq r \leq \infty$. Let a continuous linear functional $f(\phi) : \mathbf{L}_1 \to \mathbb{R}$ is defined as

$$f(\phi) = \int_0^{+\infty}\phi(x)dx.$$

Now we obtain some estimates for the Green operator $\mathcal{G}$ in the space $\mathbf{X}$.

Lemma 2.1. *The following two estimates are valid*

$$\|\mathcal{G}\phi\|_{\mathbf{X}} \leq C(\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^\infty}), \tag{2.3}$$

$$\sup_{t>1} t^{\frac{1}{\alpha}(1+\mu)}\|\mathcal{G}\phi - t^{-\frac{1}{\alpha}}\Lambda_0(xt^{-1/\alpha})f(\phi)\|_{\mathbf{L}^\infty} \leq C\|\phi\|_{\mathbf{L}^{1,\mu}} \tag{2.4}$$

for all $t > 0$, provided that the right-hand sides are bounded.

Proof. We rewrite the Green function as

$$G(x,y,t) = F_1(x-y,t) + F_2(x,y,t),$$

where

$$F_1(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-K(p)t+px}\,dp \tag{2.5}$$

$$F_2(x,y,t) = \frac{1}{4\pi^2}\int_{-i\infty}^{i\infty} e^{\xi t - k(\xi)y}k(\xi)\xi^{-1}\int_{-i\infty}^{i\infty}\frac{e^{px}K(p)}{p(K(p)+\xi)}d\xi\,dp. \tag{2.6}$$

Firstly we prove the estimate

$$t^{\frac{1}{\alpha}-\frac{1}{\alpha r}}\Big(\|F_1(q,t)\|_{L^r} + t^{-\frac{\mu}{\alpha}}\|q^\mu F_1(q,t)\|_{L^r} + t^{\frac{1}{\alpha}}\|\partial_q F_1(q,t)\|_{L^r}\Big) \leq C \tag{2.7}$$

for all $t > 0$, where $\mu > 0$ and $1 \leq r \leq \infty$. Changing variables $p^\alpha t = z^\alpha$, and $\widetilde{q} = qt^{-1/\alpha}$ we get

$$|F_1(q,t)| = \Big|\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{qp}e^{-K(p)t}\,dp\Big| \leq Ct^{-1/\alpha}\Big|\int_{-i\infty}^{i\infty} e^{\widetilde{q}z}e^{-C_\alpha z^\alpha}\,dz\Big|.$$

Therefore,

$$\|F_1(q,t)\|_{L^r} \le Ct^{-\frac{1}{\alpha}+\frac{1}{\alpha r}}.$$

After the change $p^\alpha t = z^\alpha$, we have

$$|q^\mu F_1(q,t)| \le Ct^{-\frac{1}{\alpha}+\frac{\mu}{\alpha}} \big|\widetilde{q}^\mu \int_{-i\infty}^{i\infty} e^{\widetilde{q}z} e^{-C_\alpha z^\alpha} dz\big| \le Ct^{-\frac{1}{\alpha}+\frac{\mu}{\alpha}} \langle \widetilde{q}\rangle^{\mu-1-\gamma},$$

$$|\partial_q F_1(q,t)| \le Ct^{-\frac{2}{\alpha}} \big| \int_{-i\infty}^{i\infty} z e^{\widetilde{q}z} e^{-C_\alpha z^\alpha} dz\big| \le Ct^{-\frac{2}{\alpha}} \langle \widetilde{q}\rangle^{-1-\gamma},$$

where $0<\gamma<\alpha$. Therefore,

$$\begin{aligned} \|q^\mu F_1(q,t)\|_{L^r} &\le Ct^{-\frac{1}{\alpha}+\frac{\mu}{\alpha}+\frac{1}{\alpha r}} ((\int_0^1 + \int_1^\infty) d\widetilde{q} (1+\widetilde{q}^2)^{\frac{(\mu-1-\gamma)r}{2}})^{1/r} \\ &\le Ct^{-\frac{1}{\alpha}+\frac{\mu}{\alpha}+\frac{1}{\alpha r}} \end{aligned} \tag{2.8}$$

and

$$\|\partial_q F_1(q,t)\|_{L^r} \le Ct^{-\frac{2}{\alpha}+\frac{1}{\alpha r}}. \tag{2.9}$$

Estimate (2.7) is then proved. Now we prove

$$\begin{aligned} &\sup_{t>0,y>0} t^{\frac{1}{\alpha}-\frac{1}{r\alpha}+\mu_1+\gamma} \langle t\rangle^{-2\gamma} y^{-\mu_1\alpha} (\|F_2(x,y,t)\|_{\mathbf{L}^r} \\ &+ t^{-\frac{\mu}{\alpha}} \|x^\mu F_2(x,y,t)\|_{\mathbf{L}^r} + t^{\frac{1}{\alpha}} \|\partial_x F_2(x,y,t)\|_{L^r}) \le C, \end{aligned} \tag{2.10}$$

where $\mu\ge 0, \gamma>0$, $0\le\mu_1<\frac{1}{\alpha}(1-\frac{1}{r}-\gamma\alpha)$ and $1\le r\le\infty$.

We have, by definition, $\kappa(q) = C_1|q|^{\frac{1}{\alpha}}$. Changing variables $p^\alpha t = z^\alpha$ and $\xi t = q$, we obtain the estimate

$$|F_2(x,y,t)| \le Ct^{-1/\alpha} \big| \int_{-i\infty}^{i\infty} dz e^{\widetilde{x}z} K(z) z^{-1} \int_{-i\infty}^{i\infty} dq \quad q^{-1}\kappa(q) \frac{e^{q-C_1 q^{\frac{1}{\alpha}}\widetilde{y}}}{K(z)+q}\big|. \tag{2.11}$$

We move the contours of integration with respect to z as follows

$$\mathcal{C}_1 = \{z=\rho e^{\pm i\beta_1}, \rho\ge 0, \beta_1 = \frac{\pi}{2}+\epsilon_1\} \tag{2.12}$$

and with respect to q by

$$\mathcal{C}_2 = \{q=e^{i\phi_2} \quad \phi_2\in[-\frac{\pi}{2},\frac{\pi}{2}]\} \cup \{q=\rho e^{\pm i\phi_2}\rho\ge 1, \quad \phi_2 = \frac{\pi}{2}+\epsilon_2\}, \tag{2.13}$$

where $\epsilon_1,\epsilon_2>0$ are small enough. It is apparent that

$$\begin{gathered} |e^q| \le C|q|^{-\gamma}, \quad |K(z)+q|^{-1} \le C|z|^{-\nu\alpha}|q|^{\nu-1}, \\ |e^{z\tilde{x}}| \le C|z\tilde{x}|^{-\mu_2}, \quad |e^{-C_{1j}q^{\frac{1}{\alpha}}\widetilde{y}}| \le C|q^{\frac{1}{\alpha}}\widetilde{y}|^{-\alpha\mu_1} \end{gathered} \tag{2.14}$$

for all $q\in\mathcal{C}_2$ and $z\in\mathcal{C}_1$, where $\nu\in[0,1]$, $\gamma\ge 0$, and $\mu_1\ge 0, \mu_2\ge 0$. Taking into account (2.14) we get

$$|F_2(x,y,t)| \le Ct^{-1/\alpha} y^{-\mu_1\alpha} \widetilde{x}^{-\mu_2} \int_{-i\infty}^{i\infty} |dz||z|^{\alpha-1-\nu\alpha-\mu_2} \int_{-i\infty}^{i\infty} |dq|\,|q|^{\frac{1}{\alpha}-2+\nu-\mu_1-\gamma}. \tag{2.15}$$

