COMPLEX CENTERS OF POLYNOMIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We present some results on the existence and nonexistence of centers for polynomial first order ordinary differential equations with complex coefficients. In particular, we show that binomial differential equations without linear terms do not have complex centers. Classes of polynomial differential equations, with more than two terms, are presented that do not have complex centers. We also study the relation between complex centers and the Pugh problem. An algorithm is described to solve the Pugh problem for equations without complex centers. The method of proof involves phase plane analysis of the polar equations and a local study of periodic solutions.

1. INTRODUCTION

Consider the differential equation

$$\dot{z} := \frac{dz}{dt} = A_N(t)z^N + A_{N-1}(t)z^{N-1} + \ldots + A_1(t)z \tag{1.1}$$

where z is complex and $A_i(t)$ are continuous functions in t. Let $z(t, c)$ be the solution of (1.1) such that $z(0, c) = c$. For a fixed real number ω, we say that $z(t, c)$ is *periodic* when $z(0, c) = z(\omega, c)$. If the functions $A_i(t)$ are periodic with period ω, then a periodic solution of equation (1.1) is a periodic function with period ω. The *multiplicity* of a periodic solution $\varphi(t)$ of (1.1) is the multiplicity of $\varphi(0)$ as a zero of the displacement function $q \mapsto z(\omega, c) - c$. Note that q is defined and analytic in an open set containing the origin. The solution $z = 0$ is called a *center* for the differential equation if all solutions starting in a neighborhood of 0 are ω-periodic. A center is called a *complex center* when the coefficients are complex functions, and it is called a *real center* when the coefficients are real functions. If the coefficients are real functions and $A_N(t)$ does not change sign, then $z = 0$ is not a center (see [1]). There are real centers when $A_N(t)$ changes sign (see [5], [6], and [15]); the problem is related to the classical center-focus problem of polynomial two-dimensional systems. The case $N = 3$, with $A_3(t) \equiv 1$, was considered recently in [7]; where it was shown that the equation could have a center at the origin if the coefficients are complex valued. Two unexpected properties of the Abel differential equation are given in [7]; there is no upper bound for the number of

periodic solutions, and the center variety is formed by infinitely many connected components. The work in [7] is mainly motivated by a problem stated in [9]. It was shown in [9] that if the coefficients are real and bounded by a constant C, then there are at most $8 \exp\{(3C + 2) \exp[\frac{3}{2}(2C + 3)^N]\}$ periodic solutions. The problem in [9] is to prove an analogue of this result for complex equations, that is, for complex z and complex coefficients A_i. This paper is partly motivated by the results of [7].

Another motivation for the work in this paper is the local study of the Pugh problem about equations with real coefficients. We recall that Pugh problem (see [14]) is to find an upper bound for the number of periodic solutions in terms of N. It was shown in [10] that there are no upper bounds for the number of periodic solutions for $N \geq 4$. Upper bounds can be found for some particular classes; (see [1] and its references). Hence, one should seek upper bounds for the number of periodic solutions in terms of N and the degrees of the real polynomial functions $A_i(t)$. A local version of Pugh problem is to find an upper bound for the multiplicity of a periodic solution in terms of N and the degrees of the polynomials $A_i(t)$. This problem was considered in [13], with $N = 4$. It was conjectured in [13], that if $A_4 \equiv 1$, $A_1 \equiv 0$, A_2 and A_3 are of degree k then the multiplicity of the origin is at most $k+3$. It was shown in [4], that when $k = 2$ then the multiplicity of the origin is at most 8 and there is a unique equation with this maximum multiplicity. It was shown in [3], that the multiplicity of the origin is at most 10 when the degrees of $A_2(t)$ and $A_3(t)$ are 2 and 3, respectively. Having determined the maximum multiplicity, the next step is to construct equations with this number of periodic solutions. This is done by making a sequence of perturbations in $A_2(t)$ and $A_3(t)$, each of which reduces the multiplicity of the origin by one; a periodic solution thus bifurcates out of the origin. This bifurcation task was considered in [3] and [4]. The local problem can be considered with the use of Groebner bases method. This task is considered in the last section; the case $N = 4$ and $A_1 \equiv 0$ is studied. The problem reduces to study the solvability of system of polynomial equations in many variables. Since the solvability in the theory of Groebner bases is over the field of complex numbers, it becomes necessary to consider equations with complex coefficients. That is to consider complex centers.

All the known equations with complex centers have linear terms. On the other hand, computations for equations without linear part demonstrate that $z = 0$ is not a center for polynomial coefficients. These remarks lead us to conjecture that polynomial differential equations without linear terms do not have centers at the origin, at least when the coefficients are polynomial functions in t.

Conjecture 1.1. *Assume that $A_j(t)$, for $j = 2, 3, \ldots, N - 1$, are polynomial functions in t. The solution $z = 0$ is not a center for the differential equation*

$$\dot{z} = z^N + A_{N-1}(t)z^{N-1} + \cdots + A_3(t)z^3 + A_2(t)z^2.$$

In the statement of the conjecture, we are not including equations in which the coefficients are trigonometric polynomials. The results of Section 5 about such equations demonstrate that the maximum possible multiplicity of the origin when the coefficients are complex is higher than the maximum multiplicity when the coefficients are restricted to be real numbers.

First, we present classes of equations with complex coefficients that do not have centers at the origin. Our main results in this direction are the following:

Theorem 1.2. *Assume that the differential equation* (1.1) *has an invariant line* $\theta = \alpha$, *and either* $A_N(t) > \gamma > 0$, *or* $A_N(t) < \gamma < 0$. *If* $\gamma \cos((N-1)\alpha) > 0$ *then the equation can have only a finite number of periodic solutions on the invariant line. In particular,* $z = 0$ *is not a center.*

Proposition 1.3. *Consider continuous functions* $A(t), B(t)$ *and* $C(t)$ *and let*

$$\lambda_1 = \int_0^\omega C(t)dt,$$

$$\lambda_2 = \int_0^\omega B(t)dt,$$

$$\lambda_3 = \int_0^\omega [A(t) + (M-L)B(t)(\int_0^t C(s)ds)]dt.$$

The solution $z = 0$ *of the differential equation*

$$\dot{z} = A(t)z^N + B(t)z^M + C(t)z^L, \tag{1.2}$$

with $1 < L < M < N = L + M - 1$, *has multiplicity* L *if and only if* $\lambda_1 \neq 0$. *The multiplicity is* M *if and only if* $\lambda_1 = 0$ *and* $\lambda_2 \neq 0$; *and it is* N *if and only if* $\lambda_1 = \lambda_2 = 0$ *and* $\lambda_3 \neq 0$. *If the origin is a center then* $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Remark 1.4. It follows from Proposition 1.3, that:

(1) If $B(t)$ and $C(t)$ are polynomial functions with small coefficients, then $z = 0$ is not a center for the differential equation

$$\dot{z} = z^N + B(t)z^M + C(t)z^L$$

with $1 < L < M < N = L + M - 1$.

(2) A polynomial differential equation with only two terms has a center at $z = 0$ only when one of the terms is linear.

Now, we describe some equations that have centers at the origin.

Proposition 1.5. *The solution* $z = 0$ *of the differential equation*

$$\dot{z} = z^N + A(t)z \tag{1.3}$$

is of multiplicity N *if and only if*

$$e^{\int_0^\omega A(t)dt} = 1, \quad \int_0^\omega e^{(N-1)\int_0^t A(s)ds}dt \neq 0.$$

The origin is a center if and only if

$$e^{\int_0^\omega A(t)dt} = 1, \quad \int_0^\omega e^{(N-1)\int_0^t A(s)ds}dt = 0.$$

Equations with centers at the origin are given in the following result; the second part is a version of the result in [7]. As it was first noticed in [7], the center variety in each of the equations considered here contains infinitely many connected components. Center variety of each of the known real centers contains a finite number of connected components.

Corollary 1.6. (1) *With* $\omega = 2\pi$, *we consider the differential equation*

$$\dot{z} = z^N + (C + \frac{2}{N-1}\tan(t+c_0))z,$$

where c_0 is a complex non-real number. The multiplicity of $z = 0$ is one if and only if $C = pi$, where p is an integer. The multiplicity is N if and only if

$$\int_0^{2\pi} e^{(N-1)pit} \sec^2(t + c_0)dt \neq 0.$$

If this integral vanishes then $z = 0$ is a center. In particular, there is a center when $(N-1)p$ is an odd number.

(2) *Suppose that C is a number and x is a real number. With $\omega = 2\pi$, the solution $z = 0$ is a center for the equation*

$$\dot{z} = z^N + \frac{1}{N-1}(C - xi\cos t)z$$

if and only if $C = pi$, where p is an integer and x is a zero of the Bessel's function J_p.

Now, we give another class of equations that have a center. The coefficients of the equation satisfy a composition type condition similar to those in [5], [6], and [15].

Proposition 1.7. *Let $p(t)$ be a differentiable ω-periodic function and $q(t)$ is a continuous function with $e^{\int_0^\omega q(t)dt} = 1$. The solution $z = 0$ is a center for the differential equation*

$$\dot{z} = q(t)\,z + \sum_{k=2}^{N} p'(t)\,f_k(p(t))\,e^{(1-k)\int_0^t q(s)ds}\,z^k,$$

where f_k are any continuous functions.

Finally, we use the method of Groebner bases to study multiplicity of periodic solutions when the coefficients are polynomial functions in t, or in $\cos t$ and $\sin t$.

Theorem 1.8. *Consider the equation*

$$\dot{z} = z^4 + A(t)z^3 + B(t)z^2,$$

where $A(t)$ and $B(t)$ are polynomial functions.
(I) In each of the following classes of coefficients, $z = 0$ is not a center and the maximum possible multiplicity is the same whether the coefficients are complex or restricted to be real.

(1) *A and B are polynomial functions in t of degrees 3 and 2, respectively.*
(2) *A and B are polynomial functions in t of degrees 5 and 1, respectively.*
(3) *A and B are polynomial functions in t of degrees 1 and 3, respectively.*

(II) In each of the following classes of coefficients, $z = 0$ is not a center and the maximum possible multiplicity for real coefficients is less than the maximum multiplicity for complex coefficients. Moreover, the origin is not a center.

(1) *A and B are homogeneous polynomial functions in $\cos t$ and $\sin t$ of degrees 1.*
(2) *A and B are homogeneous polynomial functions in $\cos t$ and $\sin t$ of degrees 2.*
(3) *A and B are homogeneous polynomial functions in $\cos t$ and $\sin t$ of degrees 3 and 1, respectively.*

In the next section, we prove Theorem 1.2. Section 3 contains the proof of Proposition 1.3. The proofs of Proposition 1.5 and Proposition 1.7 are given in Section 4. In Section 5, we consider the case $N = 4$. Several classes of equations are considered and provide evidences to support our conjecture.