To guarantee the convergence of this integrals we need to satisfy the following conditions

$$\frac{1}{\alpha} - 2 + \nu - \mu_1 > -1 \tag{2.16}$$

and

$$\begin{aligned} &\alpha - 1 - \nu\alpha - \mu_2 > -1, \\ &\alpha - [\alpha] - \nu\alpha - \mu_2 < -1. \end{aligned} \tag{2.17}$$

respectively. Under the condition $\alpha\mu_1 + \mu_2 < 2 - [\alpha]$ there exists some $\nu \geqslant 0$ such that the estimates (2.17) are valid. Therefore, we obtain

$$|F_2(p, y, t)| \le Ct^{-\frac{1-\mu_2}{\alpha}+\mu_1} x^{-\mu_2} y^{-\alpha\mu_1}. \tag{2.18}$$

From (2.18) it follows that

$$\begin{aligned} \|F_2(x, y, t)\|_{L^r} &\le Ct^{-\frac{1}{\alpha}+\frac{1}{r\alpha}+\mu_1\pm\gamma} y^{-\alpha\mu_1} \Big(\int_0^\infty x^{-\mu_2 r} dx\Big)^{1/r} \\ &\le Ct^{-\frac{1}{\alpha}+\frac{1}{r\alpha}+\mu_1\pm\gamma} y^{-\alpha\mu_1}, \end{aligned} \tag{2.19}$$

where $0 \le \mu_1 < \frac{1}{\alpha}(1 - \frac{1}{r})$. In the same manner we obtain

$$|x^\mu F_2(x, y, t)| \le Ct^{-\frac{1}{\alpha}+\frac{\mu-\mu_2}{\alpha}+\mu_1} x^{\mu-\mu_2} y^{-\alpha\mu_1},$$
$$|\partial_x F_2(p, y, t)| \le Ct^{-\frac{2-\mu_2}{\alpha}+\mu_1} y^{-\mu_1\alpha} x^{-\mu_2}.$$

Thus we obtain

$$\|x^\mu F_2(x, y, t)\|_{L^r} \le Ct^{-\frac{1+\mu}{\alpha}+\frac{1}{r\alpha}+\mu_1\pm\gamma} y^{-\mu_1\alpha}, \tag{2.20}$$
$$\|\partial_x F_2(x, y, t)\|_{L^r} \le Ct^{-\frac{2}{\alpha}+\frac{1}{r\alpha}+\mu_1} y^{-\mu_1\alpha}, \tag{2.21}$$

and therefore estimate (2.10) is proved. From estimates (2.7) and (2.10) we easily get result (2.3) of Lemma 2.1. To obtain (2.4) firstly we prove the following estimate

$$G(x, y, t) = t^{-1/\alpha} \sin\frac{\pi\{\alpha\}}{\alpha} \Lambda_0(xt^{-1/\alpha}) + y^\mu O(t^{-\frac{1+\mu}{\alpha}}) \tag{2.22}$$

for $t \to \infty$, where $x, y > 0$.

We write the representation (2.5) for the function $F_1(x, t)$ as

$$F_1(x - y, t) = F_1(x, t) + [F_1(x - y, t) - F_1(x, t)].$$

After the change of variables $p^\alpha t = z^\alpha$, we easily find that

$$F_1(x, t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{xp-K(p)t} dp = t^{-1/\alpha}\frac{1}{2\pi i}\int_{-\infty}^{i\infty} e^{zs-K(z)} dz. \tag{2.23}$$

Using the estimate $|e^{-py} - 1| \le C|py|^\mu$ for $y > 0$ and $p \in (-i\infty, i\infty)$ with $\mu_1 \in [0, 1]$ we have

$$\begin{aligned} |F_1(x - y, t) - F_1(x, t)| &\le \Big|\int_{-i\infty}^{i\infty} e^{xp-C_\alpha p^\alpha t}(e^{-py} - 1)\, dp\Big| \\ &\le Cy^\mu t^{-\frac{1+\mu}{\alpha}} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} |dz|\, |z|^\mu \big|e^{xt^{-\frac{1}{\alpha}}z-C_\alpha z^\alpha}\big| \\ &\le Cy^\mu t^{-\frac{1+\mu}{\alpha}} = y^\mu O\big(t^{-\frac{1+\mu}{\alpha}}\big). \end{aligned} \tag{2.24}$$

Therefore, from (2.23), we obtain

$$F_1(x - y, t) = t^{-1/\alpha}\frac{1}{2\pi i}\int_{-\infty}^{i\infty} e^{zs-K(z)} dz + y^\mu O(t^{-\frac{1+\mu}{\alpha}}). \tag{2.25}$$

Now we write the representation (2.6) of the function $F_2(x, y, t)$ as

$$F_2(x, y, t) = M(x, y, t) + R(x, y, t), \tag{2.26}$$

where $M(x,t) = F_2(x,0,t)$ and $R = [F_2(x,y,t) - F_2(x,0,t)]$. Considering that $|e^{-C\xi^{1/\alpha}y} - 1| \le C|\xi^{1/\alpha}y|^{\mu}$ and changing variables $\xi t = q$ and $p^{\alpha}t = z^{\alpha}$, we get

$$\begin{aligned}
&|F_2(x,y,t) - F_2(x,0,t)| \\
&\le Ct^{-\frac{1}{\alpha}-\frac{\mu_1}{\alpha}} y^{\mu_1} \int_{-i\infty}^{i\infty} |dz|\,|e^{xt^{\frac{1}{\alpha}}z}|\,|z|^{\alpha-1} \int_{-i\infty}^{i\infty} |dq||q|^{\frac{1}{\alpha}-1+\frac{\mu}{\alpha}} \frac{|e^q|}{|K(z)+q|} \\
&= y^{\mu} O(t^{-\frac{1+\mu}{\alpha}}).
\end{aligned}$$