2. Proof of Theorem 1.2

Proof. Let $A_m(t) = a_m(t) + ib_m(t)$ and $z = re^{i\theta}$. With this notation, equation (1.1) becomes

$$(\dot{r} + i\dot{\theta}r)e^{i\theta} = r^N e^{Ni\theta} P_N + r^{N-1} e^{(N-1)i\theta}(a_{N-1} + ib_{N-1}) + + \cdots + re^{i\theta}(a_1 + ib_1).$$

Multiplying both sides by $e^{-i\theta}$ and separating the real and imaginary parts, give:

$$\dot{r} = r^N P_N \cos((N-1)\theta)$$
$$+ \sum_{j=1}^{N-2} r^{N-j}[a_{N-j}\cos((N-j-1)\theta) - b_{N-j}\sin((N-j-1)\theta)]$$
$$\dot{\theta} = r^{N-1} P_N \sin((N-1)\theta)$$
$$+ \sum_{j=1}^{N-2} r^{N-j-1}[a_{N-j}\sin((N-j-1)\theta) + b_{N-j}\cos((N-j-1)\theta)].$$

Now, let us choose a real number a with

$$a > \left(\frac{2}{(N-1)\gamma\omega\cos((N-1)\alpha)}\right)^{\frac{1}{N-1}}$$

and such that the inequality

$$\dot{r} > \frac{\gamma\cos((N-1)\alpha)}{2}r^N$$

is satisfied when $r \geq a$ and $\theta = \alpha$. Consider a solution, $re^{i\theta}$ with initial condition $r_0 e^{i\alpha}$. If $r_0 \geq a$, then the inequality implies that

$$r(t) \geq \frac{1}{(r_0^{1-N} - 0.5(N-1)\gamma\cos((N-1)\alpha)t)^{\frac{1}{N-1}}}.$$

Therefore, if $r_0 \geq a$, then the solution becomes unbounded at a point t_0 with

$$t_0 \leq \frac{2r_0^{1-N}}{(N-1)\gamma\cos((N-1)\alpha)} \leq \omega.$$

Hence, the solution is undefined on the interval $[0, \omega]$. Therefore, solutions start outside the disk $r \leq a$ are not periodic, and any solution leaves the disk $r \leq a$ stays outside the disk as time increases.

If there is an infinite sequence of periodic solutions $z(t, r_n e^{i\alpha})$. The numbers r_n are inside the disk $r \leq a$. Hence, let $r_n \to r_0$. If $z(t, r_0 e^{i\alpha})$ is a periodic solution, then all solutions starting in a neighborhood of $r_0 e^{i\alpha}$ are periodic. But q is an analytic function. It follows that $q \equiv 0$. Therefore, $z = 0$ is a center for the equation (1.2). We define a real number R by

$$R = \sup\{r : z(t, re^{i\alpha}) \text{ is } periodic\}.$$

It is clear that $R < \infty$. If the solution $z(t, Re^{i\alpha})$ is defined, then it is a periodic solution. Since q is analytic, it follows from the property of continuous dependence of a solution on its initial value, that $z(t, re^{i\alpha})$ is periodic for r in a neighborhood

of R. This is contrary to the definition of R. Therefore, $z(t, Re^{i\alpha})$ is undefined on the interval $[0, \omega]$ and becomes unbounded at $t_0 \epsilon [0, \omega]$ and leaves the disk $r \leq a$. But $z(t, Re^{i\alpha})$ leaves $r \leq a$, whence $z(t, re^{i\alpha})$ leaves $r \leq a$ for $r < R$ and close to R. These solutions will not return to the disk $r \leq a$ because $\dot{r}(t) > 0$, and hence they are not periodic solutions. This is a contradiction to the assumption that $z = 0$ is a center.

If $z(t, r_0 e^{i\alpha})$ is undefined, then it leaves the disk $r \leq a$. Therefore, $z(t, r_n e^{i\alpha})$ leaves $r \leq a$ for large enough n. This is again a contradiction to the periodicity of $z(t, r_n e^{i\alpha})$.   □

Corollary 2.1. *Assume that $A_N(t) > \gamma > 0$.*

(1) *Let $A_j(t) = a_j(t) + ib_j(t)$, where a_j and b_j are real continuous functions. If there exists an integer k such that*

$$a_j \sin(j\frac{2k\pi}{N-1}) = b_j \cos(j\frac{2k\pi}{N-1})$$

for $j = 1, 2, \ldots, N - 1$, then $z = 0$ is not a center for (1.1).

(2) *If the coefficients A_j, $j = 1, 2, \ldots, N - 1$, are real functions then $z = 0$ is not a center for (1.1).*

(3) *If $N - 1$ is a multiple of 4, A_j are pure imaginary functions for all odd j, and A_j are real for all even j, then $z = 0$ is not a center for equation (1.1).*

Proof. To prove the first part, let $\alpha = \frac{2k\pi}{N-1}$. The condition in the statement implies that

$$a_j \sin(j\alpha) = b_j \cos(j\alpha),$$
$$\sin((N - j - 1)\alpha) = -\sin(\alpha), \quad \sin((N - 1)\alpha) = 0.$$

Hence, $\dot{\theta}(\alpha) = 0$. The line $\theta = \alpha$ is an invariant line. Any solution has a point on this line stays on the line as long as it is defined.