Therefore,

$$F_2(x,y,t-\tau) = M(x,t) + y^{\mu}O(t^{-\frac{1+\mu}{\alpha}}), \tag{2.27}$$

where

$$M(x,t) = t^{-1/\alpha}\frac{1}{4\pi^2}\int_{-i\infty}^{i\infty} dz e^{sz}K(z)z^{-1}\int_{-i\infty}^{i\infty} dq e^q \frac{q^{-1}\kappa(q)}{K(z)+q}, \tag{2.28}$$

with $s = xt^{-1/\alpha}$. Applying the Cauchy Theorem, we obtain

$$\begin{aligned}
&\int_{-i\infty}^{i\infty} dz e^{sz}K(z)z^{-1}\int_{-i\infty}^{i\infty} dq e^q \frac{q^{-1}\kappa(q)}{K(z)+q} \\
&= 2\pi i\int_{-i\infty}^{i\infty} dz e^{sz}e^{-K(z)} + \int_{-i\infty}^{i\infty} dz e^{sz}K(z)z^{-1}\int_{\Gamma} dq\, e^q \frac{q^{-1}\kappa(q)}{K(z)+q} \\
&= I_1 + I_2,
\end{aligned} \tag{2.29}$$

where $\Gamma = \{z \in (-\infty e^{-i\pi}, 0e^{-i\pi}) \cup (0e^{i\pi}, -\infty e^{i\pi})\}$. Using

$$\int_{-i\infty}^{i\infty} dz e^{sz}\int_{\Gamma} dq e^q \frac{1}{K(z)+q} = 0,$$

we get

$$\begin{aligned}
M(x,t) &= t^{-1/\alpha}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz e^{sz}e^{-C_\alpha z^\alpha} + t^{-1/\alpha}\frac{C_\alpha(-1)^{\frac{\{\alpha\}}{\alpha}}}{2\pi^2 i}\frac{A_1}{\pi - A_1} \\
&\quad\times \sin\big(\frac{\pi\{\alpha\}}{\alpha}\big)\int_{-i\infty}^{i\infty} dz e^{sz}z^{\{\alpha\}}\int_0^{+\infty} dq e^{-q}\frac{q^{-\frac{\{\alpha\}}{\alpha}}}{K(z)+q},
\end{aligned} \tag{2.30}$$

where

$$A_1 = (-1)^{\{\alpha\}+1}(-C_\alpha)^{-\frac{[\alpha]}{\alpha}}\Gamma(1-\{\alpha\})\Gamma(\{\alpha\})\sin\pi\{\alpha\}.$$

From (2.25), (2.27), (2.30), we obtain (2.22) and then estimate (2.4). This completes the proof of Lemma 2.1. □

We defined the space

$$\mathbf{W} = \{\phi(x,t) \in C((0,\infty); \mathbf{L}^1 \cap \mathbf{L}^{\infty,\mu}) : \|\phi\|_{\mathbf{W}} < \infty\}$$

where $\|\phi\|_{\mathbf{W}} = (\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^{\infty,\mu}})$.

Lemma 2.2. *The following two estimates are valid*

$$\begin{gathered}
\|x^{\mu}\int_0^t \mathcal{G}(t-\tau)\phi(\tau)d\tau\|_{\mathbf{L}^r} \le C\langle t\rangle^{-\frac{1}{\alpha}+\frac{1}{\alpha r}+\frac{\mu}{\alpha}}\|\langle t\rangle^{1+\gamma}\phi\|_{\mathbf{W}}, \\
\|\partial_x\int_0^t \mathcal{G}(t-\tau)\phi(\tau)d\tau\|_{\mathbf{L}^r} \le Ct^{1-\frac{1}{\alpha}+\frac{1}{\alpha r}}\langle t\rangle^{-\frac{1}{\alpha}}\|\langle t\rangle^{1+\gamma}\phi\|_{\mathbf{W}},
\end{gathered}$$

for all $t > 0$, $\mu \in (0,1)$, $r \ge 1$ and small enough $\gamma > 0$, provided that the right-hand sides are bounded.

Proof. Let $t \in [0,1]$. By estimate (2.21) of Lemma 2.1, we get

$$\|\|\partial_x F_2(x,y,t)\|_{\mathbf{L}^r_x}\|_{\mathbf{L}^1_y} \leq Ct^{-\frac{2}{\alpha}+\frac{1}{\alpha r}} \int_0^\infty \langle \widetilde{y} \rangle^{-\mu_1 \alpha}\, dy \leq Ct^{-\frac{1}{\alpha}+\frac{1}{\alpha r}}, \tag{2.31}$$

for all $t > 0$. Therefore, using (2.8) of Lemma 2.1 we obtain

$$\begin{aligned} &\|x^\mu \int_0^t \mathcal{G}(t-\tau)\phi(\tau)d\tau\|_{\mathbf{L}^r} \\ &\leq \int_0^t d\tau(\|\phi\|_{\mathbf{L}^r}\|x^\mu F_1\|_{\mathbf{L}^1} + \|x^\mu \phi\|_{\mathbf{L}^r}\|F_1\|_{\mathbf{L}^1}) + \int_0^t d\tau \|\phi\|_{\mathbf{L}^\infty}\|x^\mu F_2\|_{\mathbf{L}^r_x \mathbf{L}^1_y} \\ &< C(t^{-\frac{1}{\alpha}+\frac{1}{r\alpha}+1} + t^{-\frac{1}{\alpha}+\frac{1}{r\alpha}+\frac{\mu}{\alpha}+1})\|\phi\|_{\mathbf{W}} < C\|\phi\|_{\mathbf{W}} \end{aligned} \tag{2.32}$$

for all $t \in [0,1]$. Using (2.31) and (2.9), we have

$$\begin{aligned} &\|\partial_x \int_0^t \mathcal{G}(t-\tau)\phi(\tau)d\tau\|_{\mathbf{L}^r} \\ &\leq \int_0^t d\tau(\|\phi\|_{\mathbf{L}^r}\|\partial_x F_1\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^\infty}\|\partial_x F_2\|_{\mathbf{L}^r_x \mathbf{L}^1_y}) \\ &< Ct^{-\frac{1}{\alpha}+\frac{1}{\alpha r}+1}\|\phi\|_{\mathbf{W}} \leq t^{-\frac{1}{\alpha}+\frac{1}{\alpha r}}\|\phi\|_{\mathbf{W}} \end{aligned} \tag{2.33}$$

for all $t \in [0,1]$.

We consider now the case $t > 1$,

$$\begin{aligned} \|x^\mu \int_0^t \mathcal{G}(t-\tau)\phi(\tau)d\tau\|_{\mathbf{L}^r} \leq &\int_0^{t/2} d\tau(\|\phi\|_{\mathbf{L}^1}\|x^\mu F_1\|_{\mathbf{L}^r} + \|x^\mu\phi\|_{\mathbf{L}^1}\|F_1\|_{\mathbf{L}^r}) \\ &+ \int_{t/2}^t d\tau(\|\phi\|_{\mathbf{L}^r}\|x^\mu F_1\|_{\mathbf{L}^1} + \|x^\mu\phi\|_{\mathbf{L}^r}\|F_1\|_{\mathbf{L}^1}) \\ &+ \int_0^{t/2} d\tau\|\phi\|_{\mathbf{L}^1}\|x^\mu F_2\|_{\mathbf{L}^r_x \mathbf{L}^\infty_y} \\ &+ \int_{t/2}^t d\tau\|\phi\|_{\mathbf{L}^\infty}\|x^\mu F_2\|_{\mathbf{L}^r_x \mathbf{L}^1_y}. \end{aligned}$$

So in view of estimates (2.20), (2.21) , (2.8) and (2.9), we attain

$$\|x^\mu \int_0^t \mathcal{G}(t-\tau)\phi(\tau)d\tau\|_{\mathbf{L}^r} \leq Ct^{-\frac{1}{\alpha}+\frac{1}{r\alpha}+\frac{\mu}{\alpha}}\|\langle t\rangle^{1+\gamma}\phi\|_{\mathbf{W}}. \tag{2.34}$$

In the same way we estimate $\|\partial_x \mathcal{G}\phi\|_{\mathbf{L}^r}$ for $t > 1$,

$$\begin{aligned} \|\partial_x \int_0^t \mathcal{G}(t-\tau)\phi(\tau)d\tau\|_{\mathbf{L}^r} \leq &\int_0^{t/2} d\tau(\|\phi\|_{\mathbf{L}^1}\|\partial_x F_1\|_{\mathbf{L}^r} + \|\phi\|_{\mathbf{L}^1}\|\partial_x F_2\|_{\mathbf{L}^r\mathbf{L}^\infty}) \\ &+ \int_{t/2}^t d\tau(\|\phi\|_{\mathbf{L}^r}\|\partial_x F_1\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^\infty}\|\partial_x F_2\|_{\mathbf{L}^r_x \mathbf{L}^1_y}) \\ \leq &Ct^{-\frac{2}{\alpha}+\frac{1}{r\alpha}}\|\langle t\rangle^{1+\gamma}\phi\|_{\mathbf{W}}. \end{aligned} \tag{2.35}$$

Then, by (2.32)-(2.35) Lemma 2.2 is proved. □

Denote by

$$\mathbf{Z} = \{\phi(x) \in \mathbf{H}_2^{1,0} \cap \mathbf{H}_\infty^{0,\mu} \cap \mathbf{H}_1^{0,0}; \|\phi\|_{\mathbf{z}} < +\infty\},$$
$$\|\phi\|_{\mathbf{Z}} = (\|\langle x\rangle^\mu \phi\|_{\mathbf{L}^\infty} + \|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{H}_2^1}),$$

where $\mu \in (0,1]$. Now we prove local existence theorem. Note that the existence time $T > 0$ could be sufficiently small.

Theorem 2.3. *Let initial data $u_0 \in \mathbf{Z}$. Then for some time interval $T > 0$ there exists a unique solution $u \in \mathbf{C}([0,T];\mathbf{X})$ to problem* (1.1)*. Moreover the existence time T can be chosen as follows*

$$T = \Big(\|u_0\|_{\mathbf{Z}}\big(1 + \frac{1}{2C}\|u_0\|_{\mathbf{Z}}\big)^{-\sigma-1}\Big)^{\frac{1}{\mu_1}}, \quad \mu_1 = 1 - \frac{1}{\alpha}.$$

Proof. We apply the contraction mapping principle in a ball of a radius $\rho > 0$ in a complete metric space $\mathbf{X}_T$,

$$\mathbf{X}_{T,\rho} = \{\phi \in \mathbf{C}([0,T];\mathbf{X}) : \ \sup_{t\in[0,T]}\|u\|_{\mathbf{X}} = \|u\|_{\mathbf{X}_T} \le \rho\},$$

where $\rho = \frac{1}{2C}\|u_0\|_{\mathbf{Z}}$. For $v \in \mathbf{X}_{T,\rho}$ we define the mapping $\mathcal{M}(v)$ by

$$\mathcal{M}(v) = \mathcal{G}(t)u_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))d\tau, \tag{2.36}$$

where $\mathcal{N}(v(\tau)) = \lambda|v|^\sigma v$. We first prove that

$$\|\mathcal{M}(v)\|_{\mathbf{X}_T} \le \rho,$$

when $v \in \mathbf{X}_{T,\rho}$. We have by Lemmas 2.1 and 2.2 for $\mu_1 = 1 - \frac{1}{\alpha}$,

$$\begin{aligned} \|\mathcal{M}(v)\|_{\mathbf{X}_T} &\le \|\mathcal{G}u_0\|_{\mathbf{X}_T} + \|\int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))d\tau\|_{\mathbf{X}_T} \\ &\le C\|u_0\|_{\mathbf{Z}} + CT^{\mu_1}(1 + \|v\|_{\mathbf{X}_T})^{\sigma+1} \\ &\le \frac{\rho}{2} + CT^{\mu_1}(1+\rho)^{\sigma+1} \le \rho, \end{aligned} \tag{2.37}$$

if $T > 0$ is small enough. Therefore, the mapping $\mathcal{M}$ transforms a ball of a radius $\rho > 0$ into itself in the space $\mathbf{X}_T$. As in the proof of (2.37), we have for $w, v \in \mathbf{X}_T$

$$\begin{aligned} \|\mathcal{M}(w) - \mathcal{M}(v)\|_{\mathbf{X}_T} &\le \|\int_0^t \mathcal{G}(t-\tau)(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))d\tau\|_{\mathbf{X}_T} \\ &\le CT^{\mu_1}\|w - v\|_{\mathbf{X}_T}(1 + \|w\|_{\mathbf{X}_T} + \|v\|_{\mathbf{X}_T})^\sigma \le \frac{1}{2}\|w-v\|_{\mathbf{X}_T}, \end{aligned}$$

since $T > 0$ is small enough. Thus $\mathcal{M}$ is a contraction mapping in $\mathbf{X}_{T,\rho}$; therefore, there exists a unique solution $u \in \mathbf{X}_T$ to the problem (1.1). Theorem 2.3 is proved. $\square$

Now we define the space $\mathbf{X}[T_1,T_2] = \mathbf{C}([T_1,T_2];\mathbf{Z})$ with the norm

$$\|\psi\|_{\mathbf{X}[T_1,T_2]} = \sup_{t\in[T_1,T_2]}\|\psi(t)\|_{\mathbf{Z}}.$$