On the other hand,

$$\cos((N - 1)\alpha) = 1.$$

Hence, $A_N(t) \cos((N - 1)\alpha) = A_N(t) > \gamma$. The conditions of Theorem 1.2 are satisfied and the result follows.

The second part follows directly from the first part with $k = 0$. It is known that $z = 0$ is not a center when the coefficients are real functions (see [1]). To prove the third part, we take $k = \frac{N-1}{4}$. This implies that $\alpha = \frac{2k\pi}{N-1} = \frac{\pi}{2}$. Thus $\sin(m\alpha) = 0$ if m is even, and $\cos(m\alpha) = 0$ if m is odd.   □

Remark 2.2. Equations (1.1), with real coefficients, have been studied in [11] and [12] using the methods of complex analysis and topological dynamics. However, some results of [11] and [12] do not hold when the coefficients are allowed to be complex. For example, Abel differential equation, $N = 3$, has 3 periodic solutions when its coefficients are real. On the other hand, it may have an infinite number of periodic solutions when the coefficients are complex (see, [7]). Equations with real coefficients satisfy other properties; in particular, the real axis is an invariant set and the complex conjugate of a periodic solution is also a periodic solution. The phase portrait presented in [11], with some modifications, could be used for equations with complex coefficients. The next task is to obtain some global results on the number of periodic solutions as in [1] and [2]; we defer this exploitation to another paper.

3. PROOF OF PROPOSITION 1.3

Proof. For $0 \leq t \leq \omega$ and c in a neighborhood of 0, we write

$$z(t, c) = \sum_{n=1}^{\infty} d_n(t) c^n,$$

where $d_1(0) = 1$ and $d_n(0) = 0$ if $n > 1$. Thus

$$q(c) = (d_1(\omega) - 1)c + \sum_{n=2}^{\infty} d_n(\omega) c^n.$$

The multiplicity is K if and only if

$$d_1(\omega) = 1, d_2(\omega) = d_3(\omega) = \cdots = d_{K-1}(\omega) = 0, d_K(\omega) \neq 0.$$

The origin is a center when $d_1(\omega) = 1$ and $d_n(\omega) = 0$ for all $n > 1$. The functions $d_n(t)$ are determined by substituting the sum into the equation (1.2) and comparing coefficients of powers of c. The following recursive sequence of differential equations is obtained with the initial equations.

$$\dot{d}_n = AS_N + BS_M + CS_L,$$

where

$$S_K = \sum_{j_1 + j_2 + \cdots + j_K = n} d_{j_1} d_{j_2} \ldots d_{j_K},$$

with $K\epsilon\{N, M, L\}$. To solve these equations, we integrate repeatedly. It is clear that $\dot{d}_1 = 0$ and hence $d_1 \equiv 1$. The next non-zero equation is

$$\dot{d}_L = C(d_1)^L.$$

It gives

$$d_L(t) = \int_0^t C(s) ds.$$

The formula for d_M has two possibilities. If $M + 1 \neq 2L$, then

$$\dot{d}_M = B(d_1)^M.$$

This implies

$$d_M(t) = \int_0^t B(s) ds.$$

In the case $M + 1 = 2L$, the equation becomes

$$\dot{d}_M = B(d_1)^M + LCd_L(d_1)^{L-1}).$$

We integrate this equation to obtain

$$d_M(t) = \int_0^t B(s) ds + \frac{L}{2} (\int_0^t C(s) ds)^2.$$

Consequently, the formula for d_N has two possibilities. The equation is

$$\dot{d}_N = A(d_1)^N + BMd_L(d_1)^{M-1} + CLd_M(d_1)^{L-1}.$$

We integrate this equation to obtain $d_N(t)$. If $M + 1 \neq 2L$, then

$$d_N(t) = \int_0^t A(s) ds + (M - L) \int_0^t (B(s) \int_0^s C(u) du) ds + L \int_0^t B(s) ds \int_0^t C(s) ds.$$

In the case $M + 1 = 2L$, the formula becomes

$$d_N(t) = \int_0^t A(s)ds + (M - L) \int_0^t (B(s) \int_0^s C(u)du)ds$$
$$+ L \int_0^t B(s)ds \int_0^t C(s)ds + \frac{L^2}{6} (\int_0^t C(s)ds)^3.$$

The multiplicity is L if $\int_0^\omega C(t)dt \neq 0$. The multiplicity is M if $\int_0^\omega C(t)dt = 0$ but $\int_0^\omega B(t)dt \neq 0$. Finally, if the multiplicity is greater than M, then

$$d_N(\omega) = \int_0^\omega A(t)dt + (M - L) \int_0^\omega (B(t) \int_0^t C(s)ds)dt.$$

The assumption in the statement of Proposition 1.3, implies that $d_N(\omega) \neq 0$. Therefore, the multiplicity is at most N. $\square$

Corollary 3.1.

(1) *For any continuous functions $A(t)$ and $B(t)$ with $\int_0^\omega A(t)dt \neq 0$, the origin is not a center for the equation*

$$\dot{z} = A(t)z^N + B(t)z^M$$

when $1 < M < N$.