Theorem 2.4. *Let the initial data $u_0 \in \mathbf{Z}$ and the following a priory estimate be valid*

$$\|u\|_{\mathbf{X}[0,T)} \le C(T)\|u_0\|_{\mathbf{Z}}, \tag{2.38}$$

provided that there exists a solution $u \in \mathbf{X}[0,T)$ for some $T > 0$. Then there exists a unique global solution $u \in \mathbf{X}[0,\infty)$ to the problem (1.1)*.*

Proof. From Lemma 2.1, we have that $\mathcal{G}(t-T_1) : \mathbf{Z} \to \mathbf{X}[T_1, T_2]$ for any $T_2 > T_1 \geq 0$ and

$$\|\mathcal{G}(t-T_1)\phi\|_{\mathbf{X}[T_1,T_2]} \leq C\|\phi\|_{\mathbf{Z}}.$$

Also using Lemma 2.2 we obtain that $\int_{T_1}^t \mathcal{G}(t-\tau)\mathcal{N}(v(\tau))d\tau \in \mathbf{X}[T_1, T_2]$ for any $v \in \mathbf{X}[T_1, T_2]$, $T_2 > T_1 \geq 0$, and

$$\begin{aligned}
&\|\int_{T_1}^t \mathcal{G}(t-\tau)(\mathcal{N}(w(\tau)) - \mathcal{N}(v(\tau)))d\tau\|_{\mathbf{X}[T_1,T_2]} \\
&\leq C\|w-v\|_{\mathbf{X}[T_1,T_2]}(1 + \|w\|_{\mathbf{X}[T_1,T_2]} + \|v\|_{\mathbf{X}[T_1,T_2]})^{\sigma},
\end{aligned}$$

for all $v, w \in \mathbf{X}[T_1, T_2]$, where $\sigma > 0$. Using a priory estimates (2.38) we can prolongate the local solution given by Theorem 2.3 for all times $t > 0$. Indeed, by the contrary we can suppose that there exists a maximal existence time $T > 0$ such that $u \in \mathbf{X}[0, T)$. If we choose a new initial time $T_1 \in [0, T)$ and consider the problem (1.1) with initial data $u(T_1)$, then via a priory estimate (2.38) the norm $\|u(T_1)\|_{\mathbf{Z}}$ is bounded uniformly with respect to $T_1 \in [0, T)$. Then the existence time given by the local existence Theorem 2.3 is bounded from below uniformly with respect to $T_1 \in [0, T)$. Therefore if a new initial time $T_1 > 0$ is chosen to be sufficiently close to the maximal time T, then by virtue of the local existence Theorem 2.3 we can guarantee that there exists a unique solution $u \in \mathbf{X}[0, T]$. Now putting $u(T)$ as a new initial data at time T we can apply the local existence Theorem 2.3 and prolongate the solution $u(t)$ on some bigger time interval $[0, T + T_2]$. This contradicts to the fact that T is a maximal existence time. Hence there exists a unique solution $u \in \mathbf{X}[0, \infty)$ to the problem (1.1). Theorem 2.4 is proved. □

3. Large initial data (proof of Theorem 1.1)

To prove of Theorem 1.1 we first apply the so-called energy method to estimate the $\mathbf{L}^2(\mathbb{R})$ norm: i.e. we multiply equation (1.1) by u and integrate with respect to $x \in \mathbb{R}^+$, to get

$$\frac{d}{dt}\|u(t)\|^2_{\mathbf{L}^2} + 2\lambda \int_0^{+\infty} |u|^{\sigma+1} dx = -2\int_0^{+\infty} u\mathcal{L}u dx. \tag{3.1}$$

By the Plancherel Theorem we have

$$\begin{aligned}
2\int_0^{+\infty} u\mathcal{L}u dx &= 2\operatorname{Re}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \overline{\widehat{u}}(p)C_\alpha p^\alpha(\widehat{u}(p) - \frac{u(0)}{p})\, dp \\
&= \operatorname{Re}(VP\frac{1}{\pi i}\int_{-i\infty}^{i\infty} C_\alpha p^\alpha|\widehat{u}(p) - \frac{u(0)}{p}|^2\, dp \\
&\quad - \operatorname{Re} u(0)VP\frac{1}{\pi i}\int_{-i\infty}^{i\infty} C_\alpha p^{\alpha-1}(\widehat{u}(p) - \frac{u(0)}{p})\, dp).
\end{aligned}$$

By the Cauchy Theorem using the analyticity of $\widehat{u}(p)$ and $K(p) = C_\alpha p^\alpha$ in the right-half complex plane we attain

$$\begin{aligned}
&VP\int_{-i\infty}^{i\infty} C_\alpha p^{\alpha-1}(\widehat{u}(p) - \frac{u(0)}{p})\, dp \\
&= \frac{1}{2}\operatorname{res}_{p=0}(C_\alpha p^{\alpha-1}(\widehat{u}(p) - \frac{u(0)}{p})) + \int_{-i\infty}^{i\infty} C_\alpha p^{\alpha-1}(\widehat{u}(p) - \frac{u(0)}{p})\, dp = 0.
\end{aligned}$$

Thus from dissipative condition $\operatorname{Re} K(p) > 0$ for $\operatorname{Re} p = 0$ we get

$$2\int_0^{+\infty} u\mathcal{L}u dx = \frac{1}{\pi} VP \int_{-\infty}^{\infty} \operatorname{Re} K(ip)|\widehat{u}(ip) - \frac{u(0)}{ip}|^2 \, dp > 0.$$

Also since $\lambda > 0$ we have

$$\lambda \int_0^{+\infty} |u|^{\sigma+1} dx > 0.$$

Therefore, from equation (3.1) we obtain

$$\frac{d}{dt}\|u\|_{\mathbf{L}^2}^2 \leq 0. \tag{3.2}$$

Hence integrating with respect to time we see that

$$\sup_{t\geq 0} \|u(t)\|_{\mathbf{L}^2} \leq \|u_0\|_{\mathbf{L}^2}. \tag{3.3}$$

Now we prove that

$$\sup_{t\geq 0} \|u_x(t)\|_{\mathbf{L}^2} \leq \|u_{0x}\|_{\mathbf{L}^2}. \tag{3.4}$$

We differentiate (1.1) with respect to the space variable, multiply equation (1.1) by u_x and integrate with respect to $x \in \mathbb{R}^+$, to get

$$\frac{d}{dt}\|u_x(t)\|_{\mathbf{L}^2}^2 + 2\sigma\lambda \int_0^{+\infty} |u|^{\sigma-1}|u|_x^2 dx = -2\int_0^{+\infty} u_x \partial_x \mathcal{L}u dx. \tag{3.5}$$

Since

$$2\int_0^{+\infty} u_x \partial_x \mathcal{L}u dx = \frac{1}{\pi} VP \int_{-\infty}^{\infty} \operatorname{Re} K(ip)|p|^2|\widehat{u}(ip) - \frac{u(0)}{ip}|^2 \, dp > 0$$

and

$$\sigma\lambda \int_0^{+\infty} |u|^{\sigma-1}|u|_x^2 dx > 0$$

integrating (3.5) with respect to time we easily get (3.4).