(2) *Assume that $A(t), B(t), C(t)$, and $D(t)$ are continuous functions, and consider the differential equation*

$$\dot{z} = A(t)z^N + B(t)z^M + C(t)z^L + D(t)z,$$

with $1 < L < M < N = L + M - 1$. Let

$$A_1(t) = e^{(1-N) \int_0^t D(s)ds} A(t),$$
$$B_1(t) = e^{(1-M) \int_0^t D(s)ds} B(t),$$
$$C_1(t) = e^{(1-L) \int_0^t C(s)ds} C(t).$$

If any of the following conditions is not satisfied then the origin is not a center,

$$e^{\int_0^\omega D(t)dt} = 1,$$
$$\int_0^\omega C_1(t)dt = 0,$$
$$\int_0^\omega B_1(t)dt = 0,$$
$$\int_0^\omega [A_1(t) + (M - L)B_1(t) \int_0^t C_1(s)ds]dt = 0.$$

Proof. The first part follows directly from the above result. If $B(t) \equiv 0$, then $d_N(\omega) > \gamma\omega \neq 0$. Hence, the multiplicity is at most N.

Now we prove the second part. The first necessary condition for a center is $e^{\int_0^\omega D(t)dt} = 1$. We make the transformation $w = e^{\int_0^t D(s)ds} z$, and obtain

$$\dot{w} = A_1(t)w^N + B_1(t)w^M + C_1(t)w^L,$$

where A_1, B_1, and C_1 are as defined in the statement of the Corollary. Initial conditions and multiplicities of periodic solutions are unchanged under this transformation. Now the result follows from Proposition 1.3. $\square$

Remark 3.2. If the differential equation has a linear term, then the equations for d_k are more complicated. Instead, we will have linear differential equations in which the right-hand side depends also on d_k. We consider such cases in the next section.

4. Proof of Propositions 1.5 and 1.7

Proof of Proposition 1.5. We follow the procedure of the previous section. In this case, we obtain a sequence of linear differential equations.

$$\dot{d}_1 = Ad_1,$$
$$\dot{d}_k = Ad_k, \ 2 \le k \le (N-1),$$
$$\dot{d}_N = (d_1)^N + Ad_N$$

together with the initial conditions

$$d_1(0) = 1; \ d_k(0) = 0, \ 2 \le k \le N.$$

Solving these initial value problems, gives

$$d_1(t) = e^{\int_0^t A(s)ds},$$
$$d_k(t) \equiv 0, \ 2 \le k \le (N-1),$$
$$d_N(t) = \int_0^t A(s)ds \int_0^t e^{(N-1)\int_0^s A(u)du}ds.$$

The origin is of multiplicity N if and only if $d_1(\omega) = 1$ and $d_N(\omega) \neq 0$. These two conditions give

$$e^{\int_0^\omega A(t)dt} = 1,$$
$$\int_0^\omega e^{(N-1)\int_0^t A(s)ds}dt \neq 0.$$

If these two integrals vanish then the solution $z = 0$ is a center. This follows from the general solution of this equation. The general solution is given by

$$w(t) = e^{(1-N)\int_0^t A(s)ds}[(1-N)\int_0^t e^{(N-1)\int_0^s A(u)du}ds + C],$$

where $w = z^{1-N}$ and C is a constant. Since $w(0) = w(\omega)$, all the solutions are periodic. Hence, $z = 0$ is a center. □

It is clear that the conditions for a center are not satisfied by any real function $A(t)$. We give two equations that have centers at the origin. In the first equation, we have

$$A(t) = C + \frac{2}{N-1}\tan(t+c_0).$$

Since the zeros of the complex function cos are real, the function $A(t)$ is a periodic function of period 2π, when c_0 is a complex non-real number. The origin is of multiplicity one when

$$e^{\int_0^{2\pi}[C+\frac{2}{N-1}\tan(t+c_0)]dt} = 1.$$

This gives

$$e^{2\pi C}e^{\frac{1}{N-1}[\ln(\sec^2(2\pi+c_0))-\ln(\sec^2(c_0))]} = 1.$$

Since $\sec(t + c_0)$ is of period 2π, the condition becomes $e^{2\pi C} = 1$. Hence, $C = pi$ for an integer p. The second condition is

$$\int_0^{2\pi} e^{(N-1)pit} \sec^2(t + c_0) dt = 0.$$

If $(N-1)p$ is odd, then a change of variables $t \mapsto t - \pi$ gives

$$\int_\pi^{2\pi} e^{(N-1)pit} \sec^2(t + c_0) dt = -\int_0^\pi e^{(N-1)pit} \sec^2(t + c_0) dt = 0.$$

Therefore,

$$\int_0^{2\pi} e^{(N-1)pit} \sec^2(t + c_0) dt = 0.$$

The conditions for a center given in Proposition 1.5 are satisfied. Hence, all the solutions $z(t, c)$, with c is in a neighborhood of 0, are 2π-periodic. This proves the first part of Corollary 1.6.

The second part in Corollary 1.6, with $N = 3$, is similar to the result of [7]. In fact, it was shown in [7] that $z = 0$ is a center for the equation

$$\dot{z} = z^3 + (C_0 + C_1 e^{-it} + C_2 e^{it}) z$$

if and only if $C_0 = ki$ for an integer k, $C_1 C_2 \neq 0$, and $4i\sqrt{C_1 C_2}$ is a zero for $J_{2|k|}$.