Note that time decay estimates (3.3) and (3.4) are not optimal. To get an optimal time decay estimates we need to show that the $\mathbf{L}^1$ - norm of the solution does not grow with time. Using the idea of papers [2], [5] we multiply equation (1.1) by $S \equiv \operatorname{sign} u \equiv \frac{u}{|u|}$ and integrate with respect to x over $\mathbb{R}^+$ to get

$$\int_{\mathbb{R}^+} u_t(x,t)S(x,t)dx + \int_{\mathbb{R}^+} \mathcal{N}(u)(x,t)S(x,t)dx = -\int_{\mathbb{R}^+} S(x,t)\mathcal{L}u dx, \tag{3.6}$$

where $\mathcal{N}(u) = \lambda|u|^\sigma u$. We have

$$\int_{\mathbb{R}^+} u_t(x,t)S(x,t)dx = \int_{\mathbb{R}^+} \frac{\partial}{\partial t}|u(x,t)|dx = \frac{d}{dt}\|u(t)\|_{\mathbf{L}^1},$$
$$\int_{\mathbb{R}^+} \mathcal{N}(u)(x,t)S(x,t)dx = \lambda \int_{\mathbb{R}^+} |u|^\sigma dx \geq 0.$$

Representing the operator $\mathcal{L}u$ via the Riesz potential (see [29]) let us show that

$$\int_{\mathbb{R}^+} S(x,t)\mathcal{L}u dx \geq 0. \tag{3.7}$$

By [1], we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} p^{-\nu-1} e^{px} \, dp = \frac{1}{\Gamma(\nu+1)} x^\nu.$$

Thus for $\alpha \in (1,2)$, we get

$$\mathcal{L}u = -C\partial_x \int_0^x (x-y)^{1-\alpha}\partial_y u(y)\,dy, C>0.$$

Denote $S(x,t) = \operatorname{sign}(u(x,t))$ and represent $u(x,t) = S(x,t)|u(x,t)|$. We make a regularization

$$K''_\varepsilon(x) = \begin{cases} \partial_x^2 x^{1-\alpha}, & \text{for } x \geq \varepsilon \\ 0, & \text{for } 0 \leq x < \varepsilon, \end{cases}$$

such that $K'_\varepsilon(x) \leq 0$ and $K''_\varepsilon(x) \geq 0$ for all $x > 0$. We can easily see that

$$\partial_x \int_0^x (x-y)^{1-\alpha} u_y(y,t)\,dy = \lim_{\varepsilon\to 0} \partial_x \int_0^x K_\varepsilon(x-y)u_y(y,t)\,dy.$$

(To justify our calculation we note that the linear operator $\mathcal{L}$ in equation (1.1) is strongly dissipative, therefore by smoothing effect the solution obtain regularity $u \in \mathbf{C}^1(R^+)$ (see Theorem 2.3 and [28])).We have

$$\begin{aligned}
&\int_{\mathbb{R}^+} dx S(x,t)\partial_x \int_0^x K_\varepsilon(x-y)\partial_y u(y,t)\,dy \\
&= \int_{\mathbb{R}^+} dx S(x,t)\partial_x \int_0^x dy K_\varepsilon(x-y)S(y,t)\partial_y|u(y,t)| \\
&= K_\varepsilon(0)\int_{\mathbb{R}^+} dx S^2(x,t)\partial_x|u(x,t)| \\
&\quad + \int_{\mathbb{R}^+} dy\partial_y|u(y,t)| \int_y^{+\infty} dx S(y,t)S(x,t)\partial_x K_\varepsilon(x-y).
\end{aligned}$$

Then via the identity $S(y,t)S(x,t) = 1 - \frac{1}{2}(S(x,t)-S(y,t))^2$,

$$\begin{aligned}
&\int_{\mathbb{R}^+} dx S(x,t)\partial_x \int_0^x K_\varepsilon(x-y)\partial_y u(y,t)\,dy \\
&= K_\varepsilon(0)\int_{\mathbb{R}^+} dx S^2(x,t)\partial_x|u(x,t)| + \int_{\mathbb{R}^+} dy\partial_y|u(y,t)| \int_y^{+\infty} dx\partial_x K_\varepsilon(x-y) \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^+} dy\partial_y|u(y,t)| \int_y^{+\infty} dx(S(x,t)-S(y,t))^2\partial_x K_\varepsilon(x-y).
\end{aligned}$$

Since

$$\int_{\mathbb{R}^+} dy\partial_y|u(y,t)| \int_y^{+\infty} dx\partial_x K_\varepsilon(x-y) = -K_\varepsilon(0)\int_{\mathbb{R}^+} dy\partial_y|u(y,t)|$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^+} dx S(x,t)\partial_x \int_0^x K_\varepsilon(x-y)\partial_y u(y,t)\,dy \\
&= -\frac{1}{2}\int_{\mathbb{R}^+} dy\partial_y|u(y,t)| \int_y^{+\infty} dx(S(x,t)-S(y,t))^2\partial_x K_\varepsilon(x-y).
\end{aligned} \tag{3.8}$$

Integrating by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}^+} dy\partial_y|u(y,t)|\int_y^{+\infty} dx(S(x,t)-S(y,t))^2\partial_x K_\varepsilon(x-y)\\&=-|u(0,t)|\int_0^{+\infty} dx(S(x,t)-S(0,t))^2\partial_x K_\varepsilon(x)\\&\quad+\int_{\mathbb{R}^+} dy|u(y,t)|\int_y^{+\infty} dx K''_\varepsilon(x-y)(S(x,t)-S(y,t))^2.\end{aligned}$$

Therefore, from (3.8) using $\partial_x K_\varepsilon(x)<0$, we gain

$$\begin{aligned}&\int_{\mathbb{R}^+} dx S(x,t)\partial_x\int_0^x K_\varepsilon(x-y)\partial_y u(y,t)\,dy\\&\le -\frac{1}{2}\int_{\mathbb{R}^+} dy|u(y,t)|\int_y^{+\infty} dx K''_\varepsilon(x-y)(S(x,t)-S(y,t))^2\end{aligned}$$

and therefore since $\partial_x^2 K(x)>0$ for all $x>0$ we get

$$\int_{\mathbb{R}^+} dx S(x,t)\partial_x\int_0^x K_\varepsilon(x-y)\partial_y u(y,t)\,dy\le 0.$$

Hence we have

$$\begin{aligned}&\int_{\mathbb{R}^+} S(x,t)\mathcal{L}u dx\\&\ge -\frac{C}{2}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^+} dy|u(y,t)|\int_{\mathbb{R}^+} dx K''_\varepsilon(x-y)(S(x,t)-S(y,t))^2\ge 0.\end{aligned}$$

Thus (3.7) is true and from (3.6) we find $\frac{d}{dt}\|u(t)\|_{\mathbf{L}^1}\le 0$. From this inequality, we see that the norm $\|u(t)\|_{\mathbf{L}^1}$ is bounded for all $t\ge 0$.

We now prove that the norm $\|u_x(t)\|_{\mathbf{L}^2}\to 0$ as $t\to\infty$. Taking $\varrho\in(0,1)$ by the Plancherel theorem,

$$\begin{aligned}\int_0^{+\infty} u_x\partial_x\mathcal{L}u dx&=\frac{1}{\pi}VP\int_{-\infty}^{\infty}\operatorname{Re}K(ip)|p|^2|\widehat{u}(ip,t)-\frac{u(0)}{ip}|^2\,dp\\&\ge\int_{|p|\ge\varrho}\operatorname{Re}C_\alpha p^\alpha|\widehat{u}_x(p,t)|^2\,dp\\&\ge C\varrho^\alpha\|u_x(t)\|^2_{\mathbf{L}^2}-C\varrho^{\alpha+3}\sup_{|p|<\varrho}|\widehat{u}(p,t)|^2\\&\quad-C\rho^{\alpha+2}|u(0)|\sup_{|p|<\varrho}|\widehat{u}(p,t)|-C\varrho^{\alpha+1}|u(0)|^2\\&\ge C\varrho^\alpha\|u(t)\|^2_{\mathbf{L}^2}-C\varrho^{\alpha+3}\|u(t)\|^2_{\mathbf{L}^1}\\&\quad-C\varrho^{\alpha+2}\|u(t)\|_{\mathbf{L}^1}\|u\|_{\mathbf{L}^\infty}-C\varrho^{\alpha+1}\|u\|^2_{\mathbf{L}^\infty}\\&\ge C\varrho^\alpha\|u(t)\|^2_{\mathbf{L}^2}-C\varrho^{\alpha+3}\|u(t)\|^2_{\mathbf{L}^1}\\&\quad-C\varrho^{\alpha+2}\|u(t)\|_{\mathbf{L}^1}\|u\|^{\frac12}_{\mathbf{L}^2}\|u_x\|^{\frac12}_{\mathbf{L}^2}-C\varrho^{\alpha+1}\|u\|_{\mathbf{L}^2}\|u_x\|_{\mathbf{L}^2},\end{aligned}\tag{3.9}$$

where we have used that

$$\|u_x(t)\|^2_{\mathbf{L}^\infty}\le C\|u(t)\|_{\mathbf{L}^2}\|u_x(t)\|_{\mathbf{L}^2}.$$

Since the norms $\|u(t)\|_{\mathbf{L}^1}$, $\|u(t)\|_{\mathbf{L}^2}$ and $\|u_x(t)\|_{\mathbf{L}^2}$ are bounded, choosing $\varrho(t) = C^{-1/\alpha}(1+t)^{-1/\alpha}$, we obtain

$$\frac{d}{dt}\|u_x(t)\|^2_{\mathbf{L}^2} \le -(1+t)^{-1}\|u_x(t)\|^2_{\mathbf{L}^2} + C(1+t)^{-1-\frac{1}{\alpha}}.$$

We substitute $\|u_x(t)\|^2_{\mathbf{L}^2} = \frac{1}{2}h(t)(1+t)^{-2}$, then for $h(t)$ we have

$$h'(t) \le C(1+t)^{1-\frac{1}{\alpha}},$$

hence integration with respect to time yields $h(t) \le C(1+t)^{2-\frac{1}{\alpha}}$. Therefore we get the time decay estimate

$$\|u_x(t)\|_{\mathbf{L}^2} \le C(1+t)^{-1/(2\alpha)}.$$

and therefore

$$\|u_x(t)\|_{\mathbf{L}^\infty} \le C\|u(t)\|^{\frac{1}{2}}_{\mathbf{L}^2}\|u_x(t)\|^{\frac{1}{2}}_{\mathbf{L}^2} \le C(1+t)^{-\frac{1}{4\alpha}}.$$

We substitute this estimate in (3.9) to obtain

$$\frac{d}{dt}\|u_x(t)\|^2_{\mathbf{L}^2} \le -(1+t)^{-1}\|u_x(t)\|^2_{\mathbf{L}^2} + C(1+t)^{-1-\frac{1}{\alpha}-\frac{1}{2\alpha}}.$$

Again after the change $\|u_x(t)\|^2_{\mathbf{L}^2} = \frac{1}{2}h(t)(1+t)^{-2}$ we get

$$\|u_x(t)\|_{\mathbf{L}^2} \le C(1+t)^{-\frac{1}{2\alpha}-\frac{1}{4\alpha}}.$$

We can repeat this consideration to get the optimal time decay estimate

$$\|u_x(t)\|_{\mathbf{L}^2} \le C(1+t)^{-\frac{3}{2\alpha}} \tag{3.10}$$

for all $t > 0$.

We now prove that the norm $\|u(t)\|_{\mathbf{L}^2} \to 0$ as $t \to \infty$. Taking $\varrho \in (0,1)$ by the Plancherel theorem we get

$$\begin{aligned}
&\int_0^{+\infty} u\mathcal{L}u dx \\
&= \frac{1}{\pi}VP\int_{-\mathbf{i}\infty}^{i\infty} \operatorname{Re} K(ip)(\widehat{u}(ip,t) - \frac{u(0)}{ip})\overline{\widehat{u}}\, dp \\
&\ge \int_{|p|\ge\varrho} \operatorname{Re} C_\alpha p^\alpha |\widehat{u}(p,t)|^2\, dp - u(0)\int_{-i\infty}^{i\infty} \operatorname{Re} C_\alpha p^{\alpha-1}\overline{\widehat{u}}\, dp \\
&\ge \varrho^\alpha\|u(t)\|^2_{\mathbf{L}^2} - \varrho^{\alpha+1}\sup_{|p|<\varrho}|\widehat{u}(p,t)|^2 - |u(0)|\varrho^\alpha \sup_{|p|<\varrho}|\widehat{u}(p,t)| - \varrho^{\alpha-1}|u(0)|^2 \\
&\ge \varrho^\alpha\|u(t)\|^2_{\mathbf{L}^2} - \varrho^{\alpha+1}\|u(t)\|^2_{\mathbf{L}^1} - |u(0)|\varrho^\alpha\|u(t)\|_{\mathbf{L}^1} - \varrho^{\alpha-1}|u(0)|^2.
\end{aligned}$$