Now, we prove the second part of Corollary 1.6. The first condition for a center in Proposition 1.5 is

$$e^{\int_0^{2\pi} (C - xi \sin t) dt} = e^{2C\pi}.$$

But, $e^{2C\pi} = 1$ if and only if $C = pi$, for an integer p. The second condition for a center becomes

$$\int_0^{2\pi} e^{pit - xi \sin t} dt = \int_0^{2\pi} \cos(pt - x \sin t) dt + i \int_0^{2\pi} \sin(pt - x \sin t) dt.$$

The imaginary part of this quantity is zero; it is an integral of an odd 2π−periodic function over the interval $[0, 2\pi]$. The real part is $2\int_0^\pi \cos(pt - x \sin t) dt$. This integral is the integral form of the Bessel's function. Therefore, the real part is zero if x is a zero of the Bessel's function J_p. We recall that the integral form and power series expansion of J_p are defined by

$$J_p(x) = \int_0^\pi \cos(pt - x \sin t) dt = \sum_{j=0}^\infty (-1)^j \frac{(\frac{x}{2})^{p+2j}}{j!(p+j)!}.$$

The power series representation of Bessel's function is used in [7]. Consequently, their proof is much longer than our proof.

Proof of Proposition 1.7. With the change of variables $z \mapsto e^{\int_0^t q(s)ds} z$, the equation becomes

$$\dot{z} = \sum_{k=2}^N p'(t) f_k(p(t)) z^k.$$

From the expansion of Section 3, we can show inductively that the coefficients d_k are functions of $p(t)$. This implies that $z(t, c)$ is a function of $p(t)$. Therefore, in a neighborhood of the origin, all solutions are ω-periodic and $z = 0$ is a center. $\square$

Remark 4.1. If the coefficient of z^N in the statement of Proposition 1.7 satisfies the condition

$$p'(t)\, f_N(p(t)) = e^{(N-1)\int_0^t q(s)ds}$$

then the coefficient of z^N equals 1. This condition is satisfied when $p(t) = \frac{e^{2\pi i t}}{2\pi i}$, $f_N \equiv 1$, and $q = 2\pi i$. The period in this case is $N - 1$.

5. The Pugh problem

The use of computer algebra has led to significant progress in the investigation of the properties of polynomial differential systems. In this section, we describe an application of computer algebra to find the maximum possible multiplicity of periodic solutions of polynomial differential equations. In particular, we present solutions to the local Pugh problem [14] and Shahshahani conjecture [13]. We mention that Pugh problem was listed as a part of problem 13 in Steve Smale list of 18 open problems for the next century. This problem was considered as a version of Hilbert sixteenth problem. Hilbert sixteenth problem is to estimate the number of limit cycles of polynomial two-dimensional systems. Research related to Hilbert sixteenth problem has derived enormous benefit from the availability of computer algebra.

Consider the differential equation

$$\dot{z} = z^4 + A(t)z^3 + B(t)z^2 \tag{5.1}$$

We show that this local problem can be solved using the method of Groebner bases. We give an automatic means of finding the maximum possible multiplicity of a periodic solution. The algorithm involves computing Groebner bases. Computer algebra systems, such as Maple, can be used to implement this algorithm. If the equation does not have a complex center, then the local Pugh problem is solvable by our procedure. The algorithm for computing the maximum possible multiplicity is then described in this section. We apply the algorithm for equations in which the coefficients $A(t)$ and $B(t)$ are polynomial functions in t, and in $\cos t$ and $\sin t$. The case $N = 4$ is considered. However, the method works for any N. In the case that the coefficients are polynomial functions in t, we show that the maximum possible multiplicity is the same whether the coefficients are complex or are restricted to be real. When the coefficients are trigonometric polynomials, cases are described where the maximums are not equal. Moreover, we show that the origin is not a complex center in each of the equations considered; these results provide evidences to support our conjecture.

We follow the same procedure of Section 3. For equation (5.1), $d_1(t) \equiv 1$ and the equations satisfied by the $d_n(t)$ (for $n > 1$) are

$$\dot{d}_n = \sum_{i+j+k+l=n} d_i d_j d_k d_l + A \sum_{i+j+k=n} d_i d_j d_k + B \sum_{i+j=n} d_i d_j. \tag{5.2}$$

These equations were integrated by parts repeatedly and the formulae for d_n, with $n \leq 8$, are given in [4]. The calculations become extremely complicated as n increases. It is impossible to accomplish these computations by hand except in the simplest cases.

The next step, in computing the multiplicity, is to consider the quantities

$$\eta_n = d_n(\omega)$$

Note that η_n is a polynomial function in the coefficients of the polynomials A and B. The multiplicity of the origin is k if

$$\eta_2 = \eta_3 = \cdots = \eta_{k-1} = 0, \eta_k \neq 0.$$

We write $\mu_{\max}(\mathcal{C})$ for the maximum possible multiplicity of $z = 0$ for equations in a class $\mathcal{C}$. For the class of equations in which the coefficients are polynomial functions of degree m, Pugh problem is to find $\mu_{\max}$ in terms of m.