Since the norms $\|u(t)\|_{\mathbf{L}^1}$, $\|u(t)\|_{\mathbf{L}^2}$ are bounded and due to (3.10)

$$|u(0)| \le \|u(t)\|_{\mathbf{L}^\infty} \le C\|u(t)\|^{\frac{1}{2}}_{\mathbf{L}^2}\|u_x(t)\|^{\frac{1}{2}}_{\mathbf{L}^2} \le C(1+t)^{-\frac{3}{4\alpha}},$$

choosing $\varrho(t) = C^{-1/\alpha}(1+t)^{-1/\alpha}$, we obtain

$$\frac{d}{dt}\|u(t)\|^2_{\mathbf{L}^2} \le -(1+t)^{-1}\|u(t)\|^2_{\mathbf{L}^2} + C(1+t)^{-1-\frac{1}{2\alpha}}.$$

We substitute $\|u(t)\|^2_{\mathbf{L}^2} = h(t)(1+t)^{-2}$, then for $h(t)$ we have

$$h'(t) \le C(1+t)^{1-\frac{1}{2\alpha}},$$

hence integration with respect to time yields

$$h(t) \le C(1+t)^{2-\frac{1}{2\alpha}}.$$

Therefore, we have the time decay estimates

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{-\frac{1}{4\alpha}},$$
$$|u(0)| \le C(1+t)^{-\frac{3}{4\alpha}-\frac{1}{8\alpha}}.$$

We can repeat this consideration to get

$$\frac{d}{dt}\|u(t)\|^2_{\mathbf{L}^2} \le -(1+t)^{-1}\|u(t)\|^2_{\mathbf{L}^2} + C(1+t)^{-1-\frac{1}{\alpha}}.$$

Therefore, we obtain the optimal estimate

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{-\frac{1}{2\alpha}} \tag{3.11}$$

for all $t > 0$. Also from (3.10) and (3.11) we get the optimal time decay of the $\mathbf{L}^\infty(\mathbb{R}^+)$-norm of the solutions

$$\|u(t)\|_{\mathbf{L}^\infty} \le C(1+t)^{-1/\alpha} \tag{3.12}$$

for all $t > 0$.

Now we can estimate the norm $\mathbf{L}^{\infty,\mu}$ $(\mathbb{R}^+)$. By the integral formula (2.2) we have

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^{\infty,\mu}} \le \|u_0\|_{\mathbf{L}^1}\|G(t)\|_{\mathbf{L}^{\infty,\mu}} + \|u_0\|_{\mathbf{L}^{\infty,\mu}}\|G(t)\|_{\mathbf{L}^1}\\ + C\int_0^t \||u(\tau)|^{\sigma+1}\|_{\mathbf{L}^1}\|G(t-\tau)\|_{\mathbf{L}^{\infty,\mu}}d\tau\\ + C\int_0^t \||u(\tau)|^{\sigma+1}\|_{\mathbf{L}^{\infty,\mu}}\|G(t-\tau)\|_{\mathbf{L}^1}d\tau\end{aligned}$$

Hence using Lemmas 2.1 and 2.2,

$$\begin{aligned}&\|u(t)\|_{\mathbf{L}^{\infty,\alpha+1}}\\ &\le C\langle t\rangle^{-\frac{1-\mu}{\alpha}} + C\int_0^t \|u(\tau)\|^\sigma_{\mathbf{L}^\infty}\langle t-\tau\rangle^{-\frac{1-\mu}{\alpha}}d\tau + C\int_0^t \|u(\tau)\|^\sigma_{\mathbf{L}^\infty}\|u(\tau)\|_{\mathbf{L}^{\infty,\mu}}d\tau\\ &\le C\langle t\rangle^{-\frac{1-\mu}{\alpha}} + C\int_0^t \|u(\tau)\|_{\mathbf{L}^{\infty,\mu}}\langle\tau\rangle^{-\frac{\sigma}{\alpha}}d\tau.\end{aligned}$$

Hence for the function $h(t) = \sup_{0\le\tau\le t}\|u(\tau)\|_{\mathbf{L}^{\infty,\mu}}$, we get the inequality

$$h(t) \le C\langle t\rangle^{-\frac{1-\mu}{\alpha}} + C\int_0^t \langle\tau\rangle^{-\frac{\sigma}{\alpha}}h(\tau)d\tau$$

and since $\sigma > \alpha$ by the Gronwall's lemma it follows that

$$\|u(t)\|_{\mathbf{L}^{\infty,\mu}} \le C\langle t\rangle^{-\frac{1-\mu}{\alpha}} \quad \forall t > 0. \tag{3.13}$$

From a priory estimates, due to Theorem 2.4 there exists a unique solution

$$u(x,t) \in C([0,\infty); \mathbf{H}^{1,0}_2 \cap \mathbf{H}^{0,\mu}_\infty \cap \mathbf{H}^{0,0}_1)$$

to the problem (1.1), such that

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-1/\alpha}, \|u(t)\|_{\mathbf{L}^{\infty,\mu}} \le C\langle t\rangle^{-\frac{1-\mu}{\alpha}} \text{ and } \|u(t)\|_{\mathbf{L}^2} \le C\langle t\rangle^{-\frac{1}{2\alpha}}. \tag{3.14}$$

By Lemma 2.2 we have

$$\sup_{t>1} t^{\frac{1}{\alpha}(1+\mu)} \|\mathcal{G}\phi - t^{-\frac{1}{\alpha}} \Lambda_0(xt^{-1/\alpha}) f(\phi)\|_{\mathbf{L}^\infty} \le C\|\phi\|_{\mathbf{L}^{1,\mu}},$$

where $\mu \in [0,1]$. Substituting this formula into (2.2) we obtain

$$u(x,t) = t^{-1/\alpha} A\Lambda_0\Big(\frac{x}{t^{\frac{1}{\alpha}}}\Big) + R(x,t), \tag{3.15}$$

where by (3.14),

$$A = \int_0^{+\infty} u_0 dy + \int_0^{+\infty} d\tau \int_0^{\infty} \mathcal{N}(u)\, dy < \infty,$$

and

$$\begin{aligned} R(x,t) &= O(t^{-\frac{1+\mu}{\alpha}})\Big(\int_0^{+\infty} y^\mu u_0(y)\, dy + \int_0^{+\infty} d\tau \int_0^{+\infty} y^\mu \mathcal{N}(u)\, dy \Big) \\ &\quad + \int_0^{+\infty} \tau^{\frac{2\mu}{\alpha}} O\big((t-\tau)^{-\frac{1+\mu}{\alpha}}\big) d\tau \int_0^{+\infty} \mathcal{N}(u)\, dy \\ &= O(t^{-\frac{1+\mu}{\alpha}}). \end{aligned}$$

Thus the asymptotic (1.4) is valid. Theorem 1.1 is then proved.

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