Now, we give the formulae of η_n with $n \leq 4$, for equation (5.1). We use a tilde over a function to denote its indefinite integral:

$$\widetilde{f}(t) = \int_0^t f(s)ds$$

Proposition 5.1. *For equation* (5.1)*, the quantities* $\eta_2, \eta_3,$ *and* η_4 *are as follows.*

$$\eta_2 = \int_0^\omega B dt, \quad \eta_3 = \int_0^\omega A dt, \quad \eta_4 = \omega + \int_0^\omega (\widetilde{B}A)dt$$

Proof. The formulae were obtained by solving equations (5.2) recursively. The computations leading to the formulae proceed by sequences of judiciously chosen integrations by parts; for any functions f and g, we make use of the identity

$$\widetilde{f\widetilde{g}} = \widetilde{f}\widetilde{g} - \widetilde{\widetilde{f}g}.$$

These are elementary though together they form a complicated web. To obtain η_n, we reduce $d_n(\omega)$ modulo the ideal generated by $d_2(\omega), \ldots, d_{n-1}(\omega)$. $\square$

The Algorithm. We shall write $\mu_{\max}(\text{real})$ when the coefficients of the polynomials A and B are real numbers, and $\mu_{\max}(complex)$ when the coefficients are complex numbers. It follows from the result in [1], that $\mu_{\max}(\text{real}) < \infty$. We call the set of equations that have this maximum multiplicity, the maximum variety, $V_{\max}$. Similarly, we define $V_{\max}(\text{real})$ and $V_{\max}(complex)$.

To find $\mu_{\max}$, first we integrate recursively to compute the functions $d_n(t)$. Then we consider the expressions $d_n(\omega)$, which are polynomial functions in the coefficients of A and B. To obtain η_n we reduce $d_n(\omega)$ modulo the ideal generated by $d_2(\omega), \ldots, d_{n-1}$. We stop until the system $\eta_2 = \eta_3 = \cdots = \eta_k = 0$ has no real solutions. In this case $\mu_{\max} = k$. From the theory of Groebner bases, the Groebner basis of the ideal $\langle \eta_2, \eta_3, \ldots, \eta_k \rangle$ is $\langle 1 \rangle$ if and only if the system $\eta_2 = \eta_3 = \cdots = \eta_k = 0$ has no complex solutions. So, we have to verify that this maximum multiplicity can be attained by certain real values of coefficients. The procedure gives an upper bound for $\mu_{\max}$. However, for the equations in which the coefficients are polynomial functions of t that we will consider, $\mu_{\max}(\text{real}) = \mu_{\max}(complex) < \infty$.

The algorithm for computing $\mu_{\max}$ can be summarized as follows:

- Input: functions $A(t)$ and $B(t)$ which are polynomials in t, or in $\cos t$ and $\sin t$.
- Integrate to compute $d_n(t)$.
- Compute $d_n(\omega)$.
- Find η_n by reducing $d_n(\omega)$ modulo $\langle \eta_2, \eta_3, \ldots, \eta_{n-1} \rangle$.
- Stop when the Groebner basis $\langle \eta_2, \eta_3, \ldots, \eta_k \rangle$ is $\langle 1 \rangle$.
- Output: $\mu_{\max} = k$.

For the details related to Groebner bases, we refer to [8]. We use the software package Maple 8 to compute Groebner bases. Some of the Groebner bases are not given, but they are available upon request.

Now, we consider the first class of coefficients. Here, we assume that $\omega = 1$. Let

$$B(t) = C_1 + at + bt^2,$$
$$A(t) = C_2 + ct + dt^2 + et^3.$$

If $\eta_2 = \eta_3 = 0$, then

$$12C_1 + 6a + 4b + 3c = 0, \quad 12C_2 + 6d + 4e + 3f = 0.$$

This class of equations was used in [3] to construct equations with 10 real periodic solutions. We substitute the values of C_1 and C_2 from these equations and then compute the Groebner basis of the ideal $\langle \eta_4, \eta_5, \ldots, \eta_{10} \rangle$. This basis is $\langle 1 \rangle$. Therefore, the maximum possible multiplicity is 10. To find the equations that have this maximum multiplicity, we compute the Groebner basis $\langle \eta_4, \eta_5, \ldots, \eta_9 \rangle$. This set of equations has 15 solutions, counting multiplicity, and at least one of the solutions is real. We summarize the result for this class in the following lemma.

Lemma 5.2. *If $A(t)$ and $B(t)$ are polynomial functions in t of degrees 3 and 2, respectively, then $\mu_{\max}(\text{real}) = \mu_{\max}(\text{complex}) = 10$. Moreover, $V_{\max}$ is a zero-dimensional ideal.*

The Groebner basis is computed with the graded reverse lexicographic term order and with respect to the list $[b, c, d, a, e]$; this term order usually gives more compact Groebner basis. Then, the basis is changed to the lexicographic term order, which is the most suitable to eliminate variables from a set of equations. The last equation in this basis is a polynomial in e of degree 15; it has at least one real solution. The other variables are given explicitly as functions of e.

The next class has the coefficients

$$B(t) = C_1 + at,$$
$$A(t) = C_2 + bt + ct^2 + dt^3 + et^4 + ft^5.$$

Let $2C_1 + a = 0$ and $60C_2 + 30b + 20c + 15d + 12e + 10f = 0$ for $\eta_2 = \eta_3 = 0$. We compute the Groebner basis as in the first case. This gives $\langle \eta_4, \eta_5, \ldots, \eta_{10} \rangle = \langle 1 \rangle$. Moreover, $\langle \eta_4, \eta_5, \ldots, \eta_9 \rangle$ has a polynomial equation in f of degree 6, which has two real solutions and four complex non-real solutions. The other variables are given as functions of f. The result for this class are given in the following lemma.

Lemma 5.3. *If $A(t)$ and $B(t)$ are polynomial functions in t of degrees 5 and 1, respectively, then $\mu_{\max}(\text{real}) = \mu_{\max}(\text{complex}) = 10$. Moreover, $V_{\max}$ is a zero-dimensional ideal.*

In the last class of polynomials in t, we let

$$A(t) = C_1 + bt + ct^2 + dt^3,$$
$$B(t) = C_2 + at.$$

If $\eta_2 = \eta_3 = 0$, then

$$12C_1 + 6b + cb + 3d = 0, 2C_2 + a = 0.$$

We substitute the values of C_1 and C_2 from these equations and then compute the Groebner basis of the ideal $\langle \eta_4, \eta_5, \ldots, \eta_8 \rangle$. This basis is $\langle 1 \rangle$. Therefore, the

maximum possible multiplicity is 8. To find the equations that have this maximum multiplicity, we compute the Groebner basis $\langle \eta_4, \eta_5, \ldots, \eta_7 \rangle$. The last equation, which is a polynomial in a, has two real solutions and four complex non-real solutions. The other variables are given in terms of a.

Lemma 5.4. *If $A(t)$ and $B(t)$ are polynomial functions in t of degrees 3 and 1, respectively, then $\mu_{\max}(\text{real}) = \mu_{\max}(complex) = 8$. Moreover, $V_{\max}$ is a zero-dimensional ideal.*

$$\langle 8695641600\, b - 6086949120\, d - 773773\, a^5 + 3151791360\, a^2,$$
$$8695641600 c + 13043462400 d + 773773\, a^5 - 3151791360\, a^2,$$
$$-50295245\, a^4 + 37439568\, d^2 + 118513886400\, a,$$
$$-120772800\, d + 70343\, da^3, 773773\, a^6 - 3151791360\, a^3 + 3130430976000 \rangle$$

Now, we consider classes of coefficients which are polynomial functions in $\cos t$ and $\sin t$. Here, we take $\omega = 2\pi$.

Lemma 5.5. *If $A(t)$ and $B(t)$ are homogeneous polynomial functions in $\cos t$ and $\sin t$ of degree 1 or 2, then $\mu_{\max}(\text{real}) = 6$ and $\mu_{\max}(complex) = 7$. The sets $V_{\max}(\text{real})$ and $V_{\max}(complex)$ are not zero-dimensional ideals.*

In the case $A(t) = c\cos t + d\sin t$, and $B(t) = a\cos t + b\sin t$. It is clear that $\eta_2 = \eta_3 = 0$. From Maple, we have:

$$\langle \eta_4, \eta_5, \eta_6 \rangle = \langle a^2 + b^2, bc - ad - 2 \rangle.$$

This set has complex solutions but does not have a real solution. Moreover

$$\langle \eta_4, \eta_5, \eta_6, \eta_7 \rangle = \langle 1 \rangle.$$

For the other case $A(t) = c\cos^2 t + d\cos t \sin t + c_1 \sin^2 t$ and $B(t) = a\cos^2 t + b\cos t \sin t + a_1 \sin^2 t$. We have $\eta_2 = a + a_1$ and $\eta_3 = c + c_1$. If $a_1 = -a$ and $c_1 = -c$, then Maple gives

$$\langle \eta_4, \eta_5, \eta_6 \rangle = \langle 4a^2 + b^2, bc - ad - 8 \rangle,$$

and

$$\langle \eta_4, \eta_5, \eta_6, \eta_7 \rangle = \langle 1 \rangle.$$

A similar argument proves the Lemma.

If one of the coefficients $A(t)$ and $B(t)$ contains only terms with of even degrees and the other coefficient contains only terms of odd degrees, then it follows from 5.1 Proposition that $\mu_{\max}(\text{real}) = \mu_{\max}(complex) = 4$. When the coefficients are homogeneous polynomials of degree 3, the Groebner basis becomes very large.

Lemma 5.6. *Let $A(t)$ be a homogeneous polynomial of degree 3, and $B(t)$ is a homogeneous polynomial of degree 1. The solution $z = 0$ is not a center. Moreover, $\mu_{\max}(\text{real}) = 8$, and $\mu_{\max}(complex) = 10$.*

We take $B(t) = a\cos t + b\sin t$. Using the identities $\sin^3 t = \sin t(1 - \cos^2 t)$ and $\cos^3(t) = \cos t(1 - \sin^2 t)$, we can write a homogeneous polynomial of degree 3 in the following form:

$$A(t) = c\cos t + d\sin t + e\cos t \sin^2 t + f\sin t \cos^2 t.$$

The Groebner basis of $\langle \eta_4, \eta_5, \eta_6, \eta_7 \rangle$ has a real solution; we can take $a = d = 1, c = f = 0, e^2 = 432$, and $36\, b = e$. On the other hand, $\langle \eta_4, \ldots, \eta_8 \rangle$ does not have a real

solution but $\langle \eta_4, \ldots, \eta_9 \rangle \neq \langle 1 \rangle$, and $\langle \eta_4, \ldots, \eta_{10} \rangle = \langle 1 \rangle$. On the other hand, η_8 has the form

$$\eta_8 = \pi\left((5/8)\,a^4 + (5/8)\,b^4 + (5/4)\,a^2\,b^2 + (1/96)\,e^2 + (1/96)\,f^2\right).$$

It is clear that $\eta_8 \neq 0$, when the coefficients are restricted to be real numbers.

